Convergence rates for the Vlasov-Fokker-Planck equation and uniform in time propagation of chaos in non convex cases.

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I. Introduction

Processes

Particle system $((X_t^{i,N}, V_t^{i,N}))_{i=1,\dots,N}$, with $X_t^{i,N}, V_t^{i,N} \in \mathbb{R}^d$

$$\begin{cases} dX_t^{i,N} = V_t^{i,N} dt \\ dV_t^{i,N} = \sqrt{2} dB_t^i - V_t^{i,N} dt - \nabla U(X_t^{i,N}) dt - \frac{1}{N} \sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}) dt \\ \mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}} \end{cases}$$

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Underdamped Langevin diffusion (Non linear particle)

$$\begin{cases} d\bar{X}_t = \bar{V}_t dt \\ d\bar{V}_t = \sqrt{2}dB_t - V_t dt - \nabla U(\bar{X}_t)dt - \nabla W * \bar{\mu}_t(\bar{X}_t)dt \\ \bar{\mu}_t = Law(\bar{X}_t) \end{cases} \qquad \text{(NL)}$$

with

$$\nabla W * \bar{\mu}_t(x) = \int_{\mathbb{R}^d} \nabla W(x-y)\bar{\mu}_t(dy)$$

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To quantify this "more or less", we compare the law of any subset of k particles within the N particles system to the law of k independent non-linear particles.

Assumptions on the confinement potential

Assumption

The potential U is non-negative and there exist $\lambda > 0$ and $A \geq 0$ such that

$$\forall x \in \mathbb{R}^d, \qquad \frac{1}{2}\nabla U(x) \cdot x \geq \lambda \left(U(x) + \frac{|x|^2}{4} \right) - A.$$

Furthermore, there is a constant $L_U > 0$ such that

$$\forall x, y \in \mathbb{R}^d \times \mathbb{R}^d, \qquad |\nabla U(x) - \nabla U(y)| \leq L_U |x - y|.$$

Assumptions on the confinement potential

The double-well potential given by

$$U(x) = \begin{cases} (x^2 - 1)^2 & \text{if } |x| \leq 1, \\ (|x| - 1)^2 & \text{otherwise.} \end{cases}$$

satisfies the previous assumptions.

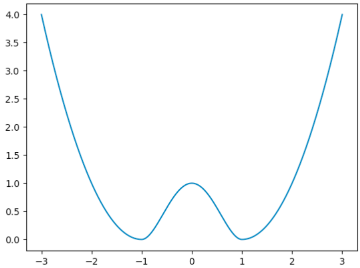

FIGURE – Double well potential

Assumptions on the interaction potential

Assumption

$\nabla W(0) = 0$ *and there exists* $L_W \leq \lambda/8$ *such that*

$$\forall x, y \in \mathbb{R}^d \times \mathbb{R}^d, \qquad |\nabla W(x) - \nabla W(y)| \leq L_W |x - y|.$$

In particular $|\nabla W(x)| \leq L_W |x|$ *for all* $x \in \mathbb{R}^d$*.*

Distance

L1-Wasserstein and L2-Wasserstein distances

Definition

Let μ and ν be two probability measures on $\mathbb{R}^{2d}$. We define

$$\mathcal{W}(\mu,\nu) = \inf_{\Gamma\in\Pi(\mu,\nu)} \int |x-\tilde{x}| + |v-\tilde{v}|\Gamma(d(x,v)d(\tilde{x},\tilde{v}))$$

$$\mathcal{W}_2(\mu,\nu) = \left(\inf_{\Gamma\in\Pi(\mu,\nu)} \int |x-\tilde{x}|^2 + |v-\tilde{v}|^2\Gamma(d(x,v)d(\tilde{x},\tilde{v}))\right)^{1/2}$$

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where the infimum is chosen on all couplings of μ and ν.

Likewise, for μ and ν two probability measures on $\mathbb{R}^{2d}$ and a measurable function $h:\mathbb{R}^{2d}\times\mathbb{R}^{2d}\to\mathbb{R}$, we define

$$\mathcal{W}_h(\mu,\nu) = \inf_{\Gamma\in\Pi(\mu,\nu)} \int h(x,v,\tilde{x},\tilde{v})\,\Gamma\left(d(x,v)\,d(\tilde{x},\tilde{v})\right).$$

Convergence

Theorem

Let $U \in \mathcal{C}^1\left(\mathbb{R}^d\right)$ satisfy the previous assumption. There is an explicit $c^W > 0$ such that, for all $W \in \mathcal{C}^1\left(\mathbb{R}^d\right)$ satisfying $L_W < c^W$, there is an explicit $\tau > 0$ such that for all probability measures ν_0^1 and ν_0^2 on $\mathbb{R}^{2d}$ with a finite second moment, there are explicit constants $C_1, C_2 > 0$ such that for all $t \geqslant 0$,

$$\mathcal{W}_1\left(\bar{\nu}_t^1, \bar{\nu}_t^2\right) \leq e^{-\tau t} C_1 , \qquad \mathcal{W}_2\left(\bar{\nu}_t^1, \bar{\nu}_t^2\right) \leq e^{-\tau t} C_2$$

where $\bar{\nu}_t^1$ and $\bar{\nu}_t^2$ are the probability densities of solutions of (NL) with respective initial distributions $\bar{\nu}_0^1$ and $\bar{\nu}_0^2$.
Furthermore, we have existence and unicity of -as well as convergence towards - a stationary solution.

Propagation of chaos

Theorem

Let $\mathcal{C}^0 > 0$ and $a > 0$. Let $U \in \mathcal{C}^1(\mathbb{R}^d)$ satisfy the previous assumption. There is an explicit $c^W > 0$ such that, for all $W \in \mathcal{C}^1(\mathbb{R}^d)$ satisfying $L_W < c^W$, there exist explicit $B_1, B_2 > 0$, such that for all probability measures ν_0 on $\mathbb{R}^{2d}$ (under some initial moment assumption depending on $\mathcal{C}^0$ and a) and for all $t \geq 0$,

$$\mathcal{W}_1\left(\nu_t^{k,N}, \bar{\nu}_t^{\otimes k}\right) \leq \frac{kB_1}{\sqrt{N}}, \qquad \mathcal{W}_2^2\left(\nu_t^{k,N}, \bar{\nu}_t^{\otimes k}\right) \leq \frac{kB_2}{\sqrt{N}},$$

for all $k \in \mathbb{N}$, where $\nu_t^{k,N}$ is the marginal distribution at time t of the first k particles $\left((X_t^1, V_t^1), ..., (X_t^k, V_t^k)\right)$ of an N particle system (PS) with initial distribution $(\nu_0)^{\otimes N}$, while $\bar{\nu}_t$ is the probability densities of (NL) with initial distribution ν_0.

Extension of

Convergence rate :

• Andreas Eberle. *Reflection couplings and contraction rates for diffusions*. Probab. Theory Relat. Fields (2016)

• Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. *Couplings and quantitative contraction rates for Langevin dynamics*. Ann. Probab. (2019)

• Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. *Quantitative Harris-type theorems for diffusions and McKean-Vlasov processes*. Trans. Am. Math. Soc.(2019)

Propagation of chaos :

• Alain Durmus, Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. *An elementary approach to uniform in time propagation of chaos*. Proc. Amer. Math. Soc. (2020)

II. Proof of convergence

Coupling

Consider two clouds of particle with different starting shape

$$\begin{cases} d\bar{X}_t^1 = \bar{V}_t^1 dt \\ d\bar{V}_t^1 = \sqrt{2} dB_t^1 - \bar{V}_t^1 dt - \nabla U(\bar{X}_t^1) dt - \nabla W * \mu_t^1(\bar{X}_t^1) dt \\ d\bar{X}_t^2 = \bar{V}_t^2 dt \\ d\bar{V}_t^2 = \sqrt{2} dB_t^2 - \bar{V}_t^2 dt - \nabla U(\bar{X}_t^2) dt - \nabla W * \mu_t^2(\bar{X}_t^2) dt \\ \mu_t^1 = Law(\bar{X}_t^1), \ \mu_t^2 = Law(\bar{X}_t^2) \end{cases}$$

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Then, denoting $\nu_t^i = Law((\bar{X}_t^i, \bar{V}_t^i))$

$$\mathcal{W}_1(\nu_t^1, \nu_t^2) = \inf_{\Gamma \in \Pi(\mu_t, \nu_t)} \mathbb{E}_\Gamma \left(|\bar{X}_t^1 - \bar{X}_t^2| + |\bar{V}_t^1 - \bar{V}_t^2| \right)$$

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Idea behind coupling arguments : instead of considering the infimum over all couplings, construct one.

Coupling

To construct a coupling, play with the randomness. Here, the Brownian motions.

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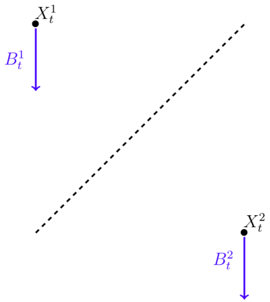

FIGURE – Synchronous coupling

Choosing $B^1 = B^2$:

• the Brownian noise is canceled out in the infinitesimal evolution of the difference $(Z_t, W_t) = (\bar{X}_t^1 - \bar{X}_t^2, \bar{V}_t^1 - \bar{V}_t^2)$,

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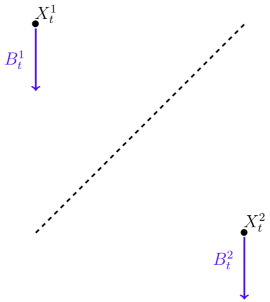

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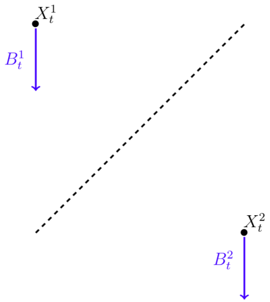

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• Here : contraction when $Z_t + W_t = 0$

Coupling

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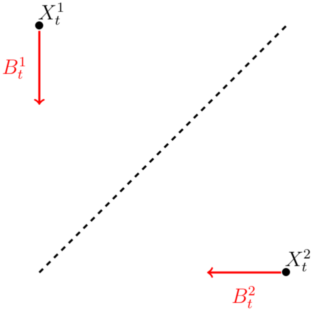

FIGURE – Reflection coupling

Writing

$$e_t = \begin{cases} \frac{Z_t+W_t}{|Z_t+W_t|} & \text{if } Z_t+W_t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

we consider
$dB_t^2 = \left(Id - 2e_te_t^T\right) dB_t^1$:

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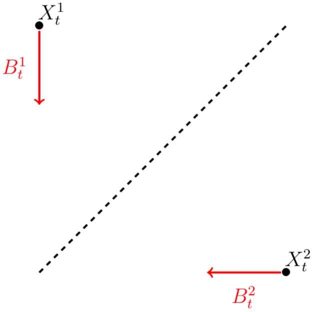

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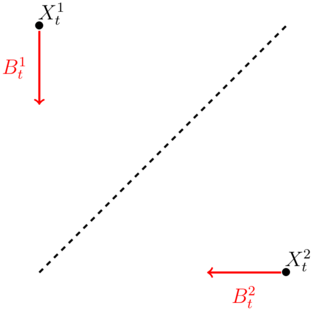

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- this maximizes the variance of the noise in the desired direction,
- requires a modification of the distance by some concave function.

Three behaviors

• When any of the particle ventures at infinity (i.e $|\bar{X}_t|$ or $|\bar{V}_t|$ becomes sufficiently big), the friction and confinement potential will tend to bring the particle back,
$\implies$ use a Lyapunov function (i.e H such that $\frac{d}{dt}\mathbb{E}H \leq B - \gamma\mathbb{E}H$).

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• When the particles are near the space

$$\left\{ \left(\bar{X}_t^1, \bar{X}_t^2, \bar{V}_t^2, \bar{V}_t^2\right) \in \mathbb{R}^{4d}, \bar{X}_t^1 - \bar{X}_t^2 + \bar{V}_t^1 - \bar{V}_t^2 = 0 \right\},$$

the L^1 distance will naturally contract,

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• Otherwise, the particles are in a compact set,
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Step 2 : Consider

$$\begin{aligned}\rho((x_1, v_1),(x_2, v_2)) &= f\left(\alpha|x_1 - x_2| + |x_1 - x_2 + v_1 - v_2|\right)\left(1 + \epsilon H(x_1, v_1) + \epsilon H(x_2, v_2)\right) \\ &= f(r)G\end{aligned}$$

such that

$$\mathcal{C}_1\rho((x_1, v_1),(x_2, v_2)) \geq |x_1 - x_2| + |v_1 - v_2|.$$

f is nondecreasing, non negative, concave, and constant for r greater than a threshold.

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Step 3 : Coupling and calculate the dynamics of $\rho((\bar{X}_t^1, \bar{V}_t^1),(\bar{X}_t^2, \bar{V}_t^2))$.

- In a contracting region of space, synchronous coupling.
- Near that space, reflection coupling.
- "At infinity", use the Lyapunov function.

Construction of the coupling

$$\begin{cases} d\bar{X}_t^1 = \bar{V}_t^1 dt \\ d\bar{V}_t^1 = -\bar{V}_t^1 dt - \nabla U\left(\bar{X}_t^1\right) dt - \nabla W * \mu_t\left(\bar{X}_t^1\right) dt + \sqrt{2} sc\left(Z_t, W_t\right) dB_t^{sc} \\ \qquad + \sqrt{2} rc\left(Z_t, W_t\right) dB_t^{rc} \\ \mu_t = \text{Law}\left(\bar{X}_t^1\right) \\ d\bar{X}_t^2 = \bar{V}_t^2 dt \\ d\bar{V}_t^2 = -\bar{V}_t^2 dt - \nabla U(\bar{X}_t^2) dt - \nabla W * \tilde{\mu}_t(\bar{X}_t^2) dt + \sqrt{2} sc\left(Z_t, W_t\right) dB_t^{sc} \\ \qquad + \sqrt{2} rc\left(Z_t, W_t\right)\left(Id - 2 e_t e_t^T\right) dB_t^{rc} \\ \tilde{\mu}_t = \text{Law}(\bar{X}_t^2), \end{cases}$$

with

$$rc^2 + sc^2 = 1,$$

$$rc\left(z, w\right) = 0 \text{ if } |z + w| \leq \frac{\xi}{2} \text{ or } \alpha|z| + |z + w| \geq R_1 + \xi,$$

$$rc\left(z, w\right) = 1 \text{ if } |z + w| \geq \xi \text{ and } \alpha|z| + |z + w| \leq R_1.$$

Calculating the dynamics

We have

$$\forall t \geq 0,\ e^{ct}\rho_t \leq \rho_0 + \int_0^t e^{cs} K_s ds + M_t,$$

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$$\begin{aligned} K_t = & \left(cf\left(r_t\right) + \left(\alpha\frac{d|Z_t|}{dt} + (L_U + L_W)|Z_t|\right) f'\left(r_t\right)\right) G_t \\ & + 4\left(f''\left(r_t\right) G_t + 24\epsilon \max\left(1, \frac{1}{2\alpha}\right) r_t f'\left(r_t\right)\right) rc\left(Z_t, W_t\right)^2 \\ & + \epsilon\left(2B - \gamma H\left(\bar{X}_t^1, \bar{V}_t^1\right) - \gamma H(\bar{X}_t^2, \bar{V}_t^2)\right) f\left(r_t\right) \\ & + L_W f'\left(r_t\right) \mathbb{E}\left(|Z_t|\right) G_t + \epsilon L_W\left(6 + 8\lambda\right)\left(\mathbb{E}\left(|\bar{X}_t^1|\right)^2 + \mathbb{E}(|\bar{X}_t^2|)^2\right) f\left(r_t\right). \end{aligned}$$

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$$\forall t \geq 0,\ e^{ct}\rho_t \leq \rho_0 + \int_0^t e^{cs} K_s ds + M_t,$$

where M_t is a continuous local martingale and

$$\begin{aligned} K_t = &\left(cf\left(r_t\right) + \left(\alpha \frac{d|Z_t|}{dt} + (L_U + L_W)|Z_t| \right) f'\left(r_t\right) \right) G_t \\ &+4\left(f''\left(r_t\right) G_t + 24\epsilon \max\left(1, \frac{1}{2\alpha}\right) r_t f'\left(r_t\right) \right) rc\left(Z_t, W_t\right)^2 \\ &+ \epsilon\left(2B - \gamma H\left(\bar{X}_t^1, \bar{V}_t^1\right) - \gamma H(\bar{X}_t^2, \bar{V}_t^2)\right) f\left(r_t\right) \\ &+ L_W f'\left(r_t\right) \mathbb{E}\left(|Z_t|\right) G_t + \epsilon L_W\left(6 + 8\lambda\right) \left(\mathbb{E}\left(|\bar{X}_t^1|\right)^2 + \mathbb{E}(|\bar{X}_t^2|)^2 \right) f\left(r_t\right). \end{aligned}$$

Reflection coupling : choose f sufficiently concave, to have those two lines nonpositive

Calculating the dynamics

We have

$$\forall t \geq 0,\ e^{ct}\rho_t \leq \rho_0 + \int_0^t e^{cs}K_s ds + M_t,$$

where M_t is a continuous local martingale and

$$\begin{aligned} K_t = & \left(cf(r_t) + \left(\alpha \frac{d|Z_t|}{dt} + (L_U + L_W)|Z_t|\right) f'(r_t)\right) G_t \\ & + 4\left(f''(r_t)\, G_t + 24\epsilon \max\left(1, \frac{1}{2\alpha}\right) r_t f'(r_t)\right) rc(Z_t, W_t)^2 \\ & + \epsilon\left(2B - \gamma H\left(\bar{X}_t^1, \bar{V}_t^1\right) - \gamma H(\bar{X}_t^2, \bar{V}_t^2)\right) f(r_t) \\ & + L_W f'(r_t)\, \mathbb{E}(|Z_t|)\, G_t + \epsilon L_W (6 + 8\lambda)\left(\mathbb{E}\left(|\bar{X}_t^1|\right)^2 + \mathbb{E}(|\bar{X}_t^2|)^2\right) f(r_t). \end{aligned}$$

Synchronous coupling : when the deterministic drift is contracting, this line alone will be sufficiently small.

Calculating the dynamics

We have

$$\forall t \geq 0,\ e^{ct}\rho_t \leq \rho_0 + \int_0^t e^{cs} K_s ds + M_t,$$

where M_t is a continuous local martingale and

$$\begin{aligned} K_t = &\left(cf(r_t) + \left(\alpha \frac{d|Z_t|}{dt} + (L_U + L_W)|Z_t| \right) f'(r_t) \right) G_t \\ &+ 4\left(f''(r_t) G_t + 24\epsilon \max\left(1, \frac{1}{2\alpha}\right) r_t f'(r_t) \right) rc(Z_t, W_t)^2 \\ &+ \epsilon \left(2B - \gamma H\left(\bar{X}_t^1, \bar{V}_t^1 \right) - \gamma H(\bar{X}_t^2, \bar{V}_t^2) \right) f(r_t) \\ &+ L_W f'(r_t) \mathbb{E}(|Z_t|) G_t + \epsilon L_W (6 + 8\lambda) \left(\mathbb{E}\left(|\bar{X}_t^1| \right)^2 + \mathbb{E}(|\bar{X}_t^2|)^2 \right) f(r_t). \end{aligned}$$

Translates the effect the Lyapunov function has in bringing back processes that would have ventured at infinity.

Calculating the dynamics

We have

$$\forall t \geq 0,\ e^{ct}\rho_t \leq \rho_0 + \int_0^t e^{cs}K_s ds + M_t,$$

where M_t is a continuous local martingale and

$$\begin{aligned} K_t = & \left(cf\left(r_t\right) + \left(\alpha \frac{d|Z_t|}{dt} + (L_U + L_W)|Z_t|\right) f'\left(r_t\right)\right) G_t \\ & + 4\left(f''\left(r_t\right) G_t + 24\epsilon \max\left(1, \frac{1}{2\alpha}\right) r_t f'\left(r_t\right)\right) rc\left(Z_t, W_t\right)^2 \\ & + \epsilon\left(2B - \gamma H\left(\bar{X}_t^1, \bar{V}_t^1\right) - \gamma H(\bar{X}_t^2, \bar{V}_t^2)\right) f\left(r_t\right) \\ & + L_W f'\left(r_t\right) \mathbb{E}\left(|Z_t|\right) G_t + \epsilon L_W \left(6 + 8\lambda\right)\left(\mathbb{E}\left(|\bar{X}_t^1|\right)^2 + \mathbb{E}(|\bar{X}_t^2|)^2\right) f\left(r_t\right). \end{aligned}$$

Contains the non linearity, tackled by taking L_W sufficiently small.

Calculating the dynamics

Conclude using Gronwall's lemma.

III. Proof of propagation of chaos

Coupling

We consider the following coupling

$$
\begin{cases}
d\bar{X}_t^i = \bar{V}_t^i dt \\
d\bar{V}_t^i = -\bar{V}_t^i dt - \nabla U\left(\bar{X}_t^i\right) dt - \nabla W * \bar{\mu}_t\left(\bar{X}_t^i\right) dt + \sqrt{2} rc\left(Z_t^i, W_t^i\right) dB_t^{rc,i} \\
\qquad\qquad + \sqrt{2} sc\left(Z_t^i, W_t^i\right) dB_t^{sc,i} \\
\bar{\mu}_t = \mathcal{L}\left(\bar{X}_t^i\right) \\
dX_t^{i,N} = V_t^{i,N} dt \\
dV_t^{i,N} = -V_t^{i,N} dt - \nabla U(X_t^{i,N}) dt - \frac{1}{N}\sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}) dt \\
\qquad\qquad + \sqrt{2}\left(rc\left(Z_t^i, W_t^i\right)\left(Id - 2e_t^i e_t^{i,T}\right) dB_t^{rc,i} + sc\left(Z_t^i, W_t^i\right) dB_t^{sc,i}\right),
\end{cases}
$$

Distance

Define

$$
\begin{aligned}
r_t^i =& \alpha|Z_t^i| + |Q_t^i|, \\
\tilde{H}(x,v) =& \int_0^{H(x,v)} \exp\left(a\sqrt{u}\right) du \\
\rho_t =& \frac{1}{N}\sum_{i=1}^N f\left(r_t^i\right)\left(1 + \epsilon\tilde{H}\left(\bar{X}_t^i, \bar{V}_t^i\right) + \epsilon\tilde{H}(X_t^{i,N}, V_t^{i,N})\right. \\
& \left. + \frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}\left(\bar{X}_t^j, \bar{V}_t^j\right) + \frac{\epsilon}{N}\sum_{j=1}^N \tilde{H}(X_t^{j,N}, V_t^{j,N})\right) \\
:=& \frac{1}{N}\sum_{i=1}^N f\left(r_t^i\right) G_t^i := \frac{1}{N}\sum_{i=1}^N \rho_t^i.
\end{aligned}
$$

Distance

We get

$$de^{ct}\rho_t^i \leq e^{ct}K_t^i dt + dM_t^i$$

with...

Distance

$$
\begin{aligned}
K_t^i =& f'\left(r_t^i\right) G_t^i \left(\alpha \frac{d|Z_t^i|}{dt} + (L_U + L_W)|Z_t^i| + (\epsilon C_{f,1} + C_{f,2}) r_t^i rc^2 \left(Z_t^i, W_t^i\right)\right) + 2cf\left(r_t^i\right) G_t^i \\
&+ 4f''\left(r_t^i\right) G_t^i rc^2\left(Z_t^i, W_t^i\right) + |\nabla W * \bar\mu_t\left(\bar X_t^i\right) - \frac{1}{N}\sum_{j=1}^N \nabla W\left(\bar X_t^i - \bar X_t^j\right)| f'\left(r_t^i\right) G_t^i \\
&+ \epsilon f\left(r_t^i\right)\left(4\tilde B - \frac{\gamma}{16}\tilde H\left(\bar X_t^i, \bar V_t^i\right) - \frac{\gamma}{16}\tilde H(X_t^{i,N}, V_t^{i,N}) - \frac{\gamma}{16N}\sum_{j=1}^N \tilde H\left(\bar X_t^j, \bar V_t^j\right)\right. \\
&\qquad \left. - \frac{\gamma}{16N}\sum_{j=1}^N \tilde H(X_t^{j,N}, V_t^{j,N})\right) \\
&+ L_W \frac{\sum_{j=1}^N |Z_t^j|}{N} f'\left(r_t^i\right) G_t^i - cf\left(r_t^i\right) G_t^i - \epsilon f\left(r_t^i\right)\left(\frac{\gamma}{16}\bar H_i \exp\left(a\sqrt{\bar H_i}\right)\right. \\
&\quad \left. + \frac{\gamma}{16} H_i^N \exp\left(a\sqrt{H_i^N}\right) + \frac{\gamma}{16N}\sum_{j=1}^N \bar H_j \exp\left(a\sqrt{\bar H_j}\right) + \frac{\gamma}{16N}\sum_{j=1}^N H_j^N \exp\left(a\sqrt{H_j^N}\right)\right) \\
&+ \epsilon L_W (6 + 8\lambda) f\left(r_t^i\right)\left(\frac{\sum_{j=1}^N |X_t^{j,N}|}{N}\right)^2 \exp\left(a\sqrt{H_i^N}\right) \\
&\qquad - \frac{\gamma\epsilon}{8} f\left(r_t^i\right)\left(H_i^N \exp\left(a\sqrt{H_i^N}\right) + \frac{1}{N}\sum_{j=1}^N H_j^N \exp\left(a\sqrt{H_j^N}\right)\right).
\end{aligned}
$$

Thank you