"More efficiency in finite element methods"
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METRIC-BASED MESH ADAPTATION

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Is Computational Mechanics sure enough?

Code analysis:
- Verification
- Manufactured solutions, convergence order
- calibration, validation

Solution analysis:
- Solution quality
- Mesh convergence
- Richardson/mesh convergence order
- Margins
- Certification.

Impact of mesh strategies on these?
An example:
Steady flow around an airfoil, compressible Navier-Stokes
Mach=1.2, Reynolds=500

<table>
<thead>
<tr>
<th>Embedded reft.</th>
<th>mesh 1</th>
<th>mesh 2</th>
<th>mesh 3</th>
<th>mesh 4</th>
</tr>
</thead>
<tbody>
<tr>
<td># of nodes</td>
<td>800</td>
<td>3114</td>
<td>12284</td>
<td>48792</td>
</tr>
<tr>
<td>Numerical order</td>
<td></td>
<td></td>
<td><strong>0.94</strong></td>
<td><strong>1.14</strong></td>
</tr>
</tbody>
</table>
adaptative refl. | mesh 1 | mesh 2 | mesh 3 | mesh 4
--- | --- | --- | --- | ---
# of nodes | 800 | 3114 | 11938 | 40965
numerical order | 1.75 | 1.92

\[
|U_3 - u|_{L^2} \leq \frac{1}{3}|U_2 - U_3|_{L^2} = 6.00 \times 10^{-5} \quad |U_3 - U_4|_{L^2} = 5.637 \times 10^{-5}.
\]
Standpoint of the theory presented

Choice of a simplified context :
Finite-element-like
Continuous $P_1$
But with (Aerospace) industrial problematics.

Exploration and design of new methods in this context
Plan of this lecture

Recalls on $P_1$ context
Convergence to discontinuous functions
(a) Mesh adaptative $L^2$ interpolation (more deeply treated by F. Alauzet)
Recalls on error-based mesh adaptation for PDE:
- (b) Adjoint based correction of Giles-Pierce,
- (c) Duality based adaptation of Becker-Rannacher,
- Association of (a) + (b) + (c), by Venditti-Darmofal,
Attempt of synthesis with interpolation analysis.
Properties and limit of mesh adaptation
High order P1-INTERPOLATION for a discontinuity

\( u \): bounded, piecewise smooth, with a few discontinuities.

Prototype: the Heavyside function + a smooth function, on \([0, 1]\).

Lemma: For a uniform refinement, the order of accuracy in \( L^2 \) of the P1 interpolation is only \( 1/2 \). Conversely, there exist adaptative refinements for which the order of accuracy of P1 interpolation is 2, i.e. \(|u - \Pi_h u| \leq KN^{-2}\).

Idea of the proof: Divide the interval around discontinuity into eight intervals of same size and divide other intervals into two. Total mesh size is only increased by a factor \( 2 + 8/N \) and error is 4 times smaller.
Can we look for the best mesh, at least with a model?

“Continuous metrics” approach: for a local mesh size $m(x)$, the $L^2$ norm of the P1-interpolation error is modelized as:

$$
\int_0^1 |e_M(x)|^2 ds = \int_0^1 (m^2 |\delta^{-2}(u(x + \delta) - 2u(x) + u(x - \delta))|)^2 ds.
$$

where $\delta$ is smaller than $m$.

$\delta^{-2}(u(x + \delta) - 2u(x) + u(x - \delta))$ :  
- is close to $\frac{\partial^2 u}{\partial x^2}$,  
- or of the order of $\delta^{-2}$,  
- bounded in $L^{1/2}$ independantly of $\delta$.  

Optimal mesh for a given number of nodes

\[ \int m(x)^{-1} \, dx = N. \quad (1) \]

\[ m_{opt}(x) = C(N).\left|\left(\delta^{-2}(u(x + \delta) - 2u(x) + u(x - \delta))\right)(x)\right|^{\frac{2}{5}}. \]

Further the resulting error in \( L^2 \) writes:

\[ \text{error} = \frac{2}{N^2} \left( \int \left|\delta^{-2}(u(x + \delta) - 2u(x) + u(x - \delta))\right|^{\frac{2}{5}} \right)^{\frac{5}{2}} < \frac{K}{N^2} \]

which gives second-order accuracy.
Discontinuity capturing: Numerical illustration:

Two examples: smooth arctangent, discontinuous Heavyside.
Convergence to the continuous: Heavyside

Abscissae: number of nodes; ordinates: interpolation error, Dashes: uniform refinement, line: adaptive refinement.
Convergence to the continuous: Arctangent

Uniform refinement: late capturing
Adaptative refinement: early capturing
Early capturing/late capturing

Uniform refinement needs $N_S$ nodes, where $N_s$ is the inverse of the size of the smallest detail (1D).

A good adaptative refinement needs $N_d$ nodes, where $N_d$ is (1D) the number of details (for example: the function is monotone on $N_d$ intervals).

$$N_d << N_S.$$
Preliminary conclusion:

- Optimal metric adaptative formulation applies to smooth and discontinuous functions and produces second-order accuracy(*),

- The answer is a continuous function.

- Second-order discontinuity capturing $\iff$ early smooth capturing,

- (*) extendable to higher-order.
Two-dimensional $P_1$ interpolation of function

$u \in H^2(\Omega)$

$\mathcal{T}_h$ a triangulation/tetrahedrization of $\Omega$

$\Pi_h$ the $P_1$ interpolation from values at vertices:

\[ \forall S_i, \text{ vertex}, \quad \Pi_h u(S_i) = u(S_i) \]

\[ \forall K, \text{ element}, \quad \Pi_h u|_K \in P_1(K). \]

There exist $K_1 > 0$ and $K_2 > 0$ such that:

\[ |u - \Pi_h u|_{H^1} \leq K_1 h |u|_{H^2} \]

and:

\[ |u - \Pi_h u|_{L^2} \leq K_2 h^2 |u|_{H^2}. \]

We focus on $L^2$ convergence.
Optimal anisotropic mesh

We modelize a mesh as a **continuous medium**, with an anisotropic property, the **local metric** ((*) (George, Hecht, Fortin, Habashi, ...):

\[
\mathcal{M}_{x,y} = R_{\mathcal{M}}^{-1} \begin{pmatrix} (m_\xi)^{-2} & 0 \\ 0 & (m_\eta)^{-2} \end{pmatrix} R_{\mathcal{M}},
\]

The metric defines a Riemannian topology: the length \( L_{\mathcal{M}}(\vec{c}d) \) of a vector \( \vec{c}d \) in metric \( \mathcal{M} \) is defined as follows:

\[
L_{\mathcal{M}}(\vec{v}) = \int_0^1 \sqrt{\vec{v}.\mathcal{M} \vec{v}} (x'\vec{c} + (1 - x')\vec{d}) \ dx',
\]

and a class of meshes: A **mesh satisfies a metric if any of its edges has a unit length for this metric**.

**Number of nodes:**

\[
N_{\mathcal{M}} = \int m_\xi^{-1} m_\eta^{-1} \ dx \ dy.
\]
Continuous metrics method for P1 interpolation (2D)

For any $M$, any function $u$: **local P1-interpolation error**:

$$\mathcal{E}_M = \int \left( \left| \frac{\partial^2 u}{\partial \xi^2} \right| m_{\xi}^2 + \left| \frac{\partial^2 u}{\partial \eta^2} \right| m_{\eta}^2 \right)^2 dxdy$$

where $\xi$ and $\eta$ are directions of diagonalization of the Hessian of $u$.

**Discontinuous case:**

use $(u(\xi + \delta, \eta) - 2u(\xi, \eta) + u(\xi - \delta, \eta))/\delta^2$, bounded in $L^2$.

**Optimal metric problem:**

$$\min_M \mathcal{E}_M \quad \text{under the constraint} \quad N_M = N.$$
Solving the optimal problem

Step 1: Pointwise optimization

Given at a point a mesh density $d = (m_\xi m_\eta)^{-1}$, the optimal direction and strength of stretching give:
- optimal stretching direction:
  $\mathcal{R}_M = \mathcal{R}_u$ diagonalising the Hessian of $u$,
- optimal stretching strength: $m_\xi / m_\eta = (|u_{\eta\eta}| / |u_{\xi\xi}|)^{1/2}$.

Step 2: constrained global optimization

$$\min_d \quad \mathcal{E}_d = \int \left( d^{-1} |\frac{\partial^2 u}{\partial \xi^2}| + |\frac{\partial^2 u}{\partial \eta^2}| \right)^2 \ dx \ dy$$

under the constraint $\int d \ dx \ dy = N$. 
Optimality

\[ M_{\text{opt}} = \frac{C}{N} \mathcal{R}^{-1} \begin{pmatrix} \left| \frac{\partial^2 u}{\partial \eta^2} \right|^{-5/6} \left| \frac{\partial^2 u}{\partial \xi^2} \right|^{1/6} & 0 & 0 \\ 0 & \left| \frac{\partial^2 u}{\partial \xi^2} \right|^{-5/6} \left| \frac{\partial^2 u}{\partial \eta^2} \right|^{1/6} & 0 \\ 0 & 0 & \left| \frac{\partial^2 u}{\partial \eta^2} \right|^{-5/6} \left| \frac{\partial^2 u}{\partial \xi^2} \right|^{1/6} \end{pmatrix} \mathcal{R} . \]

with:

\[ C = \int \left( \left| \frac{\partial^2 u}{\partial \xi^2} \right| \cdot \left| \frac{\partial^2 u}{\partial \eta^2} \right| \right)^{2/6} d\xi d\eta . \]
Minimal error

Minimal error: \[ \varepsilon_{\text{opt}} = \frac{4C^2}{N^2} \int \left( \left| \frac{\partial^2 u}{\partial \xi^2} \right|^{1/3} \left| \frac{\partial^2 u}{\partial \eta^2} \right|^{1/3} \right) dxdy \]

Second-order accuracy for discontinuous solution is predicted.
Fixed-point adaptation principle

Interpolation error reduction:

- Interpolate $u^{(n)}$ on $\mathcal{M}^{(n)}$
- Find $\mathcal{M}^{(n+1)}$ minimizing $\mathcal{E}(\mathcal{M}, u^{(n)})$
Convergence properties for discontinuous fields

<table>
<thead>
<tr>
<th>Conv. order</th>
<th>Isotropic</th>
<th>Anisotropic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theory</td>
<td>$\leq 1$  (*)</td>
<td>$\leq 2$</td>
</tr>
<tr>
<td>Optimal Metric Theory</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Optimal Metric Num. exp. Heavyside 2D</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Fixed-point adaptation principle (end’d)

1. Interpolation error reduction

   - Interpolate \( u^{(n)} \) on \( M^{(n)} \)
   - Find \( M^{(n+1)} \) minimizing \( E(M, u^{(n)}) \)

2. Can we extend this to PDE?
   Hessian-based (in fact: interpolation-based) approximation error reduction:

   - Solve PDE for \( u^{(n)} \) on \( M^{(n)} \)
   - Find \( M^{(n+1)} \) minimizing \( E(M, u^{(n)}) \)
Interpolation/Hessian based approaches, concluded

- $L^\infty$ or error distribution method very frequently used.

- $L^p$ or “continuous metric optimization” mathematically established.

- Anisotropy is well taken into account.

- $H^1$ theory is currently developed.

Extension to PDE:

- Are PDE errors enough controlled through \textit{a priori} estimates?

Goal-oriented PDE errors:

Best mesh for an accurate evaluation of a scalar $j(u)$?
PDE-Approximation error analysis

\[ Au = f : \quad - \Delta u - f = 0 \quad \text{in } \Omega \quad u|_{\partial \Omega} = 0 \]

Prop If \( u \in H^2 \), there exists \( K \leq 0 \) such that:

\[ ||u - u_h||_{H^1} \leq Kh^1||u||_{H^2} \]

Prop If \( u \in H^2 \), there exists \( K \leq 0 \) such that:

\[ |u - u_h|_2 \leq Kh^2||u||_{H^2} \]
P1-Galerkin a posteriori error analysis

\[ Au = f \]

\[ V = H^1_0, \quad V' = (H^1_0)', \quad A : V \to V', \quad u \mapsto (\phi \mapsto (\nabla u, \nabla \phi)) \]

- Assume \( u_h \) is any function of \( H^1_0 \), then:

\[ u_h - u = A^{-1}(Au_h - f) \quad \text{a posteriori estimate} \]

N.B.1. \( Au_h \) is in \( V' \).

N.B.2. Correction: \( A^{-1} \) can be approximated by \( A_h^{-1} \).
A posteriori goal-oriented corrector (Giles-Pierce)

\[ u_h - u = A^{-1}(Au_h - f) \]
\[ j(u) = (u, g) \]
\[ j_h = (u_h, g) = j(u) + (u_h - u, g) = j(u) + O(h^\alpha) \]
\[ = j(u) + (Au_h - f, p) \]
\[ p = A^{-*}g ; \quad p_h = A^{-*}_h g \]

\[ \Rightarrow \tilde{j}_h = (u_h, g) - (Au_h - f, p_h) = j(u) + O(h^{2\alpha}) . \]

This method is quite approximation-independent and have shown fourth-order convergence of drag and lift in 2D for second-order approximations.
**A posteriori goal-oriented mesh refinement (Becker-Rannacher)**

\[ j(u) - j(u_h) = (g, u - u_h) = a(u - u_h, z) \]

we use the two equations into this expression:
\[ a(u, z) - a(u_h, z) = a(u, z - i_h z) - a(u_h, z - i_h z) + a(u, i_h z) - a(u_h, i_h z) \]

the two last terms vanish and
\[ a(u, z - i_h z) - a(u_h, z - i_h z) = (f, z - i_h z) - a(u_h, z - i_h z) \]

putting \( \rho(u_h, \delta z) = (f, \delta z) - a(u_h, \delta z) \) we get
\[ j(u) - j(u_h) = \rho(u_h, z - i_h z) . \]

This residual with weight \( \delta z \) can be decomposed as follows:
\[ \rho(u_h, \delta z) = \sum_{K \in \mathcal{T}_h} (f + \Delta u_h, \delta z)_K + \sum_{E \in \mathcal{E}} ([\frac{\partial}{\partial n} u_h], \delta z)_E \]
It is then useful to start majorations:

$$\rho(u_h, \delta z) \leq \sum_{K \in \mathcal{T}_h} \| f + \Delta u_h \|_K \| \delta z \|_K + \sum_{E \in \mathcal{E}} \| [\partial_n u_h] \|_E \| \delta z \|_E$$

in which the cell residuals and edge residuals are respectively weighted by $\| \delta z \|_K$ and $\| \delta z \|_E$.

The $\| \delta z \|$'s need be approximated: two levels of interpolation of $z_h$ are used instead. This can be coarse-fine or two different degrees of interpolation.

Then $\rho(u_h, \delta z)$ is compared with a tolerance “TOL” for local refinement decision.
A posteriori mesh refinement (Becker-Rannacher)

\[ u_h - u = A^{-1}(Au_h - f) \text{ a posteriori estimate} . \]

\[ \rho(u_h, \delta z) \leq \sum_{K \in \tau_h} \| f + \Delta u_h \|_K \| \delta z \|_K + \sum_{E \in \mathcal{E}} \| \partial_n u_h \|_E \| \delta z \|_E \]

We assume “z” is unknown but smooth:

\[ \| u - u_h \| \leq \text{const.} \left( \sum_{K \in \tau_h} \eta^2_K \right)^{1/2} \quad r = 0 \text{ or } 1 \]

with:

\[ \eta_K = h_K \rho_K . \]

**Error balancing** : The local error indicator \( \eta_K \) is equilibrated by refining or coarsening the elements \( K \) of \( \tau_h \) according to:

\[ \eta_K = \frac{TOL}{\sqrt{N \text{const.}}} , \quad N \text{ number of elements.} \]
A posteriori superconvergent goal-oriented mesh refinement

Venditti-Darmofal

\[ j(u_h) - j(u) - (Au_h - f, p_h) = -(Au_h - f, p_h - p) \]
\[ j(u_h) - j(u) - (Au_h - f, p_h) = -(u_h - u, A^*p_h - g) \]

- For the evaluation of \( u - u_h \) and \( p - p_h \), a two-level strategy is applied:
  . solutions \( u_H \) and \( P_H \) are computed on the current coarse grid,
  . a fine grid is obtained by uniform division of coarse grid elements,
  . two extensions from coarse to fine are built (1) linear interpolation \( L^H_h \) and (2) quadratic reconstruction \( Q^H_h \)

\[
\delta u = Q^H_h u_H - L^H_h u_H \\
\delta p = Q^H_h p_H - L^H_h p_H
\]

\[ \forall K \in \tau_H : \varepsilon_K = \frac{1}{2} \sum_j \left| \left[ R^p_h (L^H_h p_H) \right]_j^T [\delta u]_j \right| + \left| \left[ \delta p \right]_j^T [R_h (L^H_h u_H)]_j \right| \]

\[ \forall j: \text{fine node of } K, \left[ R^p_h \right]_j^K, \left[ R^p_h \right]_j^K \text{ fine nodal residuals averaged on } K. \]
Venditti-Darmofal, cont’d

- Apply the Becker-Rannacher “TOL” isotropic strategy but on RHS. This gives a maximal mesh density “$m_\xi$”.

- Introduce the **stretching** specified by Hessian-based analysis applied to a sensor, the Mach number:
  - optimal stretching direction: $\mathcal{R}_M = \mathcal{R}_{\text{Mach}}$,
  - optimal stretching strength: $m_\xi/m_\eta = (\text{Mach}_\eta|/|\text{Mach}_\xi|)^{1/2}$.

The “approximations” which have been done:

$$u - u_h \approx Q_h^H u_H - L_h^H u_H$$

$$A u_h \approx A_h L_h^H u_H$$ (idem adjoint).
Figure 3: RAE 2822 Airfoil test case: $Re = 6.5 \times 10^6$, $M_{\infty} = 0.725$, $\alpha = 2.46^\circ$. Comparison of final adapted grids using the proposed output-based method on the lift (top), and drag (middle), and using pure Hessian-based adaptation (bottom).
P1-Galerkin a priori analysis: Aubin-Nitsche by equalities

"Au = f ; A_h u_h = f_h \Rightarrow u_h - u = A_h^{-1}(f_h - A_h u)"

\[ V = H_0^1 \cap H^2, \quad V' = (H_0^1 \cap H^2)', \quad f \in L^2 \]
\[ A : H_0^1 \rightarrow (H_0^1)', \quad u \mapsto (\phi \mapsto (\nabla u, \nabla \phi)) \]

\(V_h\) usual continuous \(P^1\) discretization,
\(\Pi_h\) nodal interpolation from \(V \cup V_h\) to \(V_h\)
\[(Au_h, \Pi_h \phi) - (f, \Pi_h \phi) = 0 \]
\[\Pi^*_h : \forall v \in V', (\Pi^*_hv, \phi)_{V' \times V} = (v, \Pi_h \phi)\]
\[\Pi^*_h Au_h = \Pi^*_hf\]
\[\Pi^*_h A \Pi_h (u_h - u) = \Pi^*_hf - \Pi^*_hA \Pi_h u = \Pi^*_hA(u - \Pi_h u)\]

where \(\Pi^*_hA\) can be inverted \(V_h \rightarrow \Pi^*_hA(V)\).

Note that:
\[u - u_h = u - \Pi_h u + \Pi_h u - u_h = (1 - (\Pi^*_hA)^{-1} \Pi^*_hA)(u - \Pi_h u)\).
Goal-oriented a priori analysis

Evaluate $j(u) = (u, g)$:

\[
(u - u_h, g) = (u - \Pi_h u, g) + (\Pi_h u - u_h, g)
\]

\[
= (u - \Pi_h u, g) - ((\Pi_h^* A)^{-1}\Pi_h^* A(u - \Pi_h u), g)
\]

\[\forall \phi_h \ , \ (\phi_h, g) = (\Pi_h^* A\phi_h, p_h) \quad (\Pi_h^* A \text{ inversible dans } V_h)\]

\[
\Rightarrow (u - u_h, g) = (u - \Pi_h u, g) - (\Pi_h^* A(u - \Pi_h u), p_h)
\]

\[
\Rightarrow (u - u_h, g) = (u - \Pi_h u, g) - (A(u - \Pi_h u), \Pi_h p_h)
\]

\[
\Rightarrow (u - u_h, g) = (u - \Pi_h u, g) - (A(u - \Pi_h u), p_h)
\]

\[
(u - u_h, g) = (u - \Pi_h u, g - A^* p_h)
\]
Goal-oriented a priori analysis, variational

\[ a(u_h, \Pi_h \phi) = (f, \Pi_h \phi) = a(u, \Pi_h \phi) \]
\[ a(u_h - \Pi_h u, \Pi_h \phi) = a(u - \Pi_h u, \Pi_h \phi) \]
\[ a(u_h - \Pi_h u, \phi_h) = a(u - \Pi_h u, \phi_h) \]

Evaluate \( j(u) = (u, g) \):

\[ (u - u_h, g) = (u - \Pi_h u, g) + (\Pi_h u - u_h, g) \]
\[ \forall \phi_h, \quad a(\phi_h, p_h) = (\phi_h, g) \]
\[ (u - u_h, g) = (u - \Pi_h u, g) - a(u_h - \Pi_h u, p_h) \]
\[ (u - u_h, g) = (u - \Pi_h u, g) - a(u - \Pi_h u, p_h) \]
\[ (u - u_h, g) = (u - \Pi_h u, g - A^* p_h) \]
Goal-oriented a priori analysis, end’d

\[(u - u_h, g) = (u - \Pi_h u, g - A^* p_h)\]

Two difficulties arise:
- \(g - A^* p_h\) is not \(L^2\) in elliptic case,
- \(g - A^* p_h\) can be small and be influenced by mesh.

A possible strategy is second case consists in keeping on the implicit error \(u_h - \Pi_h u\) in the analysis, using another device for reducing the interpolation error in \((g, u)\). Then we get:

\[\mathcal{E}_M = \int \left( |\frac{\partial^2 u}{\partial \xi^2}| m_\xi^2 + |\frac{\partial^2 u}{\partial \eta^2}| m_\eta^2 \right) |A^* p_h| \ dxdy\]

The term \(|A^* p_h|\) is a weighting for the Hessian-based metric.
APPLICATION TO EULER FLOWS

\[ \frac{\partial}{\partial t} \int_C U dV + \oint_{\partial C} F(U) \cdot n dS = 0 \]

*U* vector of conserved variables, *F* flux of *U* across the bounding surface \( \partial C \) with outward unit normal *n* of any control volume *C*.

\[
U = \begin{pmatrix}
\rho \\
\rho u \\
\rho v \\
\rho w \\
\rho E
\end{pmatrix}; \quad F(U) = \begin{pmatrix}
\rho u \\
\rho uu + p_i x \\
\rho uv + p_i y \\
\rho uw + p_i z \\
\rho uH
\end{pmatrix}
\]

\( \rho, p, E \) fluid density, thermodynamic pressure, and total energy per unit mass. \( u, v, \) and \( w \): Cartesian components of the velocity vector \( u \) and \( H \) total enthalpy \( H = E + \frac{p}{\rho} \).

\[
p = \rho (\gamma - 1) \left[ E - \frac{1}{2} (u^2 + v^2 + w^2) \right]
\]

\( \gamma \) ratio of specific heats at constant pressure and volume.
$W \in V = [H^1(\Omega)]^5$ conservation variables, $\mathcal{F}(W) = \mathbf{F}(U)$ flux functions:

$$W = \begin{pmatrix} \rho \\ u \\ v \\ w \\ p \end{pmatrix}; \quad \mathcal{F}(W) = \begin{pmatrix} \rho u \\ \rho uu + p_i x \\ \rho uv + p_i y \\ \rho uw + p_i z \\ \rho uH \end{pmatrix}.$$ 

The field $W$ satisfies, for any non-scalar test function $\phi$ of $V$:

$$\int_{\Omega} \phi \nabla \cdot \mathcal{F}(W) d\Omega - \int_{\partial \Omega} \phi \mathcal{F}(W).n d\partial \Omega = 0.$$ 

The new notation $\bar{\mathcal{F}}(W)$ holds for a boundary flux evaluation which uses both values of $W$ and specified boundary conditions.

Discretization:

A particular P1 Finite-Element Galerkin formulation
Tetrahedrization $T_h$, $V_h$ subset of $V = [H^1(\Omega)]^5$:

$$V_h = \{ \phi_h, \phi_h \text{ is continuous}, \phi_h|_T \text{ is linear } \forall T \in T_h \}$$

Interpolation operator:

$$\Pi_h : V \rightarrow V_h ; \phi \mapsto \Pi_h \phi,$$

$$\Pi_h \phi(i) = \phi(i) \forall i \text{, vertex of } T_h.$$ 

The discrete formulation writes:

$$\int_\Omega \phi_h \nabla \mathcal{F}_h(W_h) d\Omega - \int_{\partial \Omega} \phi_h \bar{\mathcal{F}}_h(W_h).n d\partial \Omega = 0$$

where $\mathcal{F}_h(W)$ is the interpolate of $\mathcal{F}$, i.e. $\mathcal{F}_h(W_h) = \Pi_h \mathcal{F}(W_h)$, and same for $\bar{\mathcal{F}}_h(W_h)$. 
Error analysis for the Euler discrete model

After some computations we get an equation for the implicit error $W_h - \Pi_h W$:

$$W_h - \Pi_h W \doteq (A_h(W))^{-1} \quad \text{RHS} \quad + O(h^3)$$

with:

$$A_h(W)(W_h - \Pi_h W) = \int_{\Omega} \phi_h \nabla.(\Pi_h \frac{\partial F_h}{\partial W}(W_h - \Pi_h W))d\Omega - \int_{\partial\Omega} \phi_h(\Pi_h \frac{\partial F_h}{\partial W}(W_h - \Pi_h W)).n d\partial\Omega.$$  

and:

$$\text{RHS} = \int_{\Omega} \phi \nabla.(F(W) - \Pi_h F(W))d\Omega - \int_{\partial\Omega} \phi(\bar{F}(W) - \Pi_h \bar{F}(W)).n d\partial\Omega$$
After integration by parts:

\[
RHS = - \int_\Omega (\nabla \phi). (\mathcal{F}(W) - \Pi_h \mathcal{F}(W)) d\Omega \\
+ \int_{\partial \Omega} \phi (\mathcal{F}(W) - \Pi_h \mathcal{F}(W)). nd \partial \Omega - \\
\int_{\partial \Omega} \phi (\bar{\mathcal{F}}(W) - \Pi_h \bar{\mathcal{F}}(W)). nd \partial \Omega
\]

In case of a goal-oriented analysis, \( \phi = p_h \):

\[
|RHS_2| \leq - \int_\Omega |\nabla p_h|. |\mathcal{F}(W) - \Pi_h \mathcal{F}(W)| d\Omega \\
+ \int_{\partial \Omega} |p_h|. |\mathcal{F}(W) - \Pi_h \mathcal{F}(W)|. nd \partial \Omega - \\
\int_{\partial \Omega} |p_h|. |\bar{\mathcal{F}}(W) - \Pi_h \bar{\mathcal{F}}(W)|. nd \partial \Omega
\]
Numerical application (in progress):

- Hessian strategy can be applied to interpolation error on each of the 15 fluxes

- Each of these 15 interpolation errors is weighted by a gradient of a component of adjoint state

- Final sum is symmetric positive definite and produces a candidate optimal anisotropic mesh
CONCLUDING REMARKS

A priori and a posteriori estimates are both incomplete by their nature. Then part of them are guessed from existing calculations.

Both produce formulas that are close to each other.

The synthesis is presented on particular “P1-exact” schemes.

Finding good formula for taking anisotropy in account in a general manner or for other particular contexts is an open problem.