# Convection

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## Introduction

Math

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# Characteristic

*Let* $\Omega$ *be an open bounded set with boundary* $\Gamma$ *Lipschitz; we denote by n its exterior normal ; the system*

$$\phi_{,t} + u\nabla\phi + a\phi = f \text{ in } Q = \Omega\times]0,T[ \tag{57}$$

$$\phi(x,0) = \phi^0(x) \quad \forall x \in \Omega \tag{58}$$

$$\phi(x,t) = g(x,t) \quad \forall (x,t) \in \Sigma = ((x,t) : u(x,t).n(x) < 0) \tag{59}$$

*has a unique solution in* $C^0(0,T;L^2(\Omega))$ *when* $\phi^0 \in L^2(\Omega)$, g $\in C^0(0,T;L^2(\Gamma))$ *and* $a, u \in L^\infty(Q)$, Lipschitz in x, $f \in L^2(Q)$.

# Characterics

We complete the proof by constructing the solution. Let $X(\tau)$ the solution of

$$\frac{d}{d\tau}X(\tau) = u(X(\tau),\tau) \text{ if } X(\tau) \in \Omega \tag{60}$$

$$= 0 \text{ otherwise}$$

with the boundary condition

$$X(t) = x. \tag{61}$$

If $u$ is the velocity of the fluid, then $X$ is the trajectory of the fluid particle that passes $x$ at time $t$. With $u \in L^\infty(Q)$ problem (60)-(61) has a unique solution. As $X$ depends on the parameters $x, t$, we denote the solution $X(x, t; \tau)$ ; it is also the "characteristic" of the hyperbolic equation (57).

# Time discretization for Convection-Diffusion

**Time discretization for Convection-Diffusion** In this paragraph we shall analyze some schemes obtained by using finite difference methods to discretise $\partial\phi/\partial t$ and the usual variational methods for the remainder of the equation. Let us consider equation (67) with (70) and assume, for simplicity from now on and throughout the chapter that $\kappa_{ij} = \nu\delta_{ij}, \nu > 0$, $u \in L^\infty(Q)$ and

$$\nabla.u = 0 \text{ in Q } u.n = 0 \text{ on } \Gamma\times]0,T[ \tag{71}$$

$$\phi = 0 \text{ on } \Gamma\times]0,T[ \tag{72}$$

As before, $\Omega$ is also assumed regular.
The reader can extend the results with no difficulty to the case $u.n \neq 0$, with the help of paragraph 2.
As usual, we divide ]0,T[ into equal intervals of length $k$ and denote by $\phi^n(x)$ an approximation of $\phi(x, nk)$.

**3.3.1. Implicit Euler scheme :**
We search $\phi_h^{n+1} \in H_{oh}$, the space of polynomial functions of degree $p$ on a triangulation of $\Omega$, continuous and zero on $\Gamma$ such that for all $w_h \in H_{0h}$ we have

$$\frac{1}{k}(\phi_h^{n+1} - \phi_h^n, w_h) + (u^{n+1}\nabla\phi_h^{n+1}, w_h) + \nu(\nabla\phi_h^{n+1}, \nabla w_h) = (f^{n+1}, w_h) \tag{73}$$

This problem has a unique solution because this is an $N \times N$ linear system, $N$ being the dimension of $H_{0h}$ :

$$(A + kB)\Phi^{n+1} = kF + I\Phi \tag{74}$$

where $A_{ij} = (w^i, w^j) + \nu k(\nabla w^i, \nabla w^j)$, $B_{ij} = (u^{n+1}\nabla w^i, w^j)$, $F_i = (f, w^i)$, $I = (w^i, w^j)$ and where $\{w^i\}$ is a basis of $H_{0h}$ and $\phi_i$ the coefficients of $\phi_h$ on this basis. This system has a unique solution because the kernel of $A + kB$ is empty :

$$0 = \psi^T(A + kB)\psi = \psi^T A\psi \Rightarrow \psi = 0. \tag{75}$$

**Proposition 9 :**

*If* $\phi \in L^2(0, T; H^{p+1}(\Omega))$ *and* $\phi_{,t} \in L^2(0, T; H^p(\Omega))$, *we have*

$$(|\phi_h^n - \phi(nk, .)|_0^2 + \nu k|\nabla(\phi_h^n - \phi(nk, .))|_0^2)^{\frac{1}{2}} \leq C(h^p + k) \tag{76}$$

*where p is the degree of the polynomial approximation for* $\phi_h^n$.

**3.3.2. Leap frog scheme :**

The previous scheme requires the solution of a non-symmetric matrix for each iteration. This is a scheme which does not require that kind of operation but which works only when $k$ is $O(h)$.

$$\frac{1}{2k}(\phi_h^{n+1} - \phi_h^{n-1}, w_h) + (u^n \nabla \phi_h^n, w_h) + \frac{\nu}{2}(\nabla[\phi_h^{n+1} + \phi_h^{n-1}], \nabla w_h) \tag{85}$$
$$= (f^{n+1}, w_h) \quad \forall w_h \in H_{0h}, \quad \phi_h^{n+1} \in H_{0h}$$

To start (85), we could use the previous scheme.

**Proposition 10 :**
*The scheme (85) is marginally stable , i.e. there exists a C such that*

$$|\phi_h^n|_o \leq C|f|_{0,Q}(1 - C|u|_\infty \frac{k}{h})^{-\frac{1}{2}} \tag{86}$$

*for all k such that*

$$k < \frac{h}{C|u|_\infty}. \tag{87}$$

### 3.3.3. Adams-Bashforth scheme :

It is important to make the dissipation terms implicit as we have done for $\Delta\phi$ because the leap-frog scheme is only marginally stable (cf. Richtmeyer-Morton [197]). In the same way, if $\Sigma \neq \oslash$ $(u.n \neq 0)$, it is necessary to make implicit the integral on $\Gamma - \Sigma$. For this reason, we consider the Adams-Bashforth scheme of order 3 which is explicit when we use mass lumping and which has a better stability than the leap-frog scheme.

$$\frac{1}{k}(\phi_h^{n+1} - \phi_h^n, w_h) = \frac{23}{12}b(\phi_h^n, w_h) - \frac{16}{12}b(\phi_h^{n-1}, w_h) + \frac{5}{12}b(\phi_h^{n-2}, w_h)$$

$\forall w_h \in H_{0h}$ where

$$b(\phi_h, w_h) = -[(u\nabla\phi_h, w_h) + \nu(\nabla\phi_h, \nabla w_h) - (f, w_h)]$$

The stability and convergence of the scheme can be analysed as in §3.3.5.

### 3.3.4 The $\theta-$ schemes :

In a general way, let $A$ and $B$ be two operators and the equation in time be

$$u_{,t} + Au + Bu = f; u(0) = u^0$$

We consider a scheme with three steps

$$\frac{1}{k\theta}(u^{n+\theta} - u^n) + Au^{n+\theta} + Bu^n = f^{n+\theta}$$

$$\frac{1}{(1-2\theta)k}(u^{n+1-\theta} - u^{n+\theta}) + Au^{n+\theta} + Bu^{n+1-\theta} = f^{n+1-\theta}$$

$$\frac{1}{k\theta}(u^{n+1} - u^{n+1-\theta}) + Au^{n+1} + Bu^{n+1-\theta} = f^{n+1}$$

# Time-space Approximation

In this section we analyze schemes for the convection-diffusion equation (53)-(55) which still converge when $\nu = 0$ without generating oscillations. This classification is

somewhat arbitrary because the previous schemes can be made to work when $\nu = 0$ or when $\varphi$ is irregular. But we have put in this section schemes which have been generalized to nonlinear equations (Navier-Stokes and Euler equations for example).
Let us list the desirable properties for a scheme to work on nonsmooth functions $\phi$ :
- convergence in $L^\infty$ norm,
- positivity: $\varphi > 0 \Rightarrow \varphi_h > 0$,
- convergence to the stationary solution when $t \to \infty$,
- localization of the solution if $\nu = 0$ ( that is to say that the solution should not depend upon whatever is downstream of the characteristic when $\nu = 0$).

## 4.1. Discretisation of the total derivative:

### 4.1.1 Discretisation in time.

We have seen that if $X(x, t; \tau)$ denotes the solution of

$$\frac{dX}{d\tau}(\tau) = u(X(\tau), \tau); \quad X(t) = x \tag{93}$$

then

$$\phi_{,t} + u\nabla\phi = \frac{\partial}{\partial\tau}\phi(X(x, t; \tau), \tau)|_{\tau=t} \tag{94}$$

Thus, taking into account the fact that $X(x,(n+1)k;(n+1)k) = x$, we can write:

$$(\phi_{,t} + u\nabla\phi)^{n+1} \cong \frac{1}{k}[\phi^{n+1}(x) - \phi^n(X^n(x))] \tag{95}$$

where $X^n(x)$ is an approximation of $X(x,(n+1)k;nk)$.
We shall denote by $X_1^n$ an approximation $0(k^2)$ of $X^n(x)$ and by $X_2^n$ an approximation $0(k^3)$ (the differences between the indices of $X$and the exponents of $k$ are due to the fact that $X^n$ is an approximation of $X$ obtained by an integration over a time $k$ ; thus a scheme $0(k^\alpha)$ gives a precision $0(k^{\alpha+1})$).
For example

$$X_1^n(x) = x - u^n(x)k \text{ (Euler scheme for (93))} \tag{96}$$

$$X_2^n(x) = x - u^{n+\frac{1}{2}}(x - u^n(x)\frac{k}{2})k \text{ ( Second order Runge-Kutta)} \tag{97}$$

modified near the boundary so as to get $X_i^n(\Omega) \subset \Omega$. To obtain this inclusion one can use (96) or (97) inside the elements so that one passes from $x$ to $X^n(x)$ by a broken line rather than a straight line (see (18)(19)).

This yields two schemes for (53) :

$$\frac{1}{k}(\phi^{n+1} - \phi^n o X_1^n) - \nu \Delta \phi^{n+1} = f^{n+1} \tag{98}$$

$$\frac{1}{k}(\phi^{n+1} - \phi^n o X_2^n) - \frac{\nu}{2}\Delta(\phi^{n+1} + \phi^n) = f^{n+\frac{1}{2}} \tag{99}$$

**Lemma 1 :**
*If u is regular and if $X_i^n(\Omega) \subset \Omega$, the schemes (98) and (99) are $L^2-$stable and converge in $0(k)$ and $0(k^2)$ respectively.*

*Proof :*
Let us show consider (98). We multiply by $\phi^{n+1}$ :

$$|\phi^{n+1}|^2 + \nu k |\nabla \phi^{n+1}|^2 \leq (|f^{n+1}|k + |\phi^n o X_1|)|\phi^{n+1}| \tag{100}$$

But the map $x \to X$ preserves the volume when $u^n$ is solenoidal $(\nabla . u = 0)$. So from (96) :

$$|\phi^n o X_1^n|_0^2 = \int_{X_1^n(\Omega)} \phi^n(y)^2 det[\nabla X_1^n]^{-1} dy \leq |\phi^n|_{0,\Omega}^2 (1 + ck^2) \tag{101}$$

Hence $\phi^n$ verifies

$$||\phi^n||_\nu \leq c[|f|_{0,Q} + |\phi^o|_{0,\Omega}] \tag{102}$$

To get an error estimate one proceeds as in the beginning of the proof of proposition 9 by using (95).

**Remark :**
$X_i^n(\Omega) \subset \Omega$ is necessary because $u.n = 0$. Otherwise one only needs $X_i^n(\Omega) \cap \partial\Omega \subset \Sigma$.

### 4.1.2 Approximation in space.

Now if we use the previous schemes to approximate the total derivative (scheme (98) of order 1 , scheme (99) of order 2) and if we discretise in space by a conforming polynomial finite element we obtain a family of methods for which no additional upwinding is necessary and for which the linear systems are symmetric and time independent .
Take for example the case of (98) :

$$\int_\Omega \phi_h^{n+1} w_h + k\nu \int_\Omega \nabla\phi_h^{n+1} \nabla w_h = k \int_\Omega f^{n+1} w_h + \int_\Omega \phi_h^n(X_1^n(x)) w_h(x)$$

$$\forall w_h \in H_{0h} \quad \phi_h^{n+1} \in H_{0h} \tag{103}$$

where $H_{0h}$ is the space of continuous polynomial approximation of order 1 on a triangulation of $\Omega$, and zero on the boundaries.

**Proposition 11 :**
*If $X_1(\Omega) \subset \Omega$, the scheme* (103) *is $L^2(\Omega)$ stable even if $\nu = 0$.*

*Proof :*
One simply replaces $w_h$ by $\phi_h^{n+1}$ in (103) and derives upper bounds :

$$|\phi_h^{n+1}|_0^2 \leq \int_\Omega |\phi_h^{n+1}|^2 + k\int_\Omega \nu \nabla\phi_h^{n+1}\nabla\phi_h^{n+1} \tag{104}$$
$$= k\int_\Omega f^{n+1}\phi_h^{n+1} + \int_\Omega \phi_h^n(X_1^n(x))\phi_h^{n+1}(x)$$
$$\leq (k|f^{n+1}|_0 + |\phi_h^n o X_1^n(.)|_0)|\phi_h^{n+1}|_0$$
$$\leq (|\phi_h^n|_0(1+\frac{c}{2}k^2) + k|f^{n+1}|_0)|\phi_h^{n+1}|_0$$

The last inequality is a consequence of(101).
Finally by induction one obtains

$$|\phi_h^n|_{0,\Omega} \leq (1+\frac{c}{2}k^2)^n(|\phi_h^o|_{0,\Omega} + \sum k|f^n|_{0,\Omega}) \tag{105}$$

**Remark :**

By the same technique similar estimates can be found for (105) but the norms on $\phi_h^n$ et $\phi_h^o$ will be $||.||_\nu$ (cf. (79)).

**Proposition 12** .

*If $H_{0h}$ is a $P^1$ conforming approximation of $H_0^1(\Omega)$ then the $L^2(\Omega)$ norm of the error between $\phi_h^n$ solution of (103) and $\phi^n$ solution of (98) is $0(h^2/k+h)$. Thus the scheme is $0(h^2/k+k+h)$ .*

*Proof :*

One subtracts (98) from (103) to obtain an equation for the projected error:

$$\epsilon_h^{n+1} = \phi_h^{n+1} - \Pi_h \phi^{n+1}, \tag{106}$$

where $\Pi_h\ \phi^{n+1}$ is an interpolation in $H_{0h}$ of $\phi^{n+1}$. One gets

$$\int_\Omega \epsilon_h^{n+1} w_h + k\nu \int_\Omega \nabla \epsilon_h^{n+1} \nabla w_h - \int_\Omega \epsilon_h^n o X_1^n w_h = \tag{107}$$

$$\int_\Omega (\phi^{n+1} - \Pi_h \phi^{n+1}) w_h + \nu k \int_\Omega \nabla(\phi^{n+1} - \Pi_h \phi^{n+1}) \nabla w_h$$

$$- \int_\Omega (\phi^n - \Pi_h \phi^n) o X_1^n w_h$$

From (107), with $w_h = \epsilon_h^{n+1}$ we obtain

$$||\epsilon_h^{n+1}||_\nu^2 \leq (||\epsilon_h^n||_\nu + ||\phi^{n+1} - \Pi_h \phi^{n+1}||_\nu + |\phi^n - \Pi_h \phi^n|_0)||\epsilon_h^{n+1}||_\nu$$

therefore

$$||\epsilon_h^{n+1}||_\nu \leq ||\epsilon_h^n||_\nu + C(h^2 + \nu k h)$$

**Remark :**

By comparing with (103), we see that $\epsilon_h$ and $\phi_h$ are solutions of the same problem but for $\epsilon_h$, $f$ is replaced by :

$$\frac{1}{k}(\phi^{n+1} - \Pi_h \phi^{n+1}) - \nu \Delta_h (\phi^{n+1} - \Pi_h \phi^{n+1}) - \frac{1}{k}(\phi^n - \Pi_h \phi^n) o X_1^n,$$

where $\Delta_h$ is an approximation of $\Delta$. We can bound independently the first and the last terms. By working a little harder (Douglas-Russell [69]) one can shown that the error is, in fact, $0(h + k + \min(h^2/k, h))$.

**Proposition 13 :**

*With the second order scheme in time (99) and a similar approximation in space one can build schemes* $0(h^2 + k^2 + \min(h^3/k, h^2))$ *with respect to the* $L^2$ norm:

$$(\phi_h^{n+1}, w_h) + \frac{\nu k}{2}(\nabla(\phi_h^{n+1} + \phi^n), \nabla w_h) = (\phi_h^n o X_2^n, w_h) + k(f^{n+\frac{1}{2}}, w_h) \tag{108}$$

$$\forall w_h \in H_{0h}; \phi_h^{n+1} \in H_{0h}$$

*where* $H_{0h}$ *is a* $P^2$ *conforming approximation of* $H_0^1(\Omega)$.

*Proof :*
The proof is left as an exercise.

**The case** $\nu = 0$ :
We notice that (103) becomes

$$\int \phi_h^{n+1} w_h = \int_\Omega \phi_h^n o X_2^n w_h + k \int_\Omega f^{n+1} w_h \quad \forall w_h \in H_{0h} \tag{109}$$

$$\phi_h^{n+1} \in H_{0h}$$

That is to say

$$\phi_h^{n+1} = \Pi_h(\phi_h^n o X_1^n) + k\Pi_h f^{n+1} \tag{110}$$

where $\Pi_h$ is a $L^2$ projection operator in $W_{0h}$. Scheme (108) becomes :

$$\phi_h^{n+1} = \Pi_h(\phi_h^n o X_2^n) + \frac{k}{2}\Pi_h(f^{n+1} + f^n) \tag{111}$$

If $f = 0$ the only difference between the schemes are in the integration formula for the characteristics. Notice also that the numerical diffusion comes from the $L^2$ projection at each time step. Thus it is better to use a precise integration scheme for the characteristics and use larger time steps. Experience shows that $k \approx 1.5h/u$ is a good choice.

Notice that when $\nu$ and $f$ are zero one solves

$$\phi_{,t} + u\nabla\phi = 0 \quad \phi(x,0) = \phi^0(x) \tag{112}$$

-(since we have assumed $u.n = 0$, no other boundary condition is needed).

Since $\nabla.u =$ 0, we deduce from (112) that (conservativity)

$$\int_\Omega \phi(t,x) = \int_\Omega \phi^0(x) \quad \forall t \tag{113}$$

On the other hand, from (109), with $w_h =$ 1

$$\int_\Omega \phi_h^{n+1}(x) = \int_\Omega \phi_h^n o X_1^n = \int_{X_1^n(\Omega)} \phi_h^n(y) det|\nabla X_1^n|^{-1} dy \tag{114}$$

So if $det|\nabla X_1^n| = 1$ (which requires $X_1^n(\Omega) = \Omega$), one has

$$\int_\Omega \phi_h^{n+1}(x)dx = \int_\Omega \phi_h^n(y)dy = \int_\Omega \phi^0(x)dx \tag{115}$$

We say then that the scheme is conservative. It is an important property in practice.

**The streamline upwinding method (SUPG).**

The streamline upwinding method studied in §2.4 can be applied to (123) without distinction between $t$ and $x$ but it would then yield very large linear systems. But there are other ways to introduce streamline diffusion in a time dependent convection-diffusion equation.
The simplest (Hughes [116]) is to do it in space only; so consider

$$(\phi_h^{n+1}, w_h + \tau\nabla.(uw_h)) + \frac{k}{2}(\nabla.(u[\phi_h^n + \phi_h^{n+1}]), w_h + \tau\nabla.(uw_h))$$

$$+\frac{k\nu}{2}(\nabla(\phi_h^{n+1} + \phi_h^n), \nabla w_h) - \frac{k\nu}{2}\sum_l \int_{T_l}(\Delta(\phi_h^{n+1} + \phi_h^n)\tau\nabla.(uw_h))$$

$$= k(f^{n+\frac{1}{2}}, w_h + \tau\nabla.(uw_h)) + (\phi_h^n, w_h + \tau\nabla.(uw_h)), \quad \forall w_h \in H_{oh}$$

where $T_l$ is an element of the triangulation and $u$ is evaluated at time $(n+1/2)k$ if it is time dependant; $\tau$ is a parameter which should be of order $h$ but has the dimension of a time.
With first order elements, $\Delta\phi_h = 0$ and it was noticed by Tezduyar [227] that $\tau$ could be chosen so as to get symmetric linear systems when $\nabla.u = 0$; an elementary computation shows that the right choice is $\tau = k/2$. Thus in that case the method is quite competitive, even though the matrix of the linear system has to be rebuilt at each time step when $u$ is time dependant.

The error analysis of Johnson [125] suggests the use of elements discontinuous in time, continuous in space and a mixture upwinding by discontinuity in time and streamline upwinding in space.
To this end space-time is triangulated with prisms. Let $Q^n = \Omega\times]nk,(n+1)k[$, let $W^n_{oh}$ be the space of functions $in\{x,t\}$ which are zero on $\Gamma\times]nk,(n+1)k[$ continuous and piecewise affine in $x$ and in $t$ separately on a triangulation by prisms of $Q^n$ ;
We search $\phi^n_h$ with $\phi^n_h - \phi_{\Gamma h} \in W^n_{oh}$ solution of

$$\int_{Q^n}[\phi^n_{h,t} + \nabla(u\phi^n_h)][w_h + h(w_{h,t} + \nabla(uw_h)]dxdt + \int_{Q^n}\nu\nabla\phi^n_h.\nabla w_h dxdt \qquad (136)$$

$$+\int_\Omega \phi_h^n(x,(n-1)k+0)w_h(x,(n-1)k)dx = \int_{Q^n} f(w_h + h(w_{h,t} + \nabla.(uw_h))dxdt$$

$$+\int_\Omega \phi_h^{n-1}(x,(n-1)k-0)w_h(x,(n-1)k)dx, \quad \forall w_h \in W_{oh}^n$$

Note that if $N$ is the number of vertices in the triangulation of $Q^n$, equation (136) is

an $N \times N$ linear system, positive definite but non symmetric.
One can show (Johnson et al. [127]) the following :

$$\left(\int_0^T |\phi_h^n - \phi|_{0,\Omega}^2 dt\right)^{\frac{1}{2}} \leq C(h^{\frac{3}{2}} + k^{\frac{3}{2}})||\phi||_{H^2(Q)}. \tag{137}$$