An adaptive finite element method for boundary value problems in automotive applications

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5th European Finite Element Fair
Ceci n’est pas un Renault
Aims

- Optimize control systems in cars.
- In particular, electronic stability control (ESC).
Find states $y(t) \in \mathbb{R}^{d_1}$ and controls $u(t) \in \mathbb{R}^{d_2}$ which fulfill

$$
\min_{} J(y, u) = \int_0^T L(t, y(t), u(t)) \, dt
$$

s.t.  
$$
\dot{y}(t) = f(t, y(t), u(t)), \quad I_0 y(0) = y_0, \quad I_T y(T) = y_T. 
$$

$I_0, I_T$ are diagonal matrices with zeroes or ones on the diagonals, $\text{rank}(I_0) + \text{rank}(I_T) = d_1$.

Hamiltonian:

$$
H = L(t, y, u) + \lambda^T f(t, y, u)
$$
The optimal $y$ and $u$ fulfill

$$
\dot{y} = \frac{\partial H}{\partial \lambda} = f(y),
$$

$$
\dot{\lambda} = \frac{\partial H}{\partial y} = -\left(\frac{\partial f}{\partial y}\right)^T \lambda,
$$

$$
\frac{\partial H}{\partial u} = 0,
$$

$$
l_0 y(0) = y_0, \quad l_T y(T) = y_T,
$$

$$
(l - l_0) \lambda(0) = \lambda_0, \quad (l - l_T) \lambda(T) = \lambda_T.
$$
Numerical Methods

- Collocation
- Multiple Shooting
- Use Adaptive Finite Element Methods to solve the stiff problems arising from vehicle modelling.

D. Estep et al., *The solution of a launch vehicle trajectory problem by an adaptive finite-element method*, 2001
The algebraic equation can be solved explicitly.

Join states, $y$, and costates, $\lambda$, into one new variable $\mathbf{x} \in \mathbb{R}^d$.

We get a boundary value problem for ODE

$$\dot{\mathbf{x}} = f(\mathbf{x}),$$
$$l_0 \mathbf{x}(0) = x_0, \quad l_T \mathbf{x}(T) = x_T,$$
The algebraic equation can be solved explicitly.
Join states, $y$, and costates, $\lambda$, into one new variable $x \in \mathbb{R}^d$.
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Join states, \( y \), and costates, \( \lambda \), into one new variable \( x \in \mathbb{R}^d \).

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\[
\dot{x} = f(x),
\]

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l_0 x(0) = x_0, \quad l_T x(T) = x_T,
\]
Weak Formulation

\[ V = \{ v \in C^1([0, T]) \} \]

Seek \( x \in V \) such that

\[ I_0 x(0) = x_0, \quad I_T x(T) = x_T, \]

\[ F(x, v) = \sum_{n=1}^{N} \int_{I_n} (\dot{x} - f(x), v) \, dt = 0, \quad \forall v \in V. \]
FEM problem

Mesh: $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$, $k_n = t_n - t_{n-1}$ and $I_n = (t_{n-1}, t_n)$.

Trial space: $W_k = \{ w|_{I_n} : w \in P^0(I_n) \} \times \mathbb{R}^d \times \mathbb{R}^d$, discontinuous piecewise constant functions.

Test space: $V_k = \{ v|_{I_n} : v \in P^1(I_n), v \in C^0([0, T]) \}$, continuous piecewise linear functions.

Find a function $X \in W_k$ which fulfills

$$I_0 X_0^- = x_0, \quad I_T X_N^+ = x_T,$$

$$F(X, v) = \sum_{n=1}^{N} \int_{I_n} (\dot{X} - f(X), v) \, dt + \sum_{n=0}^{N} ([X]_n, v_n) = 0, \quad \forall v \in V_k.$$
Mesh: $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$, $k_n = t_n - t_{n-1}$ and $l_n = (t_{n-1}, t_n)$.

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\begin{align*}
  l_0 X_0^- &= x_0, & l_T X_T^+ &= x_T, \\
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\end{align*}
FEM problem

- Mesh: $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$, $k_n = t_n - t_{n-1}$ and $I_n = (t_{n-1}, t_n)$.
- Trial space: $W_k = \{ w|_{I_n} : w \in P^0(I_n) \} \times \mathbb{R}^d \times \mathbb{R}^d$, discontinuous piecewise constant functions.
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Find a function $X \in W_k$ which fulfills

$$I_0 X_0^- = x_0, \quad I_T X_N^+ = x_T,$$

$$F(X, v) = \sum_{n=1}^{N} \int_{I_n} (\dot{X} - f(X), v) \, dt + \sum_{n=0}^{N} ([X]_n, v_n) = 0, \quad \forall v \in V_k.$$
Let \( \phi \) be the solution to the dual problem with functional \( G(e) \) as data.

Standard calculations give us

\[
G(e) \leq \sum_{n=1}^{N} R_n S_n,
\]

with

\[
R_n = k_n \begin{cases} 
  \| [X_0] \| + \frac{k_1}{k_1+k_2} \| [X_1] \| + k_1 \| f(X_1^-) \|_1, \\
  \frac{k_n}{k_n+k_{n+1}} \| [X_n] \| + \frac{k_n}{k_n+k_{n-1}} \| [X_{n-1}] \| + k_n \| f(X_{n}^-) \|_n, \\
  \frac{k_N}{k_{N-1}+k_N} \| [X_{N-1}] \| + \| [X_N] \| + k_N \| f(X_N^-) \|_N,
\end{cases}
\]

and

\[
S_n = C \int_{l_n} \| \ddot{\phi} \| \, dt.
\]
\[ \dot{v}_y(t) = a_{11} v_y(t) + a_{12} r(t) + b_{f1} \delta_f(t) + b_{r1} \delta_r(t), \]
\[ \dot{r}(t) = a_{21} v_y(t) + a_{22} r(t) + b_{f2} \delta_f(t) + b_{r2} \delta_r(t), \]
\[ \dot{\psi}(t) = r(t), \]
\[ \dot{X}(t) = v_x \cos(\psi(t)) - v_y(t) \sin(\psi(t)), \]
\[ \dot{Y}(t) = v_x \sin(\psi(t)) + v_y(t) \cos(\psi(t)). \]
A Lane Change Manoeuvre
Energy Optimal

\[ v_y(0) = 0, \quad v_y(T) = 0, \]
\[ r(0) = 0, \quad r(T) = 0, \]
\[ \psi(0) = 0, \quad \psi(T) = 0, \]
\[ X(0) = 0, \]
\[ Y(0) = 0, \quad Y(T) = 10. \]

\[ J(\delta_f, \delta_r) = \int_0^T \frac{1}{2} a \delta_f^2 + \frac{1}{2} b \delta_r^2 \, dt \]
Results
Future Work

- DAE
- Constraints on controls
- Parameter identification from real data