Local Time Step for a Finite Volume Scheme

I. Faille F. Nataf*, F. Willien, S. Wolf**

*: Laboratoire J. L. Lions
**: Université Pierre et Marie Curie
Outline of this talk

- Industrial context
- Bibliography
- Local Time step strategy
- Numerical results
Reservoir simulation

- Oil Reservoir
  - reservoir: hundred meters to km
  - wells: 10 cm

- Reservoir simulation
  - estimate production characteristics
  - calibrate reservoir parameters
  - help to position and manage wells and well trajectories in order to maximize the oil and gas recovery
A simple model: immiscible two phase flow

- Two phase in a porous medium $p = \omega, \sigma$
  - Mass conservation of each phase
    
    \[
    \frac{\partial}{\partial t} (\rho_p \Phi S_p) + \text{div} (\rho_p \vec{v}_p) = f_p
    \]
  - Darcy’s law for each phase
    \[
    \vec{v}_p = -K \frac{k_{rp}}{\mu_p} \text{grad} \left( P - P_{c_p} \right) - \rho_p g
    \]
- $\Phi$ porosity, $K$ permeability
- Main unknowns:
  - $P$ pressure
  - $S_p$ saturation of phase $p$
- System
  - Parabolic with respect to $P$ (compressible phases)
  - Non linear hyperbolic ($kr(S)$) or degenerate parabolic if capillary pressure ($P_c$) not neglected
- Large range of parameters
  - Permeability: from 1 mD to 10 Darcy

Relative permeability

Mass conservation of each phase

Darcy’s law for each phase
A simple model: immiscible two phase flow

- **Numerical requirements**
  - **Robust**
    - able to solve a wide range of problems: compositionnal, 3 phase flow...
  - **Locally conservative**

- **Traditional scheme**
  - **Cell centered finite Volume**
    - upwind scheme for the saturation
    - two-point or multi-points scheme for pressure
  - **Fully implicit in Time**
  - **Non linear algebraic system**
    - Newton method
    - Difficult to solve if high flow rates, small cells
Representation of wells

- **Wells**
  - Transient phenomena are initiated at wells:
    - depressurization
    - water injection
    - ....

- **Traditional approach:**
  - a priori knowledge of the singularity induced by a well
  - analytical source terms, in cells at reservoir scale

- **Detailed description of the near wellbore**
  - necessary to represent certain phenomena such as water coning
  - already available in "near well bore" simulator
  - challenge for fullfield simulation
Modeling wells on full field level

- Full field grid with refined near well bore area
Local Time stepping strategy

Locally refined grids induce decreased time step length, due to convergence problems of the Newton method.

Aim: implement a local time stepping strategy
- Parabolic equation
- Parabolic/hyperbolic coupled system
- Applicable to complex fluid flow

Requirements
- Fully Implicit Finite Volume
- Locally conservative
- Reduced cpu-time / small time steps
- Accuracy of the solution
Bibliography(1)

- Ewing, Lazarov, Vassilevski 1990
  - Parabolic problem, Cell centered Finite Difference scheme
  - At the interface between the coarse time step and refined time step zones:
    - Coarse flux = Integral of refined ones
    - Unknowns at "slave" points: constant or linear interpolation
  - Properties
    - Conservative
    - Stability and error analysis for piecewise constant interpolation, loss of accuracy: \( O(DT/h^{1/2}) \)
  - Linear system for all the unknowns between two coarse time levels: Coarse grid preconditioner of the Schur complement where the refined unknowns have been eliminated
Bibliography (2)

- Ewing, Lazarov 1994
  - Parabolic problem
  - Relax the conservation constraint to obtain stability even for linear interpolation of the unknown on the interface
  - Linear interpolation for "slave" points
  - Standard scheme for unknowns of the coarse time steps
  - Stability
  - Similar preconditioner
  - Not conservative
Bibliography(3)

- Dawson, Du, Dupont 1991
  - Parabolic problem, Finite difference scheme
  - Explicit update of interface unknowns with a large interface time step $DT$ on a large mesh $H$
  - Then, interior unknowns are updated by an implicit scheme and interpolation of the interface unknowns
  - Stability condition $DT < cH^2$
  - Not conservative
Bibliography(4)

Mlacnik, Heinemann 2001

- Reservoir simulation, compositional multiphase flow
- Two steps method:
  - one implicit computation on the whole domain with a coarse time step
    - linearly implicit in the refined time-step zone to avoid convergence problems of the non linear solver
    - fully implicit elsewhere
  - one computation in the refined zone with a Neumann boundary condition
- Conservative, fast
- But potentially limited accuracy
  - No control on the continuity of the unknowns at the interface.
Local time stepping strategy

- Based on space-time domain decomposition framework
  - Extends the approach proposed by Ewing et al (1), by generalizing the interface conditions
  - Improves the predictor-corrector method of (4)

- Interface conditions: enforce flux and unknown continuity
  - Classical Dirichlet-Neumann interface conditions, choosing the coarse grid as the master or the slave.

On the interface:

\[
\begin{align*}
\sum_{f} dt_f u_f &= DT_c u_c \\
\sum_{f} dt_f p_f &= DT_c p_c \\
p_f &= p_c \\
u_f &= u_c
\end{align*}
\]

- Dirichlet can be conditions written in the neighboring cells
- For two phase flow, continuity of P, S and phase fluxes.
Local time stepping strategy

- **Solution method**
  - **Predictor stage:**
    - one implicit computation on the whole domain with a coarse time step
    - it gives a first guess for the interface unknowns
  - **Iterative corrector stage**
    - solve alternatively the equations in the refined domain and the coarse one taking the interface condition previously computed
    - stop when both interface conditions are satisfied

- **Remarks:**
  - Numerical scheme and solution method applicable to complex compositional multiphase flow:
    In the predictor stage, linear implicit approximation in the refined time-step zone to avoid convergence problems of the non linear solver
  - Iterative corrector stage can be stopped after the resolution of the domain where a Neumann boundary condition is applied, without loosing conservation
Local time stepping strategy

- Analysis of the numerical scheme

\[
\frac{\partial p}{\partial t}(x, t) - \Delta p(x, t) = f(x, t) \quad \forall x \in \Omega \quad \forall t \in [0, T]
\]

\[
p(x, 0) = p_0(x) \quad \forall x \in \Omega
\]

\[
p(x, t) = 0 \quad \forall x \in \partial \Omega \quad \forall t \in [0, T]
\]

Theorem:
The scheme is

- unconditionally stable
- of order 1 in space and time
  - if the Dirichlet condition is localized on the interface
  - under an assumption on the aspect ratio of the interface cells if the Dirichlet condition is written at cell centers.
Numerical results

Parabolic problem (from Ewing 1994)

\[ \frac{\partial}{\partial t} p(x, t) - \frac{\partial^2 p}{\partial x^2}(x, t) = f(x, t) \quad \forall t \in [0., 0.1] \quad \forall x \in [0., 1.] \]
\[ p(x, t) = 0, \forall x \in \partial \Omega \]
\[ p(x, 0) = 0 \]

- Exact solution:

\[ p(x, t) = e^{\exp(20(t - t^2) - 37x^2 + 8x - 1)} \]

Solution at \( t=0.1 \)
Parabolic problem

interface \( x = 0.25 \)

\[ \text{dt} = 0.002 \quad \text{DT} = 0.02 \]

\[ \text{dx} = 0.01 \quad \text{DX} = 0.1 \]

Error with the exact solution

Coarse DT

Fine dt

after convergence (iterations)

after one iteration
Immiscible two phase flow

- Simple 1D problem, one well and one reservoir

- Constant initial pressure, depressurization at x=0

- Algorithm

Well

Reservoir

Sw=0.1

Sw=1

Linearly implicit

Fully implicit

Coarse DT

Fully, dt

Neumann BC (Fw, Fo)

Dirichlet BC (P, S)

Fully, DT
Immiscible two phase flow

- $D_x=2$ in well zone, $D_x=9$ in reservoir zone
- $DT/dt = 20$
- 2 to 5 iterations per coarse time step
Immiscible two phase flow

- $Dx=2$ in well zone, $Dx=9$ in reservoir zone
- $DT/dt = 10$
- 2 to 5 iterations per coarse time step

→ Good results compared to the fine step solution
Conclusion

- **Local time step strategy**
  - Dirichlet/Neumann interface conditions
  - Iterative solution method, with coarse grid predictor
  - Conservative
  - Applicable to compositional multiphase flow

- **Numerical results on simple 1D tests**
  - Good behavior of the scheme and solution method

- **Currently being implemented in a 3D reservoir simulator.**