

Remark, on Stokes equation

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Equation

The Stress of the fluid

Stokes Equations

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The Stress of the fluid

Denote u the velocity field et p the pressure field

Then the classical mechanical stress σ^* of the fluid :

$$\sigma^*(\mathbf{u}, p) = 2\mu\mathbb{D}(\mathbf{u}) - p I_d, \quad \mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u}) \quad (1)$$

Or in math formulation

$$\sigma^\bullet(\mathbf{u}, p) = \mu\nabla\mathbf{u} - p I_d \quad (2)$$

So σ is one of this two stress tensor,

Remark: if $\nabla \cdot \mathbf{u} = 0$ then $\nabla \cdot 2\mathbb{D}(\mathbf{u}) = \nabla \cdot \nabla\mathbf{u} + \nabla \cdot {}^t\nabla\mathbf{u} = \nabla \cdot \nabla\mathbf{u} + \underbrace{\nabla \cdot \nabla\mathbf{u}}_{=0} = \nabla \cdot \nabla\mathbf{u}$

Stokes equations

In Domain Ω of \mathbb{R}^d , find the velocity field \mathbf{u} et the pressure field p solution of

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{u}, p) = \mathbf{f} \quad (3)$$

$$-\nabla \cdot \mathbf{u} = 0 \quad (4)$$

+ Boundary condition are defined through the variational form

Where \mathbf{f} is the density of force.

Variational form of Stokes equations

In Domain Ω of \mathbb{R}^d , find the velocity field \mathbf{u} et the pressure field p

Mechanical Variational form of Stokes equation

$$\forall \mathbf{v}, q; \quad \int_{\Omega} 2\mu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) - q \nabla \cdot \mathbf{u} - p \nabla \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma} {}^t \mathbf{n} \boldsymbol{\sigma}^*(\mathbf{u}, p) \mathbf{v}$$

Mathematical Variational form of Stokes equation

$$\forall \mathbf{v}, q; \quad \int_{\Omega} \mu \nabla \mathbf{u} : \nabla \mathbf{v} - q \nabla \cdot \mathbf{u} - p \nabla \cdot \mathbf{v} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma} {}^t \mathbf{n} \boldsymbol{\sigma}^{\bullet}(\mathbf{u}, p) \mathbf{v}$$

with Ok, but what is the difference, and remember ${}^t \mathbf{n} \boldsymbol{\sigma}^{\bullet}(\mathbf{u}, p)$ are boundary density forces \mathbf{f}_{Γ} .

Basic Boundary condition for Stokes equations

Remove or know the boundary term $\int_{\Gamma} {}^t \mathbf{n} \boldsymbol{\sigma}(\mathbf{u}, p) \mathbf{v}$

First remark

$$\int_{\Gamma} {}^t \mathbf{n} \boldsymbol{\sigma}(\mathbf{u}, p) \mathbf{v} = \int_{\Gamma} {}^t \mathbf{f}_{\Gamma} \mathbf{v}.$$

Where \mathbf{f}_{Γ} is the boundary force density (in mechanical formulation) .

All the boundary the trick is to know ${}^t \mathbf{f}_{\Gamma} \mathbf{v}$ or to put " $\mathbf{v} = 0$ " on some component and this imply \mathbf{u} know on this component

So try, with FreeFem++

Execute Stokes-Pipe.edp

Execute Stokes-ext.edp

Navier Boundary condition of Stokes equations

$\boldsymbol{\tau}$ the tangent , \mathbf{n} the normal, on Γ , g a given function, remember the boundary force $\mathbf{f}_\Gamma = {}^t\mathbf{n}\boldsymbol{\sigma}(\mathbf{u}, p)$.

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad (5)$$

$$\mathbf{f} \cdot \boldsymbol{\tau} = \beta u \cdot \boldsymbol{\tau} + \mathbf{g} \cdot \boldsymbol{\tau} \quad (6)$$

This imply add in V.F. in RHS:

$$- \int_{\Gamma} \beta \mathbf{u} \cdot \boldsymbol{\tau} \mathbf{v} \cdot \boldsymbol{\tau} + \mathbf{g} \cdot \boldsymbol{\tau} = - \int_{\Gamma} \beta {}^t\mathbf{u} (\boldsymbol{\tau} {}^t\boldsymbol{\tau}) \mathbf{v} + \mathbf{g} \cdot \boldsymbol{\tau}$$

Remark, if $\mathbf{n} \neq \mathbf{e}_i$, change $\mathbf{u} \cdot \mathbf{n} = 0$ by penalisation we have

$$0 = \frac{1}{\epsilon} \mathbf{u} \cdot \mathbf{n}; \quad \text{Add to V.F. in RHS} - \int_{\Gamma} \frac{1}{\epsilon} {}^t\mathbf{u} (\mathbf{n} {}^t\mathbf{n}) \mathbf{v}$$

Remark, Implementation of Dirichlet Boundary Conditions

Original problem is , Find $\mathbf{U} = (\mathbf{u}_i) \in \mathbb{R}^n$, such that

$$(AU = B)_i \quad \text{Dof. } i \notin \Gamma_d \quad (7)$$

$$U_i = G_i = (\Pi_h g)_i \quad \text{Dof. } i \in \Gamma_d \quad (8)$$

where A is the matrices associated to the V.F. , B the RHS of the VF without the Dirichlet Boundary Conditions.

Let us call $t_{gv} = 10^{30}$ a huge value (tres grand valeur), and $I_{\Gamma_d} = ((i \in \Gamma_d)\delta_{ij})$

$$A_{t_{gv}} = A + t_{gv} I_{\Gamma_d}, \quad B_{t_{gv}} = B + t_{gv} I_{\Gamma_d} G$$

We solve $A_{t_{gv}} U = B_{t_{gv}}$, the approximation is in $O(10^{-30})$, it's better than the number of digits 16, so it's exact not to close to 0.

[Execute Stokes-Pipe-Navier.edp](#) [Execute Stokes-ext-Navier.edp](#) [Execute Stokes-BC.edp](#)

Zero Tangent velocity, and Neumann boundary condition

if $\mathbf{u} \cdot \boldsymbol{\tau} = 0$ and at continuous level when $\nabla_{\boldsymbol{\tau}} \cdot \mathbf{u} = 0$ and $0 = \nabla \cdot \mathbf{u} = \nabla_{\boldsymbol{\tau}} \cdot \mathbf{u} + \partial_n u_n$ so
 $\partial_n u_n = 0$

so in the case

$$\mathbf{f}_r \cdot \mathbf{n} = {}^t \mathbf{n} \sigma(u, p) \mathbf{n} = p$$

and we have the following Boundary condition:

$$p = \mathbf{f}_r \cdot \mathbf{n}$$

Curve pipe

Execute Stokes-Pipe-Curve.edp

Incompressible Navier-Stokes with Newton method's

To solve $F(u) = 0$ the Newton's algorithm is

1. u^0 a initial guest
2. do
 - ▶ find w^n solution of $DF(u^n)w^n = F(u^n)$
 - ▶ $u^{n+1} = u^n - w^n$
 - ▶ if($\|w^n\| < \varepsilon$) break;

For Navier Stokes problem the algorithm is: $\forall v, q,$

$$F(u, p) = \int_{\Omega} (u \cdot \nabla) u \cdot v + \nu \nabla u : \nabla v - q \nabla \cdot u - p \nabla \cdot v + BC$$

$$DF(u, p)(w, w_p) = \int_{\Omega} (w \cdot \nabla) u \cdot v + (u \cdot \nabla) w \cdot v \\ + \int_{\Omega} \nu \nabla w : \nabla v - q \nabla \cdot w - p_w \nabla \cdot v + BC0$$

Execute `cavityNewton.edp`

Execute `NSNewtonCyl-100-mpi.edp`