Numerical inf-sup condition for finite element approximation for the Stokes equations with velocity and pressure meshes

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outline

- Stokes equation
- Discrete Stokes equation - inf-sup condition - discrete inf-sup constant
- Definition of the aspect of numerical study
  - behavior of discrete inf-sup constant
- Computation of discrete inf-sup constant
- Numerical study

Stokes equation

- Let Ω be a bounded connected open domain in $\mathbb{R}^2$ with a Lipschitz-continous boundary. The Stokes problem with a homogeneous Dirichlet boundary condition on Γ reads:

$$\begin{cases} -\nu\,\Delta\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \mathrm{div}(\mathbf{u}) = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \Gamma. \end{cases}$$

- Well Posed : For an external forces f in $H^{-1}(\Omega)$, Stokes problem is well-posed in $H_0^1(\Omega) \times L_0^2(\Omega)$ for the velocity-pressure pair where $L_0^2(\Omega) = \{q \in L^2(\Omega);\ \int_\Omega q(x)\,dx = 0\}$.
- inf-sup condition:

$$0 < \beta = \inf_{q \in L_0^2(\Omega)} \sup_{\mathbf{v} \in H_0^1(\Omega)} \frac{(-div(\mathbf{v}), q)}{|\mathbf{v}|_{H^1(\Omega)}\ \|q\|_{L^2(\Omega)}}.$$

Discretization of Stokes equation

- Mixed Finite Element Method
  - Two triangulations $\{\mathcal{T}_{h_p}\}$ and $\{\mathcal{T}_{h_{\mathbf{u}}}\}$ of Ω
  - Continous piecewise Finite Element: Pk-Pk' element for $(\mathbf{u}_h, p_h)$

$$
\begin{array}{ll}
X_{h_u} & = \{\mathbf{v} \in (C^0(\Omega))^d / \forall K \in \mathcal{T}_{h_{\mathbf{u}}}, \mathbf{v} \in (P_k)^d\} \\
M_{h_p} & = \{q \in C^0(\Omega) / \forall K \in \mathcal{T}_{h_p}, q \in P'_k, \int_\Omega q \ = 0\}
\end{array}
$$

- Variationnal Form
  Find $\mathbf{u}_h \in X_{h_u}$ et $p_h \in M_{h_p}$

$$
\begin{array}{ll}
\forall \mathbf{v}_h \in X_{h_u}, & a_{h_u}(\mathbf{u}_h, \mathbf{v}_h) + b_{h_u}(\mathbf{v}_h, p_h) = (f, \mathbf{v}_h), \\
\forall q_h \in M_{h_p}, & b_{h_u}(\mathbf{u}_h, q_h) = 0
\end{array}
$$

where

$$
\begin{array}{l}
a_{h_u}(\mathbf{u}_h, \mathbf{v}_h) = \sum_{K \in \mathcal{T}_{h_u}} \int_K grad(\mathbf{u}_h).grad(\mathbf{v}_h)\, d\mathbf{x}, \\
b_{h_u}(\mathbf{u}_h, q_h) = -\sum_{K \in \mathcal{T}_{h_u}} \int_K q_h\, div(\mathbf{v}_h)\, d\mathbf{x}.
\end{array}
$$

- The choice of $\mathcal{T}_{h_u}$ will be explained after.

Discretization of Stokes equation

- Existence, Uniqueness and Stability solution
  - Saddle Point problem
  - Theorem de Babuška-Brezzi $\Rightarrow$ inf-sup condition (discrete inf-sup constant)

$$0 < \beta_{h_u,h_p} = \inf_{q_h \in M_{h_p}} \sup_{\mathbf{v}_h \in X_{h_u}} \frac{b_{h_u}(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{X_{h_u}} \|q_h\|_{M_{h_p}}}$$

  - The parameter size h tend 0

$$\Rightarrow 0 < \beta^* \leq \beta_{h_u,h_p}$$

- Discretized error :

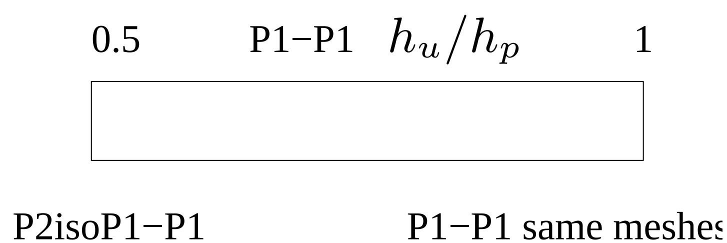
$$|u - u_h|_{H^1(\Omega)^d} + \|p - p_h\|_{L^2(\Omega)}$$

$$\leq C(\inf_{v_h \in X_{h_u}} |u - v_h|_{H^1(\Omega)^d} + \inf_{q_h \in M_{h_p}} \|p - q_h\|_{L^2(\Omega)}),$$

where C constant depend linearly on the inverse of β^*.

Definition of our study

- Classical Mixed FE for Stokes
  - Same meshes for $\mathbf{u}$ and p: P1b-P1, P2b-P1, P2-P1(Taylor-Hood)
  - Nested meshes: P2isoP1-P1 (Bercovier and Pironneau),
- Inf-sup condition is not verified:
  P1-P1 and P2-P2 with same meshes for $\mathbf{u}$ and $p \Rightarrow$ Stabilized method
- P1-P1 with different meshes: P2isoP1-P1 $\Rightarrow$ P1-P1 with $h_u/h_p = 0.5$



- Numerical Study: Behavior of discrete inf-sup constant
  - Two different meshes for $\mathbf{u}$ and p. No nested assumption.
  - ratio h_u/h_p constant.
  - FE: P1-P1 and P2-P2

Computation of discrete inf-sup constant

- Eigenproblem (D.S. Malkus, 1981):

$$B^t\, A^{-1}\, B\, \mathcal{P} = \lambda\, M\, \mathcal{P},$$

where A and B are FE matrix corresponding to $a_{h_u}(.,.)$ and $b_{h_u}(.,.)$ respectively and M is the mass matrix associated to the FE space M_{h_p}

  - $\beta^2_{h_u,h_p}$ *is the smallest Eigenvalue*
  - *computation of the inverse of A^{-1} is expensive*
    $\Rightarrow$ *Equivalent eigenproblem $A\mathcal{U} = B\mathcal{P}$*

$$\begin{pmatrix} -A & B \\ B^t & 0 \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{U}} \\ \tilde{\mathcal{P}} \end{pmatrix} = \lambda \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{U}} \\ \tilde{\mathcal{P}} \end{pmatrix}$$

  - *Solution $(\mathcal{U}', \mathcal{P}', \lambda') = (0, C(1, ..., 1)^t, 0)$ but pressure approximation satisfied $\int_\Omega p = 0 \Rightarrow$ not a correct solution*
  - *Numerically we need to find this solution for consistency*

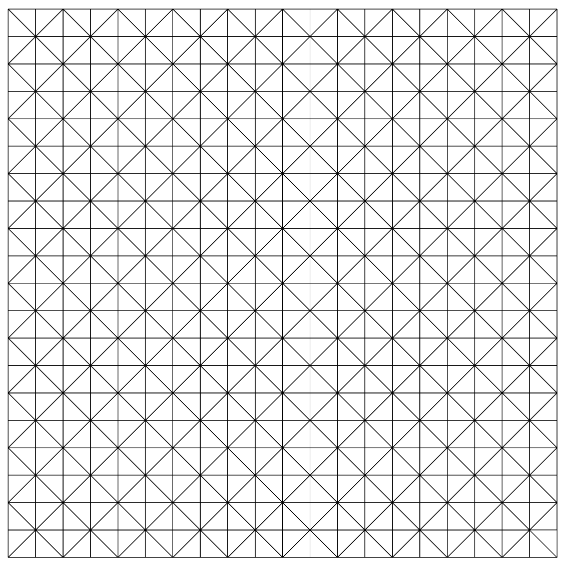
- Discretization of $b(\mathbf{u}_h, p_h) = \int_\Omega -\mathrm{div}(\mathbf{u_h})\, \mathrm{p_h}$
  - *pressure's mesh $\Rightarrow (\mathcal{U}', \mathcal{P}', \lambda')$ is not a solution*
  - *velocity's mesh $\Rightarrow (\mathcal{U}', \mathcal{P}', \lambda')$ is a solution*

Hypothesis on the Mesh

- Reminder : Behavior of β_{h_u,h_p} for h_u/h_p constant
- Structured meshes with this ratio between space steps

$$\frac{\Delta x_u}{\Delta x_p,} = \frac{\Delta y_u}{\Delta y_p} = C$$

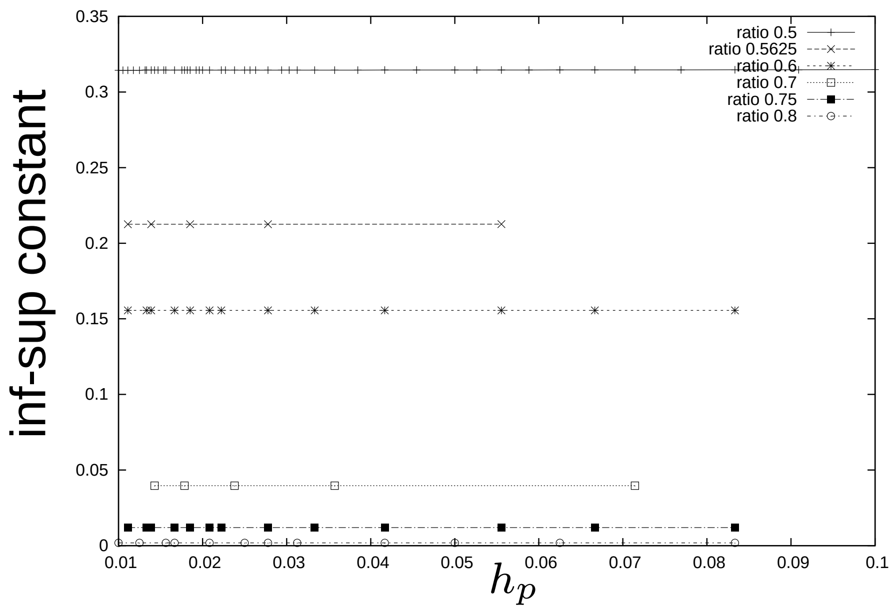
- Square $(0,1) \times (0,1)$ and rectangle $(0,1) \times (0,10)$



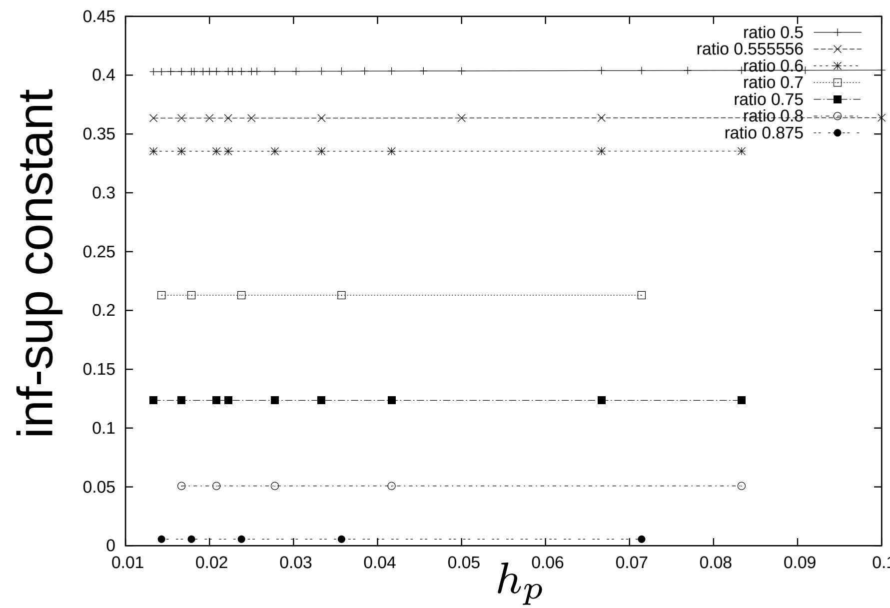
Numerical study: h_u/h_p constant

- Freefem++ software - eigensolver arpack
- Discrete inf-sup constant at h_u/h_p constant for the square

P1-P1



P2-P2

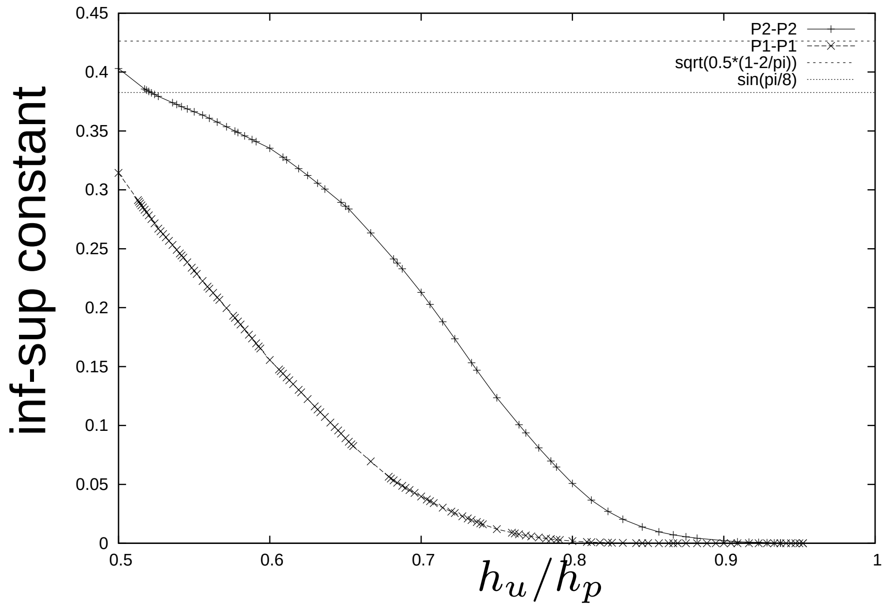


- Discrete inf-sup constant doesn't depend on h_p
- Lower bound independant of h
- Same behavior for rectangle

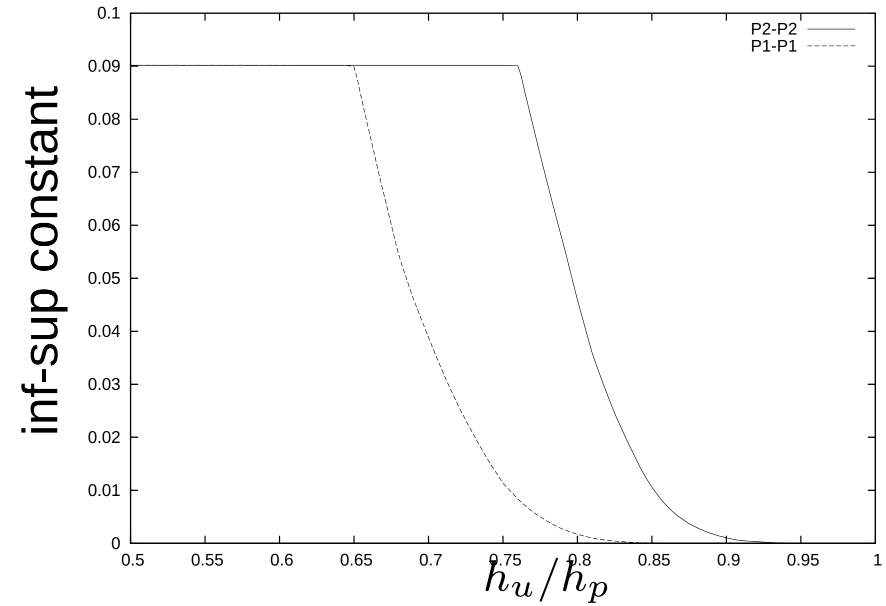
Numerical study

- Lower bound of discrete inf-sup constant for different ratio h_u/h_p

square



rectangle (ratio 10)



- *E. V. Chizhonkov and M. A. Olshanskii(2000): Discrete inf-sup constant converge to continuous inf-sup constant for rectangle with a ratio* > 4.
- *Discrete inf-sup condition satisfied for* h_u/h_p
  [0.5,0.8] for P1-P1
  [0.5,0.9] for P2-P2
- *A lower discrete inf-sup constant (P1-P1*$h_u/h_p = 0.8$ *3.35e-4). it is good for application?*

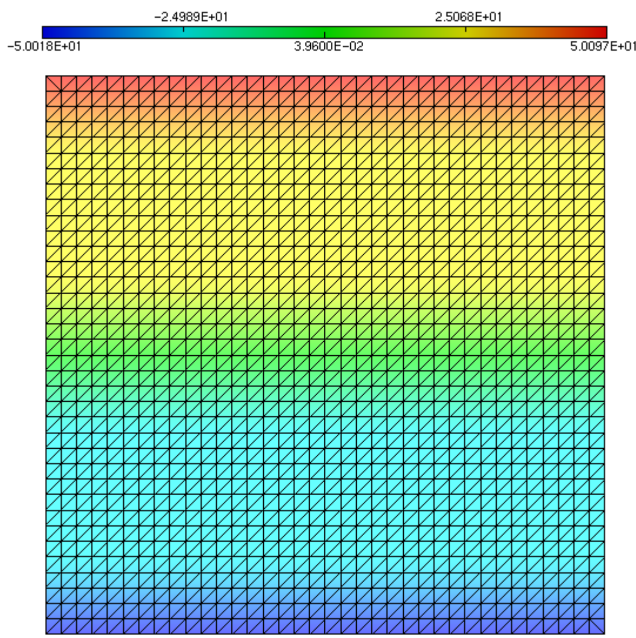
Numerical study: Discretized error

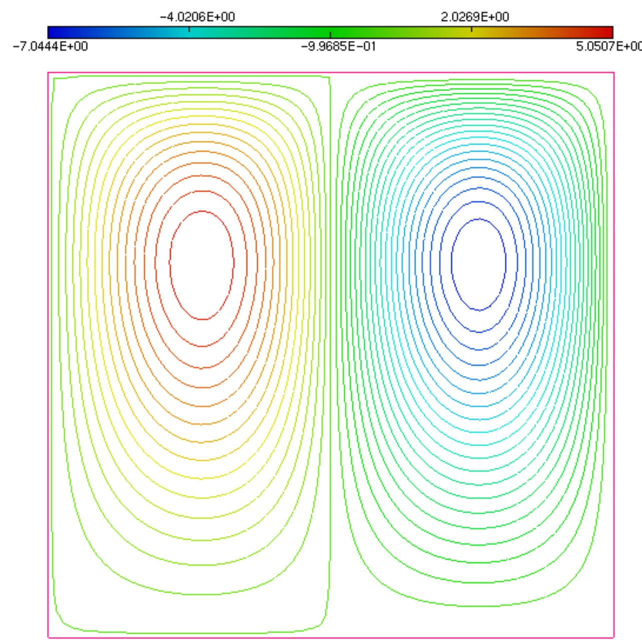
- Analytical solution: $\Omega = [0,1]^2$

$$\mathbf{u} = (\partial_y \varphi, -\partial_x \varphi), \qquad p = 100(y - 0.5),$$

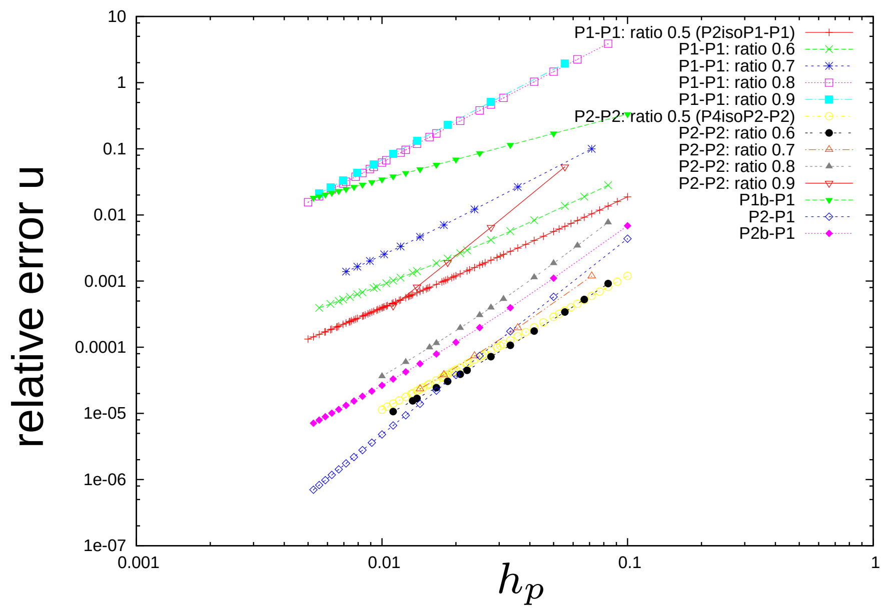
where $\varphi(x,y) = (y+1)^3 \, \sin(\pi y) \, (x+1) \, \sin(2\,\pi\, x)$.

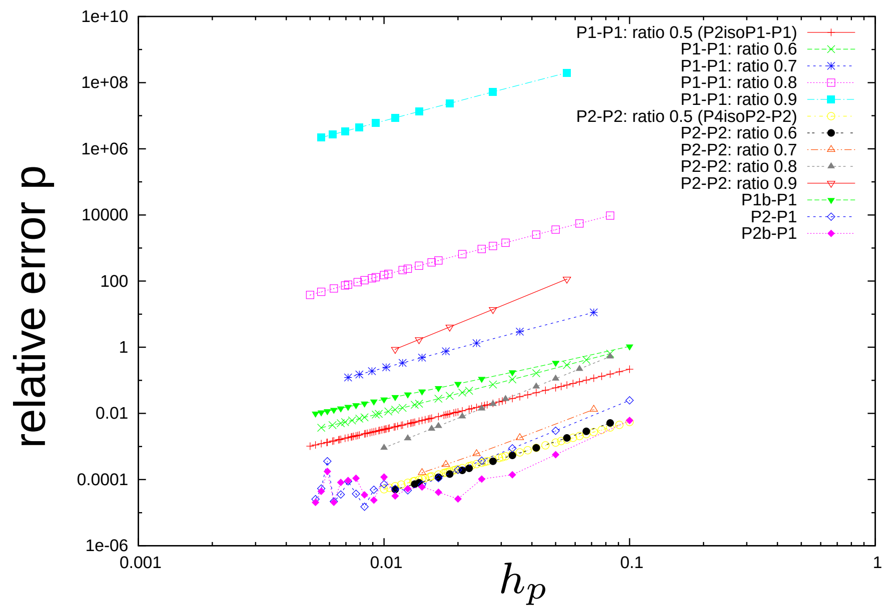
- P2-P2 with $h_u/h_p = 0.6$





Numerical study: Discretized error





- *Velocity: a relative error lower to 0.01.*
- *Pressure: difficult to obtain an good error for high ratio ($h_u/h_p = 0.8$ P1-P1)*
  - *Relative error depend linearly of $1/\beta^*$*
  - *Discrete inf-sup condition is not sufficient to obtain good convergence*
- *Bad pressure error: β^* is 100 times lower than continous inf-sup.*
  $\Rightarrow$ Discrete inf-sup constant of the same order than the continuous inf-sup constant (?).
  - *P2-P2 with ratio $h_u/h_p = 0.7 \simeq$ P2-P1*
  - *P1-P1 with ratio $h_u/h_p = 0.6 \leq$ P1b-P1$\leq$ P1-P1 with ratio $h_u/h_p = 0.7$*

Conclusion and Perspective

- Conclusion
  - Discretization of $b(u,p) = \int_\Omega -\mathrm{div}(\mathbf{u})\,\mathrm{p} \Rightarrow$ triangulation of velocity.
  - Discrete inf-sup constant doesn't depend on h_p for structured meshes
  - Discrete inf-sup condition is not sufficient to obtain good convergence
  - P1-P1: ratio h_u/h_p [0.5,0.7]
  - P2-P2: ratio h_u/h_p [0.5,0.8]

- Perspective
  - Non structured meshes
  - Three dimensional case