Mathematical Problems in Mechanics/Differential Geometry

Continuity in $H^1$-norms of surfaces in terms of the $L^1$-norms of their fundamental forms

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Abstract

The main purpose of this Note is to show how a ‘nonlinear Korn’s inequality on a surface’ can be established. This inequality implies in particular the following interesting per se sequential continuity property for a sequence of surfaces. Let $\omega$ be a domain in $\mathbb{R}^2$, let $\theta : \overline{\omega} \to \mathbb{R}^3$ be a smooth immersion, and let $\theta^k : \overline{\omega} \to \mathbb{R}^3$, $k \geq 1$, be mappings with the following properties: They belong to the space $H^1(\omega)$; the vector fields normal to the surfaces $\theta^k(\omega)$, $k \geq 1$, are well defined a.e. in $\omega$ and they also belong to the space $H^1(\omega)$; the principal radii of curvature of the surfaces $\theta^k(\omega)$ stay uniformly away from zero; and finally, the three fundamental forms of the surfaces $\theta^k(\omega)$ converge in $L^1(\omega)$ toward the three fundamental forms of the surface $\theta(\omega)$ as $k \to \infty$. Then, up to proper isometries of $\mathbb{R}^3$, the surfaces $\theta^k(\omega)$ converge in $H^1(\omega)$ toward the surface $\theta(\omega)$ as $k \to \infty$. To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).

Résumé

Continuité en norme $H^1$ de surfaces en terme des normes $L^1$ de leurs formes fondamentales. L’objectif principal de cette Note est de montrer comment on peut établir une « inégalité de Korn non linéaire sur une surface ». Cette inégalité implique en particulier la propriété de continuité séquentielle suivante, intéressante par elle-même. Soit $\omega$ un domaine de $\mathbb{R}^2$, soit $\theta : \overline{\omega} \to \mathbb{R}^3$ une immersion régulière, et soit $\theta^k : \overline{\omega} \to \mathbb{R}^3$, $k \geq 1$, des applications ayant les propriétés suivantes : Elles appartiennent à l’espace $H^1(\omega)$ ; les champs de vecteurs normaux aux surfaces $\theta^k(\omega)$, $k \geq 1$, sont définis presque partout dans $\omega$ et appartiennent aussi à l’espace $H^1(\omega)$ ; les modules des rayons de courbure principaux des surfaces $\theta^k(\omega)$ sont uniformément minorés par une constante strictement positive ; finalement, les trois formes fondamentales des surfaces $\theta^k(\omega)$ convergent dans $L^1(\omega)$ vers les trois formes fondamentales de la surface $\theta(\omega)$ lorsque $k \to \infty$. Alors, à des isométries propres de $\mathbb{R}^3$ près, les surfaces $\theta^k(\omega)$ convergent dans $H^1(\omega)$ vers la surface $\theta(\omega)$ lorsque $k \to \infty$. Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).

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1. Notations and other preliminaries

The symbols $M^n$, $S^n$, and $O^n$ respectively designate the sets of all real matrices of order $n$, of all real symmetric matrices of order $n$, and of all real orthogonal matrices $R$ of order $n$ with $\det R = 1$. The Euclidean norm of a vector $b \in \mathbb{R}^n$ is denoted $|b|$ and $|A| := \sup_{\|b\|_1} |Ab|$ denotes the spectral norm of a matrix $A \in M^n$.

Let $U$ be an open subset in $\mathbb{R}^n$. Given any smooth enough mapping $\chi : U \rightarrow \mathbb{R}^n$, we let $\nabla \chi(x) \in M^n$ denote the gradient matrix of the mapping $\chi$ at $x \in U$ and we let $\partial_i \chi(x)$ denote the $i$th column of the matrix $\nabla \chi(x)$. Given any mapping $F \in L^1(U; \mathbb{R}^n)$, we let

$$\|F\|_{L^1(U; \mathbb{R}^n)} := \int_U |F(x)| \, dx,$$

and, given any mapping $\chi \in H^1(U; \mathbb{R}^n)$, we let

$$\|\chi\|_{H^1(U; \mathbb{R}^n)} := \left\{ \int_U |\chi(x)|^2 + \sum_{i=1}^n |\partial_i \chi(x)|^2 \right\}^{1/2}.$$

A domain $U$ in $\mathbb{R}^n$ is an open and bounded subset of $\mathbb{R}^n$ with a boundary that is Lipschitz-continuous in the sense of Adams [1] or Nečas [10], the set $U$ being locally on the same side of its boundary. If $U$ is a domain in $\mathbb{R}^n$, the space $C^1(\overline{U}; \mathbb{R}^{m})$ consists of all vector-valued mappings $\chi \in C^1(U; \mathbb{R}^m)$ that, together with all their partial derivatives of the first order, possess continuous extensions to the closure $\overline{U}$ of $U$. The space $C^1(\overline{U}; \mathbb{R}^{m})$ thus also consists of restrictions to $\overline{U}$ of all mappings in the space $C^1(\mathbb{R}^n; \mathbb{R}^m)$ (for a proof, see, e.g., [13] or [7]).

Latin indices and exponents henceforth range in the set $\{1, 2, 3\}$ save when they are used for indexing sequences, Greek indices and exponents range in the set $\{1, 2\}$, and the summation convention is used in conjunction with these rules.

The notations $(a_{\alpha\beta})$, $(a^{\alpha\beta})$, $(b^\alpha_i)$, and $(g_{i\alpha})$ respectively designate matrices in $M^2$ and $M^3$ with components $a_{\alpha\beta}, a^{\alpha\beta}, b^\alpha_i$, and $g_{i\alpha}$, the index or exponent $\alpha$ and the index $i$ designating here the row index.

Complete proofs of the results announced in this Note are found in [3].

2. A nonlinear Korn inequality on a surface

Our main result is a nonlinear Korn inequality on a surface (Theorem 2.4), the proof of which relies on several preliminaries, a crucial one being the following nonlinear Korn inequality on an open subset in $\mathbb{R}^n$ recently established by Ciarlet and Mardare [6]. Its long, and sometimes technical, proof hinges in particular on a fundamental ‘geometric rigidity lemma’ due to Friesecke et al. [9] and on a general methodology reminiscent to that used in Ciarlet and Laurent [4]. See also Reshetnyak [12] for related results.

**Theorem 2.1.** Let $\Omega$ be a domain in $\mathbb{R}^n$. Given any mapping $\Theta \in C^1(\overline{\Omega}; \mathbb{R}^n)$ satisfying $\det \nabla \Theta > 0$ in $\overline{\Omega}$, there exists a constant $C(\Theta)$ with the following property: Given any mapping $\tilde{\Theta} \in H^1(\Omega; \mathbb{R}^n)$ satisfying $\det \nabla \tilde{\Theta} > 0$ a.e. in $\Omega$, there exist a vector $b = b(\Theta, \tilde{\Theta}) \in \mathbb{R}^n$ and a matrix $R = R(\Theta, \tilde{\Theta}) \in O^n$ such that

$$\|b + R\tilde{\Theta} - \Theta\|_{H^1(\Omega; \mathbb{R}^n)} \leq C(\Theta)\|\nabla \Theta^{-1} \nabla \tilde{\Theta} - \nabla \Theta^{-1} \nabla \Theta\|_{L^1(\Omega; \mathbb{R}^n)}^{1/2}.$$ 

The next two lemmas show that some classical definitions and properties pertaining to surfaces in $\mathbb{R}^3$ still hold under less stringent regularity assumptions than the usual ones (these definitions and properties are traditionally given and established under the assumptions that the immersions denoted $\theta$ in Lemma 2.2 and $\tilde{\theta}$ in Lemma 2.3 belong to the space $C^2(\overline{\omega}; \mathbb{R}^3)$). For this reason, we shall continue to use the classical terminology, e.g., normal
vector field (for \(a_3\) or \(\tilde{a}_3\), or first, second, and third fundamental forms (for \((a_{\alpha\beta})\) or \((\tilde{a}_{\alpha\beta})\), \((b_{\alpha\beta})\) or \((\tilde{b}_{\alpha\beta})\), and \((c_{\alpha\beta})\) or \((\tilde{c}_{\alpha\beta})\)), etc. If \(y = (y_\omega)\) designates the generic point in a domain \(\omega\) in \(\mathbb{R}^2\), we let \(\partial_\omega := \partial/\partial y_\omega\).

**Lemma 2.2.** Let \(\omega\) be a domain in \(\mathbb{R}^2\) and let \(\theta \in C^1(\omega; \mathbb{R}^3)\) be an immersion such that \(a_3 := \frac{a_1 \wedge a_2}{|a_1 \wedge a_2|} \in C^1(\omega; \mathbb{R}^3)\), where \(a_\alpha := \partial_\alpha \theta\). Then the functions

\[
a_{\alpha\beta} := a_\alpha \cdot a_\beta, \quad b_{\alpha\beta} := -\partial_\alpha a_3 \cdot a_\beta, \quad b^\alpha := a^{\alpha\sigma} b_{\sigma\beta}, \quad c_{\alpha\beta} := \partial_\alpha a_3 \cdot \partial_\beta a_3,
\]

where \((a^{\alpha\beta}) := (a_{\alpha\beta})^{-1}\), belong to the space \(C^0(\omega)\), and \(b_{\alpha\beta} = b_{\beta\alpha}\). Define the mapping \(\Theta : \omega \times \mathbb{R} \rightarrow \mathbb{R}^3\) by

\[
\Theta(y, x_3) := \theta(y) + x_3 a_3(y) \quad \text{for all } (y, x_3) \in \bar{\omega} \times \mathbb{R}.
\]

Then \(\Theta \in C^1(\bar{\omega} \times \mathbb{R}; \mathbb{R}^3)\). Furthermore,

\[
\det \nabla \Theta(y, x_3) = \sqrt{a(y)} \left\{ 1 - 2H(y)x_3 + K(y)x_3^2 \right\} \quad \text{for all } (y, x_3) \in \bar{\omega} \times \mathbb{R},
\]

where the functions

\[
a := \det(a_{\alpha\beta}) = |a_1 \wedge a_2|^2, \quad H := \frac{1}{2} (b_1^2 + b_2^2), \quad K := b_1^3 b_2^2 - b_1 b_2^3
\]

belong to the space \(C^0(\bar{\omega})\). Finally, let

\[
(g_{ij}) := \nabla \Theta^T \nabla \Theta.
\]

Then the functions \(g_{ij} = g_{ji}\) belong to the space \(C^0(\bar{\omega} \times \mathbb{R})\) and they are given by

\[
g_{\alpha\beta}(y, x_3) = a_{\alpha\beta}(y) - 2x_3 b_{\alpha\beta}(y) + x_3^2 c_{\alpha\beta}(y) \quad \text{and} \quad g_{\beta\alpha}(y, x_3) = \delta_{\beta\alpha}
\]

for all \((y, x_3) \in \bar{\omega} \times \mathbb{R}\).

**Sketch of proof.** Since the symmetric matrices \((a_{\alpha\beta}(y))\) are positive-definite at all points \(y \in \bar{\omega}\), the inverse matrices \((a^{\alpha\beta}(y))\) are well defined and also positive-definite at all points \(y \in \bar{\omega}\), and the functions \(a^{\alpha\beta}\) belong to the space \(C^0(\bar{\omega})\). Therefore the functions \(b_{\alpha\beta}^\alpha\) are well-defined and they also belong to the space \(C^0(\bar{\omega})\).

The symmetry \(b_{\beta\alpha} = b_{\alpha\beta}\) is clear if \(\theta \in C^2(\bar{\omega}; \mathbb{R}^3)\) since \(b_{\alpha\beta} = a_3 \cdot \partial_\alpha a_\beta\) in this case. As shown in the proof of Theorem 3 of Ciarlet and Mardare [5], this symmetry still holds under the weaker assumptions of Lemma 2.2.

Thanks to the relations \(\partial_\alpha (a_3 \cdot a_3) = 0\), the classical formulas of Weingarten, viz.,

\[
\partial_\alpha a_3 = -b_{\alpha\beta} a_\beta,
\]

still hold under the present assumptions. The expressions giving the functions \(\det \nabla \Theta\) and \(g_{ij}\) then follow from this observation. \(\square\)

**Lemma 2.3.** Let \(\omega\) be a domain in \(\mathbb{R}^2\) and let there be given a mapping \(\tilde{\theta} \in H^1(\omega; \mathbb{R}^3)\) such that \(\tilde{a}_1 \wedge \tilde{a}_2 \neq 0\) a.e. in \(\omega\), where \(\tilde{a}_\alpha := \partial_\alpha \tilde{\theta}\), and such that

\[
\tilde{a}_3 := \frac{\tilde{a}_1 \wedge \tilde{a}_2}{|\tilde{a}_1 \wedge \tilde{a}_2|} \in H^1(\omega; \mathbb{R}^3).
\]

Then the functions

\[
\tilde{a}_{\alpha\beta} := \tilde{a}_\alpha \cdot \tilde{a}_\beta, \quad \tilde{b}_{\alpha\beta} := -\partial_\alpha \tilde{a}_3 \cdot \tilde{a}_\beta, \quad \tilde{c}_{\alpha\beta} := \partial_\alpha \tilde{a}_3 \cdot \partial_\beta \tilde{a}_3
\]

are well defined a.e. in \(\omega\), they belong to the space \(L^1(\omega)\), and \(\tilde{b}_{\alpha\beta} = \tilde{b}_{\beta\alpha}\). Define the mapping \(\tilde{\Theta} : \omega \times \mathbb{R} \rightarrow \mathbb{R}^3\) by

\[
\tilde{\Theta}(y, x_3) := \tilde{\theta}(y) + x_3 \tilde{a}_3(y) \quad \text{for almost all } (y, x_3) \in \omega \times \mathbb{R}.
\]

Then \(\tilde{\Theta} \in H^1(\omega \times |\delta, \delta; \mathbb{R}^3)\) for any \(\delta > 0\). Furthermore,

\[
\det \nabla \tilde{\Theta}(y, x_3) = \sqrt{\tilde{a}(y)} \left\{ 1 - 2\tilde{H}(y)x_3 + \tilde{K}(y)x_3^2 \right\} \quad \text{for almost all } (y, x_3) \in \omega \times \mathbb{R},
\]
where
\[
\tilde{a} := \det(\tilde{a}_{ab}) = |\tilde{a}_1 \wedge \tilde{a}_2|^2, \quad \tilde{H} := \frac{1}{2}(\tilde{b}_1^2 + \tilde{b}_2^2), \quad \tilde{K} := \tilde{b}_1^2 \tilde{b}_2 - \tilde{b}_1 \tilde{b}_2^2, \quad \tilde{b}_a^a := \tilde{a}^{b\alpha} \tilde{a}_{ab},
\]
and \((\tilde{a}^{ab}) := (\tilde{a}_{ab})^{-1}\). Finally, let
\[
(\tilde{g}_{ij}) := \nabla \tilde{\Theta}^T \nabla \tilde{\Theta} \quad \text{a.e. in } \omega \times \mathbb{R}.
\]
Then the functions \(\tilde{g}_{ij} = \tilde{g}_{ji}\) belong to the space \(L^1(\omega \times ]-\delta, \delta[)\) for any \(\delta > 0\) and they are given by
\[
\tilde{g}_{ab}(y, x_3) = \tilde{a}_{ab}(y) - 2x_3 \tilde{b}_{ab}(y) + x_3^2 \tilde{c}_{ab}(y) \quad \text{and} \quad \tilde{g}_{i3}(y, x_3) = \delta_{i3}
\]
for almost all \((y, x_3) \in \omega \times \mathbb{R}\).

**Sketch of proof.** The proof is analogous to that of Lemma 2.2. The symmetry \(\tilde{b}_{ab} = \tilde{b}_{ba}\) again follow from Theorem 3 of [5]. Note that, although the functions \(\tilde{a}, \tilde{H}, \tilde{K}\) and \(\tilde{b}_a^a\) are well defined a.e. in \(\omega\) under the assumptions of Lemma 2.3, they do not necessarily belong to the space \(L^1(\omega)\). \(\square\)

We now state the announced nonlinear Korn inequality on a surface. The notations are the same as those in Lemmas 2.2 and 2.3.

**Theorem 2.4.** Let there be given a domain \(\omega\) in \(\mathbb{R}^2\), an immersion \(\theta \in C^1(\overline{\omega}; \mathbb{R}^3)\) such that \(a_3 \in C^1(\overline{\omega}; \mathbb{R}^3)\), and \(\varepsilon > 0\).

Then there exists a constant \(c(\theta, \varepsilon)\) with the following property: Given any mapping \(\tilde{\theta} \in H^1(\omega; \mathbb{R}^3)\) such that \(\tilde{a}_1 \wedge \tilde{a}_2 \neq 0\) a.e. in \(\omega\), \(\tilde{a}_3 \in H^1(\omega; \mathbb{R}^3)\), and
\[
|\tilde{H}| \leq \frac{1}{\varepsilon} \quad \text{and} \quad \tilde{K} \geq -\frac{1}{\varepsilon^2} \quad \text{a.e. in } \omega,
\]
there exist a vector \(b := b(\theta, \tilde{\theta}, \varepsilon) \in \mathbb{R}^3\) and a matrix \(R = R(\theta, \tilde{\theta}, \varepsilon) \in O_3\) such that
\[
\|b + R\tilde{\Theta} - \theta\|_{H^1(\omega; \mathbb{R}^3)} + \varepsilon \|Ra_3 - a_3\|_{H^1(\omega; \mathbb{R}^3)} \\
\leq c(\theta, \varepsilon) \left\{ \left( \tilde{a}_{ab} - a_{ab} \right) \right\}_{L^2(\omega; \mathbb{S})}^{1/2} + \varepsilon \left\{ \left( \tilde{b}_{ab} - b_{ab} \right) \right\}_{L^2(\omega; \mathbb{S})}^{1/2} + \varepsilon \left\{ \left( \tilde{c}_{ab} - c_{ab} \right) \right\}_{L^2(\omega; \mathbb{S})}^{1/2}.
\]

**Sketch of proof.** Without loss of generality, we assume that \(\varepsilon \leq 1\). Let the mappings \(\Theta : \bar{\omega} \times \mathbb{R} \to \mathbb{R}^3\) and \(\tilde{\Theta} : \omega \times \mathbb{R} \to \mathbb{R}^3\) be constructed as in Lemmas 2.2 and 2.3 from the mappings \(\theta : \bar{\omega} \to \mathbb{R}^3\) and \(\tilde{\theta} : \omega \to \mathbb{R}^3\) appearing in Theorem 2.4. Then there exists a constant \(\delta(\theta) > 0\) such that
\[
\det \nabla \Theta > 0 \quad \text{in } \bar{\Omega} \quad \text{and} \quad \det \nabla \tilde{\Theta} > 0 \quad \text{a.e. in } \Omega,
\]
where \(\bar{\Omega} = \Omega(\theta, \varepsilon) := \omega \times ]-\delta(\theta)\varepsilon, \delta(\theta)\varepsilon[\).

Theorem 2.1 then shows that there exists a constant \(c_0(\theta, \varepsilon)\) with the following property: Given any \(\varepsilon > 0\) and any mappings \(\theta\) and \(\tilde{\theta}\) satisfying the assumptions of Theorem 2.4, there exist a vector \(b := b(\theta, \tilde{\theta}, \varepsilon) \in \mathbb{R}^3\) and a matrix \(R = R(\theta, \tilde{\theta}, \varepsilon) \in O_3\) such that
\[
\|b + R\tilde{\Theta} - \theta\|_{H^1(\Omega; \mathbb{R}^3)} \leq c_0(\theta, \varepsilon) \| (\tilde{g}_{ij} - g_{ij}) \|_{L^2(\Omega; \mathbb{S})}^{1/2}.
\]

The rest of the proof is showing that there exists constants \(c_1(\theta)\) and \(c_3(\theta)\) such that
\[
\|b + R\tilde{\Theta} - \theta\|_{H^1(\Omega; \mathbb{R}^3)} \geq c_1(\theta) \varepsilon^{1/2} \left\{ \|b + R\tilde{\theta} - \theta\|_{H^1(\omega; \mathbb{R}^3)} + \varepsilon \|Ra_3 - a_3\|_{H^1(\omega; \mathbb{R}^3)} \right\}.
\]
\[
\left\| \tilde{g}_{ij} - g_{ij} \right\|_{L^1(\Omega; \mathbb{S}^2)}^{1/2} \\
\leq c_2(\epsilon) \epsilon^{1/2} \left\| (\tilde{a}_{a\beta} - a_{a\beta}) \right\|_{L^1(\omega; \mathbb{S}^2)}^{1/2} + \epsilon^{1/2} \left\| (\tilde{b}_{a\beta} - b_{a\beta}) \right\|_{L^1(\omega; \mathbb{S}^2)}^{1/2} + \epsilon \left\| (\tilde{c}_{a\beta} - c_{a\beta}) \right\|_{L^1(\omega; \mathbb{S}^2)}^{1/2}.
\]

The announced inequality then follows with \( c(\epsilon, \epsilon) := c_0(\epsilon, \epsilon)c_1(\epsilon)^{-1}c_2(\epsilon). \) \( \square \)

3. Commentary

If a mapping \( \tilde{\theta} : \omega \rightarrow \mathbb{R}^3 \) is a smooth immersion, the associated functions \( \tilde{H} \) and \( \tilde{K} \) simply represent the mean, and Gaussian, curvatures of the surface \( \tilde{\theta}(\omega) \). It is well known that these functions are also given by \( \tilde{H} = \frac{1}{\bar{2}} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \) and \( \tilde{K} = \frac{1}{\bar{2}} \left( \frac{1}{R_1 R_2} \right) \), where \( R_k \) are the principal radii of curvature along the surface \( \tilde{\theta}(\omega) \) (with the usual convention that \( |R_1(y)| \) may take the value \( +\infty \) at some points \( y \in \omega \)).

It is then easily seen that the assumptions \( |\tilde{H}| \leq \frac{1}{2} \) and \( \tilde{K} \geq -\frac{1}{\bar{2}} \) in \( \omega \) made in Theorem 2.4 imply that \( |\tilde{R}_a| \geq c \epsilon \) in \( \omega \) and that, conversely, \( |\tilde{R}_a| \geq \epsilon \) in \( \omega \) implies that \( |\tilde{H}| \leq \frac{1}{\bar{2}} \) and \( \tilde{K} \geq -\frac{1}{\bar{2}} \) in \( \omega \), for some ad hoc numerical constants \( c \) and \( \epsilon \). Hence the assumptions made on the mappings \( \tilde{\theta} \) in Theorem 2.4 have a very simple geometric interpretation: they mean that the principal radii of curvature of all the ‘admissible’ surfaces \( \tilde{\theta}(\omega) \) must stay uniformly away from zero. Naturally, such principal radii of curvature are possibly understood only in a generalized sense, viz., as the inverses of the eigenvalues of the associated matrix \( (\tilde{b}_{a\beta}) \).

Let there be given a mapping \( \theta \in H^1(\omega; \mathbb{R}^3) \) such that \( \tilde{a}_1 \land \tilde{a}_2 \neq 0 \) a.e. in \( \omega \) and \( \tilde{a}_3 \in H^1(\omega; \mathbb{R}^3) \). Then a mapping \( \tilde{\theta} : \omega \rightarrow \mathbb{R}^3 \) is said to be properly isometrically equivalent to the mapping \( \tilde{\theta} \) if there exist a vector \( b \in \mathbb{R}^3 \) and a matrix \( R \in O_3^+ \) such that \( \tilde{\theta} = b + R\tilde{\theta} \). If this is the case, then \( \tilde{\theta} \in H^1(\omega; \mathbb{R}^3) \), \( \tilde{a}_1 \land \tilde{a}_2 \neq 0 \) a.e. in \( \omega \), and \( \tilde{a}_3 \in H^1(\omega; \mathbb{R}^3) \) (with self-explanatory notations), and the two surfaces \( \tilde{\theta}(\omega) \) and \( \tilde{\theta}(\omega) \) share the same three fundamental forms in the space \( L^1(\omega; \mathbb{S}^2) \).

One application of the key inequality of Theorem 2.4 is then the following result of sequential continuity for surfaces: Let \( \theta^k \in H^1(\omega; \mathbb{R}^3) \), \( k \geq 1 \), be mappings with the following properties: The vector fields normal to the surfaces \( \theta^k(\omega) \) are well defined a.e. in \( \omega \) and they also belong to the space \( H^1(\omega; \mathbb{R}^3) \), there exists a constant \( \epsilon > 0 \) such that the principal radii of curvatures \( R^k_\alpha \) of the surfaces \( \theta^k(\omega) \) satisfy \( |R^k_\alpha| \geq \epsilon > 0 \) a.e. in \( \omega \) for all \( k \geq 1 \), and finally,

\[
\begin{align*}
(a_{a\beta}) & \quad \rightarrow \quad (a_{a\beta}) \quad \text{in} \quad L^1(\omega; \mathbb{S}^2), \\
(b_{a\beta}) & \quad \rightarrow \quad (b_{a\beta}) \quad \text{in} \quad L^1(\omega; \mathbb{S}^2), \\
(c_{a\beta}) & \quad \rightarrow \quad (c_{a\beta}) \quad \text{in} \quad L^1(\omega; \mathbb{S}^2),
\end{align*}
\]

where \( (a_{a\beta}), (b_{a\beta}), (c_{a\beta}) \) are the three fundamental forms of a surface \( \theta(\omega) \), where \( \theta \in C^1(\bar{\omega}; \mathbb{R}^3) \) is an immersion satisfying \( a_3 \in C^1(\bar{\omega}; \mathbb{R}^3) \). Then there exist mappings \( \tilde{\theta}^k \) that are properly isometrically equivalent to the mappings \( \theta^k \), \( k \geq 1 \), such that

\[
\tilde{\theta}^k \quad \rightarrow \quad \tilde{\theta} \quad \text{and} \quad \tilde{a}_3^k \quad \rightarrow \quad a_3 \quad \text{in} \quad H^1(\omega; \mathbb{R}^3).
\]

Such a sequential continuity property generalizes that previously obtained by Ciarlet [2] and by Ciarlet and Mardare [8] and Szopos [11], for the topologies of the spaces \( C^m(\omega) \), and \( C^m(\overline{\omega}) \), respectively.

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References