Dimensionality reduction for periodic magnetostatic fields

Dimensionality reduction using an edge finite element method for periodic magnetostatic fields in a symmetric domain

C.G. Albert\textsuperscript{1} O. Biro\textsuperscript{2} M.F. Heyn\textsuperscript{1} W. Kernbichler\textsuperscript{1} S.V. Kasilov\textsuperscript{1,3} P. Lainer\textsuperscript{1}

\textsuperscript{1}Fusion@ÖAW, Institute of Theoretical and Computational Physics
\textsuperscript{2}Institute of Fundamentals and Theory in Electrical Engineering
Graz University of Technology

\textsuperscript{3}Institute of Plasma Physics, National Science Center
Kharkov Institute of Physics and Technology

8th FreeFEM++ workshop, Dec 8th 2016
Who are we?

- Theoretical plasma physics group at TU Graz

General topic: **magnetic confinement fusion**
- Trap a hot plasma to allow for nuclear fusion
- Work within the **EUROfusion** framework (ITER, W7-X, ...)

C.G. Albert, O. Biro, M.F. Heyn, W. Kernbichler, S.V. Kasilov, P. Lainer, 8th FreeFEM++ workshop, Dec 8th 2016
What do we do?

- Our tasks include:
  - Understand non-axisymmetric perturbations in tokamaks
  - Compute transport and 3D equilibria in stellarators

- Our strategy:
  - Use a kinetic Monte Carlo model for the plasma
  - Couple to Maxwell’s equations solved by FEM

- More complete but slower than magnetohydrodynamics
  - optimisations needed
Tokamak and stellarator geometry

- make use of axisymmetry / periodicity
About today’s talk

- Most things are well-known
- **Goal**: calculate **3D magnetic field** from known **currents**
- Systematic way of "2.5D" reduction of **curl curl** equation
  - Starting from Maxwell’s equations
  - **symmetric** and **oscillatory** part (Fourier series)
- Generalisation to **curvilinear** coordinates
- Efficient realisation with edge elements in **FreeFEM++**
Maxwell’s equations of electrodynamics

\[
\begin{align*}
\text{div } \varepsilon \mathbf{E} &= \rho \quad (1) \\
\text{curl } \mathbf{E} + \partial_t \mathbf{B} &= 0 \quad (2) \\
\text{curl } \nu \mathbf{B} - \partial_t (\varepsilon \mathbf{E}) &= \mathbf{J} \quad (3) \\
\text{div } \mathbf{B} &= 0 \quad (4)
\end{align*}
\]

- Unknowns: Electric field \( \mathbf{E} \) and magnetic field \( \mathbf{B} \)
- Source terms: Free charge density \( \rho \), currents density \( \mathbf{J} \)
- Material parameters: Permittivity \( \varepsilon \), inverse permeability \( \nu = \mu^{-1} \)
  - Can lead to discontinous (weak) solutions for \( \mathbf{E} \) and \( \mathbf{B} \)
- Continuity equation for charges as a consequence:
  \[
  \frac{\partial \rho}{\partial t} + \text{div} \mathbf{J} = 0
  \]
Scalar and vector potential

\[ \text{div } \varepsilon \mathbf{E} = \rho \]
\[ \text{curl } \mathbf{E} + \partial_t \mathbf{B} = 0 \]
\[ \text{curl } \nu \mathbf{B} - \partial_t (\varepsilon \mathbf{E}) = \mathbf{J} \]
\[ \text{div } \mathbf{B} = 0. \]

- Simply connected domains: can find potentials \( \Phi \) and \( \mathbf{A} \) with

\[ E = -\text{grad } \Phi - \partial_t \mathbf{A}, \quad B = \text{curl } \mathbf{A} \quad (5) \]

- Equations fulfilled since \( \text{curl } \text{grad } \Phi = 0 \) and \( \text{div } \text{curl } \mathbf{A} = 0 \) \( \forall \Phi, \mathbf{A} \)

- Proof: special case of Poincaré lemma
Potential equations

\[- \text{div} \, \varepsilon \text{grad} \Phi - \text{div} \, \varepsilon \partial_t A = \rho \]  
\[\text{curl} \, \nu \text{curl} A - \partial_t \varepsilon \text{grad} \Phi + \partial_t \varepsilon \partial_t A = J\]

with \( E = - \text{grad} \Phi - \frac{\partial A}{\partial t} \), \( B = \text{curl} A \)

- **Singular** (non-unique solution) due to gauge freedom

\[ A = A' + \text{grad} \chi , \quad \Phi = \Phi' + \frac{\partial \chi}{\partial t} \]

since \( \text{curl} \, \text{grad} \chi = 0 \)
Dimensionality reduction for periodic magnetostatic fields

Textbook example: Lorenz gauge

- For constant $\varepsilon$, $\nu$, $c^2 := \nu/\varepsilon$ decouple equations by gauge

\[
\text{div} \ A + \partial_t \Phi / c^2 = 0
\]

- Wave equations follow with Laplacian $\Delta \Phi := \text{div} \ \text{grad} \ \Phi$
  and Vector Laplacian $\Delta A := \text{grad} \ \text{div} \ A - \text{curl} \ \text{curl} \ A$

\[
-\Delta \Phi - \partial_t^2 \Phi / c^2 = \rho / \varepsilon \quad (8)
\]

\[
-\Delta A + \partial_t^2 A / c^2 = J / \nu \quad (9)
\]

- Often better to stay with curl curl equation
  - $\Delta A = \Delta A_x e_x + \Delta A_y e_y + \Delta A_z e_z$ only in Cartesian coords
  - Numerical troubles of (9) in nodal basis (spurious modes)
Dimensionality reduction for periodic magnetostatic fields

Static case

\[ -\text{div} \varepsilon \text{grad} \Phi - \text{div} \varepsilon \partial_t \mathbf{A} = \rho \]  \hspace{1cm} (10)

\[ \text{curl} \nu \text{curl} \mathbf{A} - \partial_t \varepsilon \text{grad} \Phi - \partial_t \varepsilon \partial_t \mathbf{A} = \mathbf{J} \]  \hspace{1cm} (11)

- Changes of fields over time are neglected
- Relevant to find equilibrium configurations
- Equations decouple into electrostatics and magnetostatics
- In particular, Eq. (11) leads to

\[ \text{div} \mathbf{J} = 0 \]  \hspace{1cm} (12)

(continuity equation without sources)
FEM for the 3D curl-curl equation – weak form

\[ \text{curl } \nu \text{curl } A = J \]  \hspace{1cm} (13)

- Standard procedure: domain \( \Omega \) with Neumann data \( A_N \times n \) on \( \Gamma_N \)

1. Scalar multiplication by test function \( W \)
2. Do partial integration \( \Rightarrow \) weak form

\[ \int_{\Omega} \text{curl } W \cdot \nu \text{curl } A \, d\Omega = \int_{\Omega} W \cdot J \, d\Omega - \int_{\Gamma_N} \nu \, W \cdot \text{curl } A_N \times n \, d\Omega \]  \hspace{1cm} (14)

3. Discretise locally on mesh by Galerkin method
FEM for the 3D curl-curl equation – discretisation

\[ \int_{\Omega} \text{curl } W \cdot \nu \text{curl } A \, d\Omega = \int_{\Omega} W \cdot J \, d\Omega - \int_{\Gamma_N} \nu W \cdot \text{curl } A_N \times n \, d\Gamma_N \]

- **Edge** (Nédélec) elements for \( A, \ W \in H_{\text{curl}} \)
  - DOFs: integral of vector along edges
  - Stokes’ law \( \oint A \cdot d\ell = \int \text{curl } A \cdot dS \) given directly

- **Face** (Raviart-Thomas) elements for \( B = \text{curl } A \in H_{\text{div}} \)
  - DOFs: integral of vector across faces
  - Gauss’ law \( \oint A \cdot dS = \int \text{div } A \, dV \) given directly

- Either gauged (tree-cotree) or ungauged (iterative solver)
Example: Cartesian coordinates

- Prism with BCs and parameters $2\pi$-periodic in $z$
Dimensionality reduction for periodic magnetostatic fields

Reduction to 2D - symmetric part (z-independent)

- Curl splits into independent transversal $b$ and longitudinal $B_z e_z$

$$
B = \text{curl } A = \partial_y A_z e_x - \partial_x A_z e_y + (\partial_x A_y - \partial_y A_x) e_z
$$

- Two distinct equations follow from \textbf{curl curl} Eq. (13)

$$
\text{curl}_t \nu \text{curl}_t a = j \quad (15)
\text{curl}_t \nu \text{curl}_t A_z = J_z \quad (16)
$$

- Weak forms of homogenous Neumann problems:

$$
\int_\Omega \text{curl}_t w \nu \text{curl}_t a \, d\Omega_t = \int_\Omega w \cdot j \, d\Omega_t \quad (\rightarrow \text{edge elements})
$$

$$
\int_\Omega \text{curl}_t W \cdot \nu \text{curl}_t A_z \, d\Omega_t = \int_\Omega W J_z \, d\Omega_t \quad (\rightarrow \text{nodal elements})
$$
Dimensionality reduction for periodic magnetostatic fields

Reduction to 2D - oscillatory part

- All quantities oscillatory in symmetry direction, e.g. $z$

$$f(x, y, z) = \text{Re} \sum_{n \neq 0} f_n(x, y) \exp(inz)$$

- Curl also contains extra terms with $\partial_z = in$

$$B = (\partial_y A_z - inA_y)e_x + (inA_x - \partial_x A_z)e_y + (\partial_x A_y - \partial_y A_x)e_z$$

- $n \neq 0$ – why not eliminate $A_z$ by gauge transformation?

$$A \rightarrow A + \text{grad} \chi,$$

$$\chi = - \int A_z dz = - \frac{A_z}{in} \quad \text{(single harmonic)}$$
Dimensionality reduction for periodic magnetostatic fields

Reduction to 2D - oscillatory part

- Now only transversal \( \mathbf{a} \perp \mathbf{b} \) remains

\[
\mathbf{B} = -in a_y \mathbf{e}_x + in a_x \mathbf{e}_y + (\partial_x a_y - \partial_y a_x) \mathbf{e}_z
\]

- Splits into "Helmholtz" (+ means decay here) and other

\[
\text{curl}_t \nu \text{curl}_t \mathbf{a} + n^2 \nu \mathbf{a} = \mathbf{j} \quad (17)
\]

\[
- in \text{div}_t \nu \mathbf{a} = J_z \quad (18)
\]

- Eq. (18) automatically fulfilled with Eq. (17) & \( \text{div} \mathbf{J} = 0 \)

- Weak form for homogenous Neumann problem

\[
\int_{\Omega} \text{curl}_t \mathbf{w} \nu \text{curl}_t \mathbf{a} + n^2 \mathbf{w} \cdot \nu \mathbf{a} \, d\Omega_t = \int \mathbf{w} \cdot \mathbf{j} \, d\Omega_t \quad (\rightarrow \text{edge elements})
\]
Comparison symmetric – oscillatory

- Symmetric part 2D transversal equation ("Poisson")
  \[ \text{curl}_t \nu \text{curl}_t a = j \]
  - Still singular (ungauged), can add \( \text{grad}_t \chi \) to \( a \)
  - Only describes \( B_z \) component, need also other equation
- Oscillatory part 2D transversal equation ("Helmholtz")
  \[ \text{curl}_t \nu \text{curl}_t a + n^2 \nu a = j \]
  - Uniquely solvable
  - Describes full \( B \) solution using \( \text{div} B = \text{div}_t b + inB_z = 0 \)
Some basics about curvilinear coordinates

- Coordinates $x^k$ parametrize space: $r(x^1, x^2, x^3) \rightarrow$ inverse $x^k(r)$
- (Non-orthonormal) covariant and its dual (contravariant) basis
  \[ e_k = \partial_k r \quad e^k = \nabla x^k \]
- Representation of vectors in contra- and covariant components
  \[ A = \sum_k A^k e_k = \sum_k A_k e^k, \quad A^k = A \cdot e^k, \quad A_k = A \cdot e_k \]
- Jacobian is the square-root of determinant of metric tensor
  \[ J = \sqrt{g}, \quad g_{ij} = \partial_i r \cdot \partial_j r, \quad A_k = \sum_i g_{ik} A^k \]
- Differential operators ($\varepsilon_{ijk}^{} = 1: ijk=123,231,312 / -1: 321,213,132$)
  \[ \text{div}A = \frac{1}{\sqrt{g}} \sum_k \partial_k \sqrt{g} A^k \quad \text{curl}A = e_i \sum_{j,k} \frac{\varepsilon_{ijk}^{} \sqrt{g}}{g} \partial_j A_k \]
Oscillatory part in 2D coordinate space

- Careful with Fourier in curved coordinates! Assumptions:
  - Orthogonal system ($g_{ij}$ has only diagonal elements)
  - $g_{ij}$ depends only on $x^1$ and $x^2$, not on $x^3$
- Expand covariant $A$ and contravariant $J$ components

\[
A_k(x^1, x^2, x^3) = \sum_{n=-\infty}^{\infty} A_{k,n}(x^1, x^2) e^{inx^3}, \quad (19)
\]
\[
J^k(x^1, x^2, x^3) = \sum_{n=-\infty}^{\infty} J^k_n(x^1, x^2) e^{inx^3}, \quad (20)
\]

- 2D curl in coordinate space

\[
\text{curl}_2 \mathbf{a} := \frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2} = \sqrt{g} \text{curl}_t \mathbf{a}
\]
Weak form in 2D coordinate space

- Coordinate space volume element: \( d\Omega_2 := dx^1 dx^2 \)
- Coordinate space line element: \( d\Gamma_2 = \sqrt{(dx^1)^2 + (dx^2)^2} \)
- Weak form of Eq. (17) homogenous Neumann problem

\[
\int_{\Omega} \frac{g_{33}}{\sqrt{g}} \nabla \times w \cdot \nabla \times a + \nu \left( \frac{g_{22}}{\sqrt{g}} w_1 a_1 + \frac{g_{11}}{\sqrt{g}} w_2 a_2 \right) d\Omega_2 = \int_{\Omega} w \cdot j \sqrt{g} d\Omega_2
\]
Dimensionality reduction for periodic magnetostatic fields

Example: Cylindrical coordinates
Dimensionality reduction for periodic magnetostatic fields

Example: Cylindrical coordinates

- Coordinates \((R, \varphi, Z)\) symmetry coordinate: angle \(\varphi\) (ordering!)

- Weak form of Eq. (17) homogenous Neumann problem

\[
\int_{\Omega_2} R \nu \text{\text{curl}}_2 a \text{\text{curl}}_2 w + \frac{n^2}{R} \nu (w_R a_R + w_Z a_Z) \, dRdZ = \int_{\Omega_2} R w \cdot j \, dRdZ
\]

- Weighting factor follows automatically from Jacobian \(\sqrt{g}\)

- Magnetic field

\[
B^R = \frac{in}{R} a_R, \quad B^Z = -\frac{in}{R} a_Z, \quad B^\varphi = -\frac{\text{div} b}{in},
\]
Dimensionality reduction for periodic magnetostatic fields

Example: Shielding by cylinder shell with $\mu > 1$
Dimensionality reduction for periodic magnetostatic fields

FreeFEM++ implementation

```c
load "Element_Mixte"; // for 1st order edge elements
real n = 1.0; // mode number

mesh Th = square(50,50,[x+1e-31,y]); // cylinder cross-section

fespace Hrot(Th,RT1Ortho); fespace Hdiv(Th,RT1); // 1st order

Hrot [ax,ay], [wx,wy]; Hdiv [jr,jz];

func real nu(real rp, real zp) { // nu = 1/mu
  if((rp>0.4)&&(rp<0.5)&&(zp>0.2&&(zp<0.8))) return 1.0/50.0;
  return 1.0;
}

solve CurlCurl([ax,ay],[wx,wy],solver=UMFPACK) =
  int2d(Th)(nu(x,y)*(x*(dx(wy)-dy(wx))*(dx(ay)-dy(ax))
    + n^2*1.0/x*(wx*ax+wy*ay)))
  + on(1,ax=0.0,ay=0.0)
  + on(2,3,4,ax=0.0,ay=1.0*x);

plot([ax,ay],wait=true,value=true,ps="a_mu.eps");
```

C.G. Albert, O. Biro, M.F. Heyn, W. Kernbichler, S.V. Kasilov, P. Lainer,
8th FreeFEM++ workshop, Dec 8th 2016
Dimensionality reduction for periodic magnetostatic fields

A field: homogenous mag. field, $\mu = 1$ everywhere
Dimensionality reduction for periodic magnetostatic fields

\( \mathbf{b} \) field: homogenous field, \( \mu = 1 \) everywhere

---

**C.G. Albert**, O. Biro, M.F. Heyn, W. Kernbichler, S.V. Kasilov, P. Lainer, 8th FreeFEM++ workshop, Dec 8th 2016
Dimensionality reduction for periodic magnetostatic fields

**a field: shielding by cylinder shell with $\mu > 1$**
Dimensionality reduction for periodic magnetostatic fields

*b* field: shielding by cylinder shell with $\mu > 1$
A few technical issues

- Careful with $\frac{1}{R}$ terms near axis (1st order works "well enough")
  - 0th order causes troubles
- Complex numbers "emulated" now
- Find best interface FreeFEM++ ↔ Fortran
Iterations for kinetic plasma equilibria

- Formally, \( \text{curl curl} \) solver yields \( B = \hat{M}J \) with solution operator \( \hat{M} \)
- Monte Carlo kinetic code yields \( J = \hat{K}(B_0 + B) \) (noisy)
- Equilibrium field: fixed point \( B = \hat{M}\hat{K}(B_0 + B) \) or
  \[
  (\hat{M}\hat{K} - \hat{I})B = -\hat{M}\hat{K}B_0
  \]
- Eigenvalues of \( \hat{M}\hat{K} > 1 \): relaxed iterations do not help
- Trick: Arnoldi method, solve unstable part separately
- Challenge: random noise from Monte Carlo method
Dimensionality reduction for periodic magnetostatic fields

ITER-like tokamak ($B_r$, vacuum) [4]
Dimensionality reduction for periodic magnetostatic fields

ITER-like tokamak ($B_r$, kinetic equilibrium) [4]
Dimensionality reduction for periodic magnetostatic fields

Conclusion

Take-home messages:

- Magnetostatics written as singular \textbf{curl curl} equation for $A$
  - 2D eqs. \textit{ungauged} for symmetric, \textit{gauged} for oscillatory

- \textbf{Co-/contravariant} notation useful for easy \textit{generalisation}

- \textbf{FreeFEM++ very} useful for fast and easy \textit{solution}

- \textbf{Outlook}: Apply to eddy currents, fluid dynamics (Stokes), etc.

References: