Remark, on Stokes equation

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The Stress of the fluid

Denote u the velocity field et p the pressure field
Then the classical mechanical stress $\boldsymbol{\sigma}^{\star}$ of the fluid :

$$\boldsymbol{\sigma}^{\star}(\mathbf{u}, p) = 2\mu\mathbb{D}(\mathbf{u}) - p\, I_d, \qquad \mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}) \tag{1}$$

Or in math formulation

$$\boldsymbol{\sigma}^{\bullet}(\mathbf{u}, p) = \mu\nabla\mathbf{u} - p\, I_d \tag{2}$$

So $\boldsymbol{\sigma}$ is one of this two stress tensor,

Remark: if $\nabla.\mathbf{u} = 0$ then $\nabla.2\mathbb{D}(\mathbf{u}) = \nabla.\nabla\mathbf{u} + \nabla.{}^t\nabla\mathbf{u} = \nabla.\nabla\mathbf{u} + \nabla\underbrace{\nabla.\mathbf{u}}_{=0} = \nabla.\nabla\mathbf{u}$

Stokes equations

In Domain Ω of $\mathbb{R}^d$, find the velocity field $\mathbf{u}$ et the pressure field p solution of

$$\begin{aligned} \nabla.\boldsymbol{\sigma}(\mathbf{u},p) &= \mathbf{f} \qquad (3)\\ -\nabla.\mathbf{u} &= 0 \qquad (4) \end{aligned}$$

$+$ Boundary condition are defined through the variational form

Where $\mathbf{f}$ is the density of force.

Variational form of Stokes equations

In Domain Ω of $\mathbb{R}^d$, find the velocity field $\mathbf{u}$ et the pressure field p
Mechanical Variational form of Stokes equation

$$\forall \mathbf{v}, q; \quad \int_\Omega 2\mu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) - q \nabla . \mathbf{u} - p \nabla . \mathbf{v} = \int_\Omega \mathbf{f} . \mathbf{v} + \int_\Gamma {}^t\mathbf{n} \boldsymbol{\sigma}^\star(\mathbf{u}, p) \mathbf{v}$$

Mathematical Variational form of Stokes equation

$$\forall \mathbf{v}, q; \quad \int_\Omega \mu \nabla \mathbf{u} : \nabla \mathbf{v} - q \nabla . \mathbf{u} - p \nabla . \mathbf{v} = \int_\Omega \mathbf{f} . \mathbf{v} + \int_\Gamma {}^t\mathbf{n} \boldsymbol{\sigma}^\bullet(\mathbf{u}, p) \mathbf{v}$$

with Ok, but what is the difference, and remember ${}^t\mathbf{n} \boldsymbol{\sigma}^\bullet(\mathbf{u}, p)$ are boundary density forces $\mathbf{f}_\Gamma$.

Basic Boundary condition for Stokes equations

Remove or know the boundary term $\int_\Gamma {}^t\mathbf{n}\boldsymbol{\sigma}(\mathbf{u},p)\mathbf{v}$

First remark

$$\int_\Gamma {}^t\mathbf{n}\boldsymbol{\sigma}(\mathbf{u},p)\mathbf{v} = \int_\Gamma {}^t\mathbf{f}_\Gamma\mathbf{v}.$$

Where $\mathbf{f}_\Gamma$ is the boundary force density (in mechanical formulation) .
All the boundary the trick is to know ${}^t\mathbf{f}_\Gamma\mathbf{v}$ or to put "$\mathbf{v} = 0$" on some component and this imply $\mathbf{u}$ know on this component

So try, with FreeFem++
Execute Stokes-Pipe.edp

Execute Stokes-ext.edp

Navier Boundary condition of Stokes equations

$\boldsymbol{\tau}$ the tangent , $\mathbf{n}$ the normal, on Γ, g a given function, remember the boundary force $\mathbf{f}_\Gamma = {}^t\mathbf{n}\boldsymbol{\sigma}(\mathbf{u}, p)$.

$$\begin{aligned} \mathbf{u}.\mathbf{n} &= 0 && (5) \\ \mathbf{f}.\boldsymbol{\tau} &= \beta u.\boldsymbol{\tau} + \mathbf{g}.\boldsymbol{\tau} && (6) \end{aligned}$$

This imply add in V.F. in RHS:

$$-\int_\Gamma \beta \mathbf{u}.\boldsymbol{\tau}\mathbf{v}.\boldsymbol{\tau} + \mathbf{g}.\boldsymbol{\tau} = -\int_\Gamma \beta\, {}^t\mathbf{u}(\boldsymbol{\tau}\, {}^t\boldsymbol{\tau})\mathbf{v} + \mathbf{g}.\boldsymbol{\tau}$$

Remark, if $\mathbf{n} \neq \mathbf{e}_i$, change $\mathbf{u}.\mathbf{n} = 0$ by penalisation we have

$$O = \frac{1}{\epsilon} u.\mathbf{n}; \qquad \text{Add to V.F. in RHS} - \int_\Gamma \frac{1}{\epsilon}\, {}^t\mathbf{u}(\mathbf{n}\, {}^t\mathbf{n})\mathbf{v}$$

Remark, Implementation of Dirichlet Boundary Conditions

Original problem is , Find $\mathbf{U} = (\mathbf{u_i}) \in \mathbb{R}^n$, such that

$$
\begin{aligned}
(AU &= B)_i && \text{Dof.} i \notin \Gamma_d && (7)\\
U_i &= G_i = (\Pi_h g)_i && \text{Dof.} i \in \Gamma_d && (8)
\end{aligned}
$$

where A is the matrices associated to the V.F. , B the RHS of the VF without the Dirichlet Boundary Conditions.
Let us call $\mathtt{tgv} = 10^{30}$ a huge value (tres grand valeur), and $I_{\Gamma_d} = ((i \in \Gamma_d)\delta_{ij})$

$$A_{\mathtt{tgv}} = A + \mathtt{tgv}\ I_{\Gamma_d}, \qquad B_{\mathtt{tgv}} = B + \mathtt{tgv}\ I_{\Gamma_d}\ G$$

We solve $A_{\mathtt{tgv}} U = B_{\mathtt{tgv}}$, the approximation is in $O(10^{-30})$, it's better than the number of digits 16, so it's exact not to close to 0.

Execute Stokes-Pipe-Navier.edp Execute Stokes-ext-Navier.edp Execute Stokes-BC.edp

Zero Tangent velocity, and Neumann boundary condition

if $\mathbf{u}.\boldsymbol{\tau} = 0$ and at continuous level when $\nabla_{\boldsymbol{\tau}}.u = 0$ and $0 = \nabla.\mathbf{u} = \nabla_{\boldsymbol{\tau}}.\mathbf{u} + \partial_n \mathbf{u}_n$ so
$\partial_n u_n = 0$
so in the case

$$\mathbf{f}_{\Gamma}.\mathbf{n} = {}^t\mathbf{n}\sigma(u,p)\mathbf{n} = p$$

and we have the following Boundary condition:

$$p = \mathbf{f}_{\Gamma}.\mathbf{n}$$

Curve pipe

Execute Stokes-Pipe-Curve.edp

A true examples for the fun

Execute NSNewtonCyl-100-mpi.edp

Execute NSCaraCyl-100-mpi.edp
Execute PrimitiveReduite.edp