Mesh Generation, Error indicators and mesh adaptation, some comparisons

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PLAN

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The main goal of this talk is to present some numerical comparison between different techniques of mesh adaptations. The mesh generation needs a metric to define the "mesh size" constraint, and the goal of mesh adaptations is double: compute better solution at low cost.

Now, the Hessian error indicator gives the metric in a natural way, but the other error indicators give just a level of local error, so the question is how to build a metric from this local error indicator.
State of the art on automatics tet mesh generation

- Classical tet mesh: done
- The adapted (iso and aniso) mesh generation is under control in 2D
- The adapted mesh generation on surface is almost on control
- The tet iso adapted mesh generation: beginning
- The tet aniso adapted mesh generation, in progress
Voronoï cell: \[ V_i = \{ x \in \mathbb{R}^d ; \forall j \neq i ; ||x - x_i|| \leq ||x - x_j|| \} \]

And a Delaunay edge \((x_i, x_j)\) exist iff \(\dim(V_i \cup V_j) = d - 1, \ldots\)
Empty Circum Sphere

lemma : Delaunay mesh iff

\( \forall K, K' \) adjacent tet; \((\text{open circum disk of } K) \cap (\text{vertices of } K') = \emptyset \)

We can construct Delaunay mesh

Remark : This imply in 3D Delaunay mesh can be poor event with a good set point.

Take a grid \( \{0, 1, 2, 3\}^3 \) and take the four points \((1, 1, 1), (1, 2, 1), (1, 1, 2), (1, 2, 2)\) to defined a planar quad, if you just put a small noise an this 4 coordinates , this an empty sphere tet so it is a very flat Delaunay tet (call sliver tet).

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Insert a new Point $P$

$$\mathcal{T}_{n+1} = \mathcal{T}_n \setminus \mathcal{C}_n \cup \mathcal{B}_{n+1}$$

where

- $\mathcal{C}_n$ is the set of remove tet (tet such that $P$ is in circum sphere)
- $\mathcal{C}_n$ is star shape / $P_n$
- $\mathcal{B}_n$ is the set of tet construct with $\partial \mathcal{C}_n$ and $P_n$

**Fundamental Remark**: to remove the sliver element, it possible to change construction of the set $\mathcal{C}_n$ to inforce a strong star shape / $P_n$ propriety
local transformation

2d swap

2Tet → 3Tet

nTet → 2(n − 2)Tet
boundary integrity

- Algo to force a 2D edge $A, B$
  while exist edge $(a,b)$ crossing edge $A,B$
  do 2d swap edge $(a,b)$ and if after swap the new edge crossing again
  undo swap randomly.
Empty Boundary Mesh
Classical Delaunay mesh generator

- put all the given points au a big box
- insert one by one all the given points
- inforce the boundary
- loop
  - create internals points if edge are too long
  - insert one by one this new points
- make some regularization
Mesh quality :

- element quality :

\[ Q_{\text{element}} = \alpha \frac{h}{\rho} \]

with
- \( \alpha \): normalization factor
- \( h \): element diametre
- \( \rho \): in-radius

- mesh quality is measured via :

\[ Q_{\text{mesh}} = \max Q_{\text{element}} \]

- element distribution w.r.t. \( Q \)
The main IDEA for mesh generation

- The difficulty is to find a tradeoff between the error estimate and the mesh generation, because this two work are strongly different.
- To do that, we propose way based on a metric $\mathcal{M}$ and unit mesh w.r.t $\mathcal{M}$
- The metric is a way to control the mesh size.
- Remark: *The class of the mesh which can be created by the metric, is very large.*
Example of mesh adapted for 2 functions

\[
f_1 = (10 \times x^3 + y^3) + \text{atan2}(0.001, (\sin(5 \times y) - 2 \times x))
\]

\[
f_2 = (10 \times y^3 + x^3) + \text{atan2}(0.01, (\sin(5 \times x) - 2 \times y)).
\]
Metric / unit Mesh

In Euclidean geometry the length $|\gamma|$ of a curve $\gamma$ of $\mathbb{R}^d$ parametrized by $\gamma(t)_{t=0..1}$ is

$$|\gamma| = \int_0^1 \sqrt{(\gamma'(t), \gamma'(t))} dt$$

We introduce the metric $M(x)$ as a field of $d \times d$ symmetric positive definite matrices, and the length $\ell$ of $\Gamma$ w.r.t $M$ is:

$$\ell = \int_0^1 \sqrt{(\gamma'(t), M(\gamma(t))\gamma'(t))} dt$$

The key-idea is to construct a mesh where the lengths of the edges are close to 1 accordingly to $M$. 

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Unit ball

If $\mathcal{M}$ is constant (independent of $x$) $\ell = 1$ imply the study of the set of point $\mathcal{E}_\mathcal{M} = \{x \in \mathbb{R}^d / x.\mathcal{M}x = 1\}$.

So a good edge $PQ$ is such that $PQ \in \mathcal{E}_\mathcal{M}$.

Let us call, $v_i$ the $d$ unit eigenvectors and $\lambda_i$ the $d$ eigenvalues of matrix $\mathcal{M}$.

The mesh size $h_i$ in direction $v_i$ is given by $1/\sqrt{\lambda_i}$ so $\lambda_i = \frac{1}{h_i^2}$
Remark on the Metric

Let \( S \) be a surface, parametrized by

\[
F(u) \in \mathbb{R}^3 \quad \text{with} \ (u) \in \mathbb{R}^2, \ 	ext{and let} \quad \Gamma(t) = F(\gamma(t)), \ t \in [0, 1]
\]

be a curve on the surface. The length of the curve \( \Gamma \) is

\[
|\Gamma| = \int_0^1 \sqrt{\langle \Gamma'(t), \Gamma'(t) \rangle} \, dt
\]

\[
|\Gamma| = \int_0^1 \sqrt{\langle \gamma'(t), t \partial F \partial F \gamma'(t) \rangle} \, dt
\]

and on a parameteric surface the metric is

\[
\mathcal{M} = t \partial F \partial F
\]
the Metric versus mesh size

at a point \( P \),

\[
\mathcal{M} = \begin{pmatrix}
a & b \\
b & c \\
\end{pmatrix}
\]

\[
\mathcal{M} = R \begin{pmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 \\
\end{pmatrix} R^{-1}
\]

where \( R = (v_1, v_2) \) is the matrix construct with the 2 unit eigenvectors \( v_i \) and \( \lambda_1, \lambda_2 \) the 2 eigenvalues.

The mesh size \( h_i \) in direction \( v_i \) is given by \( 1/\sqrt{\lambda_i} \)

\[
\lambda_i = \frac{1}{h_i^2}
\]
Remark on metric:

If the metric is independent of position, then geometry is euclidian. But the circle in metric become ellipse in classical space.

Infact, the unit ball (ellipse) in a metric given the mesh size in all the direction, because the size of the edge of mesh is close to 1 the metric.

If the metric is dependant of the position, then you can speak about Riemmanian geometry, and in this case the sides of triangle are geodesics, but the case of mesh generation, you want linear edge.
Point insertion $P_n$ with metric

$$T_{n+1} = T_n \setminus C_n \cup B_{n+1}$$

where the anisotropic cave $C_n$ is
denote $O^T_P$ the center and $\rho^T_P$ the radius of the circum circle of $T$, in a euclidian space define with the constante metric equal to $M(P)$.

$$C_n = \left\{ T \in T_n / \sum_{P \in V^T_n} \frac{||P_n - O^T_P||M(P)}{\rho^T_P} < \#V^T_n \right\}$$

where the set point $V^T_n = \{ P_n, \text{the opposite vertex of } T \text{ with respect to } P_n \}$

Remark : $C_n$ is star-shape with respect to $P_n$. 

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A Delaunay mesh generator with metric

1) Adapte the boundary mesh. w.r. to the metric $\mathcal{M}$
2) Construct the initial mesh
3) Restore the missing edges
4) Remove the external part of the mesh
5) Split edges in $\text{int}(\text{length of edge in metric } \mathcal{M})$ sub-edges and filter the too close points
   + Insertion with the anisotropic cavity.

6) Regularization?

Example ...
Part 2

Error indicators and mesh adaptation, some comparisons
Introduction II

We compare numerically, Hessian error indicators, residual error indicators and hierarchical error indicators, and when the new mesh size $h_{n+1}$ is almost given by the following formulae:

$$h_{n+1}(x) = h_n(x) f_n(\eta_n(x))$$

- $\eta_n(x)$ is the level of error at point $x$ given by the local error indicator,
- $h_n$ is the previous “mesh size” field,
- $f_n$ is an appropriate function, generally just linear like $f_n(\eta) = \frac{\eta}{\eta_n}$, with $\eta_n$ is the mean value of $\eta_n$. 

A model problem

For the Laplace problem, find $u$ a solution of:

$$-\Delta u = f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega$$

where $\Omega$ is a domain of $\mathbb{R}^d$, $f$ is a given function such that $\int_{\Omega} f = 0$.

The variational form is:

Find $u \in H^1(\Omega)/\mathbb{R}$ such that:

$$\forall v \in H^1(\Omega)/\mathbb{R}, \quad \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx$$

where

$$H^1(\Omega) = \{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega)^d \}$$

$$H^1(\Omega)/\mathbb{R} = \{ v \in H^1(\Omega) \mid \int_{\Omega} v = 0 \}$$
Residual error indicator

For the Laplace problem

\[-\Delta u = f \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega\]

the classical error $\eta_K$ indicator [C. Bernardi, R. Verfürth] are:

$$\eta_K = \int_K h_K^2 |(f - \Delta u_h)|^2 + \int_{\partial K} h_e |[\frac{\partial u_h}{\partial n}]|^2$$

where on triangle $K$, $h_K$ is size of the longest edge, $h_e$ is the size of the current edge, $n$ the normal.

Theorem: This indicator is optimal with Lagrange Finite element

$$c_0 \sqrt{\sum_K \eta_K^2} \leq ||u - u_h||_{H^1} \leq c_1 \sqrt{\sum_K \eta_K^2}$$

where $c_0$ and $c_1$ are two constant independent of $h$, if $\mathcal{T}_h$ is a regular family of triangulation.
Hierarchical error indicator

The idea is very simple, take a Hierarchy Lagrange finite element space [R. Bank, ..]

\[ X^p_h = \{ v \in H^1(\Omega) / \forall K \in T_h \quad v|_K \in P_p(K) \} \]

by construction we have : \( X^p_h \subset X^{p+1}_h \)

Denote \( Y^{p+1}_h \) a finite element space such that \( X^{p+1}_h = X^p_h \oplus Y^{p+1}_h \) (with degrees of freedom of \( X^{p+1}_h \) not in \( X^p_h \)).

First, find \( u \in X^p_h / \mathbb{R} \) such that :

\[ \forall v_h \in X^p_h / \mathbb{R}, \quad \int_{\Omega} \nabla u_h . \nabla v_h dx = \int_{\Omega} f v_h dx \]

Second, find \( e_h \in Y^{p+1}_h \) such that

\[ \forall v_h \in Y^{p+1}_h, \quad \int_{\Omega} \nabla e_h . \nabla v_h dx = \int_{\Omega} f v_h - \nabla u_h . \nabla v_h dx \]
Hierarchical error indicator, suite

A lot of technique can be use to simplify the previous problem to compute $e_h$, here the matrix is big, the condition number of this matrix is generally good.

remark, it possible to compute a good approximation of $e_h$ where we solve just local problem.

$e_h$ give a good approximation of the real error at the place of the new degree of freedom.

See work of I. Babuška, R. Bank, O. Zienkiewicz, J. Zhu ....
A Classical Schema of mesh adaption

\[ i=0; \]

Let \( \mathcal{T}_h^i \) an initial mesh

\[ \text{loop} \]

compute \( u^i \) the solution on mesh \( \mathcal{T}_h^i \)

evaluate the level of error \( \varepsilon \)

\[ \text{si} \; \varepsilon < \varepsilon_0 \; \text{break} \]

compute the new local mesh size \( h_{i+1} \)

construct a mesh according to prescribe the mesh size.

Remark, how transform a local a posteriori local error indicator on mesh size. Theoritically we want to estimate: \( \frac{\partial \eta_K}{\partial h_i} \)

but we use \( h_{i+1} = C h_i \bar{\eta_K} \) where \( \bar{\eta_K} \) is the mean value of \( \eta_K \), and \( C \) is a user parameter.
Metric and unit Mesh I/III

In Euclidean geometry the length $|\gamma|$ of a curve $\gamma$ of $\mathbb{R}^d$ parametrized by $\gamma(t)_{t=0..1}$ is

$$|\gamma| = \int_0^1 \sqrt{\gamma'(t) \cdot \gamma'(t)} dt = \int_0^1 \|\gamma'(t)\| dt$$

where $\cdot$ is the canonical dot product of $\mathbb{R}^d$.

We introduce the metric $\mathcal{M}(x)$ as a field of $d \times d$ symmetric positive definite matrices, and the length $\ell$ of $\Gamma$ w.r.t $\mathcal{M}$ is:

$$\ell = \int_0^1 \sqrt{\gamma'(t) \cdot \mathcal{M}(\gamma(t)) \gamma'(t)} dt$$

The key-idea is to construct a mesh where the lengths of all the edges are close to 1 accordingly to $\mathcal{M}$. 
Let $S$ be a surface, parametrized by

$$F(u) \in \mathbb{R}^3 \text{ with } (u) \in \mathbb{R}^2,$$

and let $\Gamma(t) = F(\gamma(t)), t \in [0, 1]$ be a curve on the surface. The length of the curve $\Gamma$ is

$$|\Gamma| = \int_0^1 \sqrt{\Gamma'(t) \cdot \Gamma'(t)} \, dt$$

$$|\Gamma| = \int_0^1 \sqrt{\gamma'(t) . t \partial F \partial F \gamma'(t)} \, dt$$

and on a parameteric surface the metric is

$$\mathcal{M} = t \partial F \partial F$$
Metric computation with Hessien for $P_1$ Lagrange finite element

The le Cea lemma say : the error is bound with the interpolation error. In a classical Adaption way, the metric tensor $\mathcal{M}$ is constructed in order to equilibrate the error of interpolation.

Find a metric tensor $\mathcal{M}$, such that, the adapted mesh constructed from $\mathcal{M}$ minimizes the interpolation error:

$$||u - \Pi_h(u)||_X$$

where $\Pi_h$ is the finite element $P_1$ interpolation operator

the idea is $\mathcal{M} = \frac{C}{\varepsilon_0^p} |\partial_{ij}u|^p$

where $p = \frac{1}{2}, 1, 1, 2$ depend of the norm $X = L^2, L^\infty, H^1, W^\infty_1$,

and where $\varepsilon_0$ is the global level of error, the user parameter.
An error estimate in $\mathbb{R}^2$

With the $P1$ finite element the error interpolation is:

$$||u - \Pi_h u||_\infty \leq \frac{1}{6} \sup_{x,y,z \in T} |^t x\mathbf{y} \mathcal{H}(z) x\mathbf{y}| \leq \frac{1}{6} \sup_{x,y,z \in T} ^t x\mathbf{y} |\mathcal{H}(z)| x\mathbf{y}$$

where $|\mathcal{H}|$ have the same eigen-vectors and the eigen-value of $|\mathcal{H}|$ is the abs of the eigen-value of $\mathcal{H}$,

We take $M = \frac{1}{6\varepsilon_0} |\mathcal{H}|$ and where $\varepsilon_0$ is the expected error.
a not to bad solution

\[ 10^{-10}u - \Delta u = f \quad \text{in} \ \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \ \partial \Omega \]

we take a L-shape domain : \( \Omega = ]0, 1[ \setminus [\frac{1}{2}, 1[ \) and \( f = x - y. \)

Nb of Triangles = 153834, Nb of Vertices 77574
4 tests with FreeFem++

All the test start with uniform mesh $h = 1/5$.

We make 7 steps of adaptation

– Residual Error indicator ($C = 1.8$)
– Residual Hierarchical indicator ($C = 1.25$)
– isotrope Metric indicator $\varepsilon_i = 2^{-i}/100$
– anisotrope Metric indicator $\varepsilon_i = 2^{-i}/100$
Initial mesh, Iteration 0, nt=54 nv=39
Residual Error indicator

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Mesh Residual Error indicator, Iteration 5, nt=9076 nv=4687
Hierarchical Error indicator

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Mesh Hierarchical Error indicator, Iteration 5, nt=4256, nv=2244
Metric aniso

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Mesh Metric aniso Error indicator, Iteration 3, nt= 2240
nv=1197
Metric iso, Iteration 3, nt=3325 nv=1755
Mesh Metric iso Error indicator, Iteration 3
Comparaison in semi norm $H^1$
Comparaison in norm $L^\infty$
Conclusion

The anisotrope metric adaption schema gives the best result but mathematically we have not real proof.

Thank for your attention
Appendix Fine estimation of Error in 1D

Let $[a, b]$ be a interval of $\mathbb{R}$, let $u$ be a function of $[a, b]$ in $\mathbb{R}$, let us call $\Pi_1(u)$ the $P_1$ interpolation of $u$. We have

$$\Pi_1(u) : t \in [a, b] \rightarrow \frac{(b - t)u(a) + (t - a)u(b)}{b - a}$$

**Lemma 1** Let be $u, f, g \in C([a, b])^2$ such that $f(a) \leq u(a) \leq g(a)$ and $f(b) \leq u(b) \leq g(b)$ and $f'' \geq u'' \geq g''$ then we have $f \leq u \leq g$.

**Proof** : We just use that a convex function on $[a, b]$ and negative on $a$ and $b$ is negative on all $[a, b]$. Use this result with the two function $f - u$ and $u - g$ to get the proof.
$L^p$ interpolation Error in 1D

Defined $\underline{u''} = \inf_{[a,b]} u''$ and $\overline{u''} = \sup_{[a,b]} u''$. To get following the classical error estimates, we take $f = \frac{1}{2}u''(x - a)(x - b)$ and $g = \frac{1}{2}u''(x - a)(x - b)$.

$$\frac{1}{2}u''(x - a)(x - b) \leq (u - \Pi_1 u) \leq \frac{1}{2}u''(x - a)(x - b)$$ \hspace{1cm} (1)

and

$$\frac{(b - a)^2}{8}u'' \leq u - \Pi_1 u \leq \frac{(b - a)^2}{8}u''$$ \hspace{1cm} (2)

So the interpolations error in classical norm on segment $[a, b]$ are

$$\|u - \Pi_1 u\|_{L^\infty} \leq \frac{(b - a)^2}{8}\|u''\|$$ \hspace{1cm} (3)

$$\|u - \Pi_1 u\|_{L^1} \leq \frac{(b - a)^3}{12}\|u''\|$$ \hspace{1cm} (4)

$$\|u - \Pi_1 u\|_{L^2} \leq \frac{(b - a)^{5/2}}{2\sqrt{30}}\|u''\|$$ \hspace{1cm} (5)
$H^1$ interpolation Error in 1D

To compute the error estimate in semi norm $H^1$, we need a majoration of $u'$. With the Rolle theorem, it exist a point $c \in [a, b]$ such that $(u - \Pi_1 u)'(c) = 0$, thus

\[ |(u - \Pi_1 u)'(x)| \leq |u''| |x - c|, \quad x \in [a, b] \]

And by a simple calculus we have

\[ \int_a^b |x - c|^2 dx = \frac{1}{3}((b - c)^3 - (c - a)^3) \leq \frac{1}{3}(b - a)^3, \]

thus

\[ |u - \Pi_1 u|_{H^1} = \| (u - \Pi_1 u)' \|_2 \leq \frac{(b - a)^{\frac{3}{2}}}{\sqrt{3}} |u''| \] (6)

This majoration is not optimal, because we d’ont use $\int_a^b (u - \Pi_1 u)' dx = 0$, I conjecture that the optimal majoration is given by changing the quotient $\sqrt{3}$ by $2\sqrt{3}$. 

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Interpolation Error in 2D

We work on a triangle $K$ with 3 vertices call $A, B, C$ and 3 edges $a, b, c$ such that $a = [BC], b = [CA], c = [AB]$, let be $E$ the set of edge equal to $a, b, c$ and $\lambda^A, \lambda^B, \lambda^C$ are the 3 barycentric coordinates of $K$. Let a function $u$ of $K$ in $\mathbb{R}$, we can introduce the derivation in the direction of an edge $e \in E$, since $e = [PQ]$ and with denote $u_e$ the partial derivative in direction of $e$.

$$u_e = PQ. \nabla u = \frac{\partial u}{\partial PQ}.$$  (7)

Let us call $\Pi_1(u)$ as in one dimensional case the $P^1$ interpolation of a continuous function of $K$ in $\mathbb{R}$, so we have

$$\Pi_1(u) = u(A)\lambda^A + u(B)\lambda^B + u(C)\lambda^C$$

Defined $\underline{f},\eta = \inf_\eta f$ and $\overline{f},\eta = \sup_e f$ where $\eta$ can be $a, b, c$ or $K$. 

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First the derivatives of the barycentrics functions are

\[
\begin{align*}
\lambda^A_a &= 0, & \lambda^B_a &= -1, & \lambda^C_a &= 1, \\
\lambda^A_b &= 1, & \lambda^B_b &= 0, & \lambda^C_b &= -1, \\
\lambda^A_c &= -1, & \lambda^B_c &= 1, & \lambda^C_c &= 0 \\
\end{align*}
\]

(8)

**Lemma 2** Let be \(f \in C^2(K)\) such that \(f \geq 0\) on edges \(b\) and \(c\) and \(f_{aa} \leq 0\) in \(K\) then \(f \geq 0\).

**Proof** : We just use the lemma [1] on segment \([PQ]\) defined with \(P = (1 - t)A + tC\) and \(Q = (1 - t)A + tBd\) so \(PQ = t BC = ta\).
Theorem 1 Let be \( u \in C^2(K) \) then we have

\[
    u - \Pi_1(u) \leq -\frac{1}{2} \left( u_{bb,b} \lambda^A \lambda^C + u_{cc,c} \lambda^A \lambda^B + u_{aa,K} \lambda^B \lambda^C \right)
\]  

(9)

and

\[
    u - \Pi_1(u) \leq -\frac{1}{2} \left( u_{bb,b} \lambda^A \lambda^C + u_{cc,c} \lambda^A \lambda^B + u_{aa,a} \lambda^B \lambda^C + u_{aac,K} \lambda^A \lambda^B \lambda^C \right)
\]  

(10)

Proof:

- Let \( f \) be defined by

\[
    f = u - \Pi_1(u) + \frac{1}{2} \left( u_{bb,b} \lambda^A \lambda^C + u_{cc,c} \lambda^A \lambda^B + u_{aa,K} \lambda^B \lambda^C \right)
\]

We have just to prove that \( f \) is negative on \( K \).

Firstly, the computation of the second derivatives gives

\[
    (\lambda^A \lambda^B)_{aa} = (\lambda^A \lambda^C)_{aa} = 0
\]
\[(\lambda^A \lambda^B)_{cc} = (\lambda^B \lambda^C)_{aa} = (\lambda^C \lambda^A)_{bb} = -2.\]

Secondly, \(\lambda^B \lambda^C\) is zero on \(b\) and \(c\). Thus the second derivatives \(f_{bb}\) (resp \(f_{cc}\) is positive on edge \(b\) (resp edge \(c\)), the lemma 1 give \(f\) is negative on edges \(b\) and \(c\).

We have:

\[f_{aa} = u_{aa} - u_{aa,K},\]

so \(f_{aa} \geq 0\) on all \(K\). The using lemma 2 with \(-f\) proof that \(-f \geq 0\), and so we have (9).

- To prove (10), We use function \(g\) defined by

\[g = u - \Pi_1(u) + \frac{1}{2} \left( u_{bb,b} \lambda^A \lambda^C + u_{cc,c} \lambda^A \lambda^B + u_{aa,a} \lambda^B \lambda^C + u_{aac,K} \lambda^A \lambda^B \lambda^C \right).\]

Now we have just to prove that \(g\) is negative on \(K\). Since, we have \(\lambda^A \lambda^B \lambda^C = 0\) on \(\partial K\) and the lemma 1, we get

\[g \leq 0, \quad g_{aa} \geq 0, \quad \text{on } \partial K\]
and with \((\lambda^A \lambda^B \lambda^C)_{aac} = -2\), we have

\[ g_{aac} = u_{aac} - u_{aac,K} \geq 0 \]

We have \(g_{aa} \geq 0\) on \(a\) by construction, and so by integrate in direction \(c\) from the edge \(a\), \(g_{aa}\) is positive in \(K\). To finish we have to reuse the lemme \(2\) to get \(g \leq 0\) on \(K\).
the error is bound by

Let be call \( \alpha = \sup_K |u_{aa}|, \beta = \sup_b |u_{bb}|, \gamma = \sup_c |u_{cc}| \). The interpolation error in classical norm:

\[
\|u - \Pi_1(u)\|_{L^\infty} \leq \frac{1}{6} \max(\alpha, \beta, \gamma) \leq \frac{1}{6} \max(\alpha, \beta, \gamma)
\]

\[
\|u - \Pi_1(u)\|_{L^\infty} \leq \frac{1}{6} \sup_{e \in \{a, b, c\}} \sup_{x \in \partial K} \|t e^H(x)e\| + \frac{1}{27} \sup_{x \in K} |u^{(3)}(x)(a, a, c)|
\]

\[
\|u - \Pi_1(u)\|_{L^\infty} \leq \frac{1}{6} \sup_{e \in \{a, b, c\}} \sup_{x \in \partial K} \|t e^H(x)e\| + \frac{1}{27} \min_{e, e' \in \{a, b, c\}} \sup_{x \in K} |u^{(3)}(x)(e, e, e')|
\]
the error is bound by

To get (11), we have to remark that the sup of $\lambda^B\lambda^C + \lambda^A\lambda^C + \lambda^A\lambda^B$ is $\frac{1}{3}$.

\[ \|u - \Pi_1(u)\|_{L^1}^K \leq \frac{|K|}{24}(\alpha + \beta + \gamma) \] (11)

\[ \|u - \Pi_1(u)\|_{L^2}^K \leq \frac{|K|}{12\sqrt{5}}\sqrt{\alpha^2 + \beta^2 + \gamma^2 + \alpha\beta + \beta\gamma + \alpha\gamma} \] (12)