Uniform in time propagation of chaos for the generalized Dyson Brownian motion and 1D Riesz gases.

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I. Introduction

The model

1D N-particle system in mean field interaction

$$dX_t^i = \sqrt{2\sigma_N} dB_t^i - U'(X_t^i)dt - \frac{1}{N}\sum_{j\neq i} V'(X_t^i - X_t^j)dt,$$

where

- σ_N diffusion coefficient,
- $(B^i)_i$ independent Brownian motions,
- U confining potential such that either U' is Lipschitz continuous or $U'(x) = \lambda x$,
- $\exists \alpha \geq 0,\ \forall x \in \mathbb{R}^*,\ V'(x) = -\frac{x}{|x|^{\alpha+1}}$.

Motivation

The (generalized) Dyson Brownian motion

$$dX_t^i = \sqrt{\frac{2\sigma}{N}}dB_t^i - \lambda X_t^i dt + \frac{1}{N}\sum_{j\neq i}\frac{1}{X_t^i - X_t^j}dt.$$

Motivation

The (generalized) Dyson Brownian motion

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Question : What happens when $N \to \infty$?

Finding the limit

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Formally, notice $\frac{1}{N}\sum_{j=1}^N V'(X_t^i - X_t^j) = V' * \mu_t^N(X_t^i)$, where $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$.

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$$\begin{cases} dX_t = \sqrt{2\sigma} dB_t - U'(X_t)dt - V' * \bar{\rho}_t(X_t)dt, \\ \bar{\rho}_t = \mathrm{Law}(X_t), \end{cases}$$

Finding the limit

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which is linked to

$$\partial_t \bar{\rho}_t = \partial_x \left(\left(U' + V' * \bar{\rho}_t \right) \bar{\rho}_t \right) + \sigma \partial_{xx}^2 \bar{\rho}_t.$$

Usual methods

Goal : Show $\mu_t^N \to \bar{\rho}_t$.

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- Coupling methods *(McKean, Sznitman, Eberle...) :*

$$\mathcal{W}_2\left(\mu,\nu\right)^2 = \inf_{X\sim\mu, Y\sim\nu} \mathbb{E}\left(|X-Y|^2\right).$$

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Let $\rho_t^N = Law(X_t^1, ..., X_t^N)$, show $\mathcal{W}_2\left(\rho_t^N, \bar{\rho}_t^{\otimes N}\right) \to 0$.

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- Energy/Entropy estimates *(Serfaty, Jabin, Wang...)*.
- Tightness *(Rogers, Zhi, Cépa, Lépingle...)*.

Some results

Existence, uniqueness, no collisions

Theorem

Consider $N \geq 2$, and $-\infty < x_1 < ... < x_N < \infty$.

- *If $\alpha > 1$, for any $\sigma_N \geq 0$, there exists a unique strong solution $X = (X^1, ..., X^N)$ to the particle system with initial condition $X_0^1 = x_1$, ..., $X_0^N = x_N$, which furthermore satisfies $X_t^1 < ... < X_t^N$ for all $t \geq 0$, $\mathbb{P}$-a.s.*
- *The same result holds for $\alpha = 1$ and $\sigma_N \leq \frac{1}{N}$.*

"Cauchy sequence"

Lemma

Let $(\mu^N)_{N\in\mathbb{N}}$ be any sequence of independent empirical measures, such that μ_t^N is the empirical measure of the N particle system at time t. Then (for $\lambda > 0$, $\alpha = 1$, $U'(x) = \lambda x$ and $\sigma_N = \frac{1}{N}$), we have for all $t \geq 0$ and all $N, M \geq 1$

$$\mathbb{E}\left(\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2\right) \leq e^{-2\lambda t}\mathbb{E}\left(\mathcal{W}_2\left(\mu_0^N, \mu_0^M\right)^2\right) + \frac{C}{N \wedge M}.$$

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- for $U = 0$, but no longer uniform in time,
- for $\alpha \in [1, 2[$, with rate $N^{-\frac{2-\alpha}{\alpha}}$,

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Remark : The same result holds...

- for $U = 0$, but no longer uniform in time,
- for $\alpha \in [1, 2[$, with rate $N^{-\frac{2-\alpha}{\alpha}}$,
- for U' only Lipschitz continuous, but no longer uniform in time,

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Let $(\mu^N)_{N\in\mathbb{N}}$ be any sequence of independent empirical measures, such that μ_t^N is the empirical measure of the N particle system at time t. Then (for $\lambda > 0$, $\alpha = 1$, $U'(x) = \lambda x$ and $\sigma_N = \frac{1}{N}$), we have for all $t \geq 0$ and all $N, M \geq 1$

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- for $U = 0$, but no longer uniform in time,
- for $\alpha \in [1, 2[$, with rate $N^{-\frac{2-\alpha}{\alpha}}$,
- for U' only Lipschitz continuous, but no longer uniform in time,
- for the supremum, but no longer uniform in time.

Conclusion

Using independence, this implies that there exists a (deterministic) $\bar{\rho}_t$ such that

$$\mathbb{E}\left(\mathcal{W}_2(\mu_t^N, \bar{\rho}_t)^2\right) \to 0.$$

Proof of the estimate

For two sets of points $(x_i)_{i\in\{1,...,N\}}$ and $(y_j)_{j\in\{1,...,N\}}$, with $x_1 \leq ... \leq x_N$ and $y_1 \leq ... \leq y_N$, and two measures $\mu = \frac{1}{N}\sum_i \delta_{x_i}$ and $\nu = \frac{1}{N}\sum_j \delta_{y_j}$:

$$\mathcal{W}_2(\mu,\nu)^2 = \frac{1}{N}\sum_i |x_i - y_i|^2.$$

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For two sets of points $(x_i)_{i\in\{1,\dots,N\}}$ and $(y_j)_{j\in\{1,\dots,N\}}$, with $x_1 \leq \dots \leq x_N$ and $y_1 \leq \dots \leq y_N$, and two measures $\mu = \frac{1}{N}\sum_i \delta_{x_i}$ and $\nu = \frac{1}{N}\sum_j \delta_{y_j}$:

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Let

$$-\infty < X_t^1 = \dots = X_t^N < \dots < X_t^{N(M-1)+1} = \dots = X_t^{NM} < \infty$$
$$-\infty < Y_t^1 = \dots = Y_t^M < \dots < Y_t^{M(N-1)+1} = \dots = Y_t^{NM} < \infty.$$

Thus

$$\mu_t^M = \frac{1}{M}\sum_{i=1}^{M}\delta_{\tilde{X}_t^{i,M}} = \frac{1}{NM}\sum_{i=1}^{NM}\delta_{X_t^i} \quad \text{and} \quad \mu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{\tilde{Y}_t^{i,N}} = \frac{1}{NM}\sum_{i=1}^{NM}\delta_{Y_t^i},$$

and

$$\mathcal{W}_2\left(\mu_t^N,\mu_t^M\right)^2 = \frac{1}{NM}\sum_{i=1}^{NM}\left|X_t^i - Y_t^i\right|^2.$$

Proof of the estimate-2

Direct calculations yield :

$$
\begin{aligned}
d & \left(\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2\right) \\
& = -2\lambda \mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2 dt + 2\sigma\left(\frac{1}{N} + \frac{1}{M}\right) dt + dM_t \\
& \quad - \frac{2}{(NM)^2} \sum_i \left(X_t^i - Y_t^i\right) \sum_j \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) dt.
\end{aligned}
$$

Proof of the estimate-3

$$\begin{aligned}
\sum_i \left(X_t^i - Y_t^i\right) \sum_j \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - Y_t^i\right) - \left(X_t^j - Y_t^j\right)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - X_t^j\right) - \left(Y_t^i - Y_t^j\right)\right).
\end{aligned}$$

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\begin{aligned}
&\sum_i \left(X_t^i - Y_t^i\right) \sum_j \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - Y_t^i\right) - \left(X_t^j - Y_t^j\right)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - X_t^j\right) - \left(Y_t^i - Y_t^j\right)\right). \\
&\geq \sum_{i>j \text{ s.t } Y_t^i = Y_t^j} V'(X_t^i - X_t^j) \left(X_t^i - X_t^j\right) + \sum_{i>j \text{ s.t } X_t^i = X_t^j} V'(Y_t^i - Y_t^j) \left(Y_t^i - Y_t^j\right) \\
&\geq \sum_{i>j \text{ s.t } Y_t^i = Y_t^j} -1 + \sum_{i>j \text{ s.t } X_t^i = X_t^j} -1 \\
&= -\frac{M(M-1)}{2} N - \frac{N(N-1)}{2} M.
\end{aligned}
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\sum_i \left(X_t^i - Y_t^i\right) \sum_j \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - Y_t^i\right) - \left(X_t^j - Y_t^j\right)\right) \\
&= \sum_{i>j} \left(V'(X_t^i - X_t^j) - V'(Y_t^i - Y_t^j)\right) \left(\left(X_t^i - X_t^j\right) - \left(Y_t^i - Y_t^j\right)\right). \\
&\geq \sum_{i>j \text{ s.t } Y_t^i = Y_t^j} V'(X_t^i - X_t^j)\left(X_t^i - X_t^j\right) + \sum_{i>j \text{ s.t } X_t^i = X_t^j} V'(Y_t^i - Y_t^j)\left(Y_t^i - Y_t^j\right) \\
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Hence

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\mathbb{E}\left(\mathcal{W}_2\left(\mu_t^N, \mu_t^M\right)^2\right) \leq e^{-2\lambda t}\mathbb{E}\left(\mathcal{W}_2\left(\mu_0^N, \mu_0^M\right)^2\right) + \frac{C}{N \wedge M}.
$$

Thank you