GRADIENT CONJECTURE OF R. THOM AND RELATED TOPICS

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Let $f$ be a $C^2$ function in an open set $U \subset \mathbb{R}^n$ and let $\nabla f$ be its gradient in the Euclidean metric. The trajectories of $\nabla f$ are the maximal curves $x(t)$ satisfying

$$\frac{dx}{dt}(t) = \nabla f(x(t)), \quad t \in [0, \beta).$$

If $\nabla f(x_0) = 0$, then clearly it is a very challenging problem to understand the behavior of the trajectories of $\nabla f$ in the neighborhood of the point $x_0$. For a general $C^2$ (or even $C^\infty$) function its gradient trajectory may spiral, oscillate or may have infinite length. In sixties S. Lojasiewicz [L], [L1] proved the following result.

**Lojasiewicz’s theorem:** Let $f$ be a real analytic function. If $x(t)$ has a limit point $x_0 \in U$, i.e. $x(t_\nu) \to x_0$ for some sequence $t_\nu \to \beta$, then the length of $x(t)$ is finite, moreover $\beta = \infty$. Therefore $x(t) \to x_0$ as $t \to \infty$. Note that $\nabla f(x_0) = 0$, since otherwise we could extend $x(t)$ through $x_0$.

Then in early seventies R. Thom wanted to obtain a more precise statement for the gradient trajectories (of an analytic function) when approaching a critical point $x_0$. He wanted to show that a trajectory cannot spiral in some sense. Precisely he formulated it as follows [Th].

**Gradient Conjecture:** Suppose that $x(t) \to x_0$. Then $x(t)$ has a tangent at $x_0$, that is the limit of secants

$$\lim_{t \to \infty} \frac{x(t) - x_0}{|x(t) - x_0|}$$

exists.

In other words, if $\tilde{x}(t)$ is the image of $x(t)$ under the radial projection $\mathbb{R}^n \setminus \{x_0\} \ni x \mapsto \frac{x - x_0}{|x - x_0|} \in S^{n-1}$, then the conjecture claims that $\tilde{x}(t)$ has a limit.

In a joint paper with T. Mostowski, A. Parusiński [KMP] we have answered positively the conjecture. Actually we have proved a stronger result: the length of $\tilde{x}(t)$ can be uniformly bounded for trajectories starting sufficiently close to $x_0$.

The goal of my lectures will be to present main ingredients of our proof, but also related topics and generalizations to a more general context like o-minimal geometry [K], [KP]. Lojasiewicz’s theorem is a consequence of his celebrate gradient inequality [L], [L1], [LT] which states that if $f$ is analytic, then in a neighborhood of $x_0$

$$|\nabla f| \geq c|f - f(x_0)|^\rho,$$ (0.1)

for some $\rho < 1$ and $c > 0$. I will present also an alternative approach cf. [DD], to the estimate for the length of a trajectory of the gradient of an analytic function, which allows

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to obtain uniform bounds for the length in the polynomial case. In fact inequality 0.1 has a generalization to the o-minimal case and even for maps. As consequence we obtain estimates for the length (or volume) of manifolds transverse to the fibers. I will explain important notion of asymptotical critical values and characteristic exponents.

Tentative plan:

(i) Basic facts in subanalytic geometry.
(ii) Łojasiewicz’s gradient inequality, quantitative aspects in the polynomial case. Generalizations to the o-minimal case. Generalizations for the maps.
(iii) Estimates for the length of gradient trajectories.
(iv) Asymptotical critical values and characteristic exponents.
(v) Proof of the gradient conjecture.

REFERENCES


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