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Title of the thesis :

Stabilization of 1D nonlinear hyperbolic systems
by boundary controls

This is the unofficial version of the thesis, translated in English for convenience. There may be some inaccuracies and mistakes of language, for which we apologize in advance.

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Chapter 1

Introduction

1.1 Differential equations

Understanding and describing the world around us has always been a motivation for mathematics. In this approach, the appearance of the notion of derivative has been a major step and has enabled the emergence of a fundamental tool: the differential equations. The most renown differential equation is probably the fundamental principle of dynamics stated by Newton in 1687, and written:

\[ m \frac{d^2x}{dt^2} = \sum F_{\text{ext}}, \quad (1.1.1) \]

where \( m \) is the mass of the considered object, \( x \) its position vector and \( F_{\text{ext}} \) the external vector forces applied on it. This equation opened the door to the description of physical point systems and has been massively used since then and until today.

Actually, the notion of derivative was only formalized in its most rigorous form during the 19th century, but it has not prevented the rise of differential equations since the end of the 17th century and the establishment of an exceptional number of studies and solving methods. Since then, differential equations have gathered numerous areas of mathematics, from algebra to probabilities, to geometry and number theory.

1.2 Partial differential equations

The ordinary differential equations model accurately the point systems, but are inappropriate for representing the continuous systems. How to describe, for instance, the motion of a wave? Or the propagation of a vibration in a rope? For these systems we would like to introduce an infinite number of points, each of them described by a differential equation. This motivated the birth of partial differential equations, abbreviated in PDE. We can then define the derivatives with respect to a single variable for functions of several variables, \( \partial_t f(t, x) \) stands, for instance, for the time derivative of a quantity that depends for on time \( t \) and location \( x \). The conservation of the mass of a fluid with density \( \rho \) moving at a speed \( V \) can be written for instance:

\[ \partial_t \rho(t, x) = -\partial_x (\rho(t, x)V(t, x)). \quad (1.2.1) \]

In fact, PDE only appeared around the middle of the 18th century. At the time, mathematicians were looking for solving methods, often geometrical, to reduce the problems that should have led to PDE to simpler problems. PDE appeared thanks to Euler and d’Alembert around 1750 [69, 73, 87]. The first general resolution methods for such equations are due to Charpit in 1784 [92], even if his works have only been known later on and were supplemented by Lagrange et Monge. These works have enabled the arrival of the method of characteristics for first-order equations. Since then, PDE have fed numerous mathematical problems. They are everywhere in physics, mechanics, and have led to numerous mathematical theories (distributions, viscosity solutions, etc.).
1.3 Control theory

When we know how a system works, and when we can act on it, what can we make it do? This question, that Kalman asked himself in 1960, is not new and would not have been disowned by the Prince of Machiavelli.

The merit of Kalman is to have brought this question in a quantitative field renown for its extremely powerful tools: the mathematics. Control theory can be decomposed in three branches:

- **Controllability**: when my system starts from a state \( x_0 \), is it possible to bring it to a state \( x_1 \) chosen?
- **Optimal control**: if there exist trajectories bringing my system to the desired state, are some of them optimal with respect to a given criterion, and if so which ones?
- **Stabilization**: if my state or my trajectory is fixed, is it possible to make it stable, i.e. robust to perturbations that could happen?

In the two first problems, the control depends on the initial state of the system \( x_0 \), the system is said to be in open-loop. In the third problem, however, the control is a function of the state of the system at time \( t \), it is what we call a feedback law. The system is then said to be in closed-loop. From a mathematical point of view, this fundamental difference means that several questions have to be reconsidered. For instance the well-posedness of the equation \( \dot{x} = f(t, x, u(t)) \) with initial condition \( x_0 \) given is well-known when \( u \) is only a function of \( x_0 \) and \( t \). The well-posedness of \( \dot{x} = f(t, x, u(t, x)) \), in contrast, requires to be more careful with the control properties \( u \) and its dependency in \( x \). From a practical point of view, this difference means that we act on the system as a function of what we measure. And, of course, in an infinite dimensional system, where the state of the system \( x \) is a function, the question of what we allow ourselves to measure arise.

For infinite dimensional systems, i.e. for problems modelled by PDE, there exists two main types of control:

- **Internal controls.** In this case the control is directly included in the dynamics of the system and acts on a set of points inside the interior of the domain, often of non-zero measure, for instance
  \[
  \begin{align*}
  \partial_t u + \partial_x u &= f(t, x), \text{ sur } [0, 1] \\
  u(t, 0) &= u(t, 1)
  \end{align*}
  \] (1.3.1)
  where \( f \) is a control with a spatial support included in \([0, 1]\). It is the case, for instance, for a magnetic field applied to a conducting material.

- **Boundary controls.** In this case, the control acts on the limit of the domain (thus on a subset of zero measure). The dynamics of the system remains unchanged but is coupled with boundary conditions on which we act. For instance
  \[
  \begin{align*}
  \partial_t u + \partial_x u &= 0, \text{ sur } [0, 1] \\
  u(t, 0) &= f(t)
  \end{align*}
  \] (1.3.2)
  where \( f \) is a control. This is the case for instance for a river on which we can act through dams located at its extremities, or also for a highway on which we control the incoming flux using traffic lights.

For many systems it is hard to get access directly to the interior of the system to control it. For rivers for instance, one can only act with the dams upstream and downstream. Boundary controls are thus often a good physical choice, therefore we will consider this type of control in the following. Besides, we are interested in the stabilization problem, thus our controls are feedback laws, meaning that \( f(t) = f(u(t, \cdot)) \).

Finally, we will add an additional constraint: we will only consider “simple” controls, i.e. controls involving only measures on the state of the system at the boundaries. In other words

\[
 f(u(t, \cdot)) = f(u(t, \cdot)|_{\partial \Omega}),
\] (1.3.3)
where $\Omega$ is the domain. The idea behind such a constraint is twofold: first these controls were historically the most used in mathematical studies, which allows us to compare our results and our methods. Also, this constraint allows a simple implementation of the controls in practice, without having to measure quantities inside the domain (for instance inside a river), even if it means that the mathematics behind might be more complicated.

1.4 Stabilization of systems of partial differential equations

Unlike ordinary differential equations, it is difficult to study all partial differential equations as a single object class. Nevertheless one can try to classify them in three types:

- Elliptic equations.
- Parabolic equations.
- Hyperbolic equations.

Some more complicated equations are a mixture of these three types. We can then try to identify, at least locally, a predominant behavior corresponding to one of these classes. The solutions of elliptic equations are entirely determined by their boundary conditions, thus the problem of their stabilization by boundary controls does not really make sense. Regarding the parabolic equations, many results exist (e.g. [9, 32, 63, 89, 147, 189], see also [175, Section 7]), although there are still many open questions. We can also quote [67, 172, 173, 187] for examples concerning fluid equations with parabolic behaviors. Finally, the stabilization of hyperbolic equations and systems of hyperbolic equations is still far from being well understood, even in dimension 1 in space. We will be interested in these systems in this thesis.

1.4.1 Généralités sur les systèmes hyperboliques 1D

A system of quasilinear hyperbolic PDE has the following form:

$$\partial_t Y + F(Y)\partial_x Y + G(Y, x) = 0.$$  \hfill (1.4.1)

where $Y : [0, +\infty[\times[0, L] \rightarrow \mathbb{R}^n$ is the state of the system, $F$ is a function from $U$ to $\mathcal{M}_n(\mathbb{R})$, the space of square matrices of dimension $n$, such that for all $Y \in U$, $F(Y)$ is diagonalisable with distinct and real eigenvalues, $U$ is a connected and non-empty open set of $\mathbb{R}^n$, and $G$ is a function from $U \times [0, L]$ to $\mathbb{R}^n$. We suppose in addition that for any $Y \in U$, the eigenvalues of $F(Y)$ do not vanish, and that $F$ and $G$ are of class $C^1$.

**A change of coordinate** By definition, a hyperbolic system is a system that has quantities that propagate. Nevertheless these quantities can be different from the components of $Y$ when the matrix $F(Y)$ is not diagonal. To find them locally around a steady state $Y^*$, we can introduce the following change of variable: $u = S(Y - Y^*)$, with $S$ a matrix that diagonalizes $F(Y^*)$. The components of $u$ then represent locally the quantities that propagate with, for propagation speed, the eigenvalues of $F(Y^*)$. The system then takes the form

$$\partial_t u + A(u, x)\partial_x u + B(u, x) = 0.$$  \hfill (1.4.2)

where $A(0, x) = \Lambda(x)$ is the diagonal matrix of eigenvalues of $F(Y^*)$, and $B(0, x) = 0$. $A$ and $B$ are then again of class $C^1$.

**A typical phenomenon:** the shock waves. Nonlinear hyperbolic systems can present an astonishing phenomenon: even starting from a very regular initial condition at $t = 0$, the solution can spontaneously evolve towards a solution that "breaks" in finite time and presents discontinuities. These discontinuities are called shocks. We can illustrate this with the Burgers’ equation that will be presented in more detail in Section 1.7.1:

$$\partial_t u + u\partial_x u = 0.$$  \hfill (1.4.3)
For simplicity, we study this equation on the whole $\mathbb{R}$ with $u_0 \in C^1(\mathbb{R})$ as initial condition. Suppose that the solution is regular for any time, we define the characteristic curve $x(t)$ starting in $x_0$ as the only solution of

$$
x'(t) = u(t, x(t)),
$$
$$
x(0) = x_0.
$$

Then, along a characteristic $x(t)$, the solution $u$ of (1.4.3) satisfies

$$
\frac{d}{dt}(u(t, x(t))) = 0.
$$

Let us set two characteristics $(x_1(t), t)$ and $(x_2(t), t)$ starting in $x_1(0) = x_{1,0} < x_{2,0} = x_2(0)$. According to (1.4.3) and (1.4.5) these characteristics satisfy :

$$
x_1(t) = u_0(x_{1,0})t + x_{1,0} \quad \text{et} \quad x_2(t) = u_0(x_{2,0})t + x_{2,0}.
$$

Suppose now that $u_0(x_{1,0}) > u_0(x_{2,0})$, then for $t = \frac{x_{1,0} - x_{2,0}}{u_0(x_{2,0}) - u_0(x_{1,0})} > 0$, $x_1(t) = x_2(t)$. But as they are characteristics, from (1.4.5),

$$
u(t, x_1) = u_0(x_{1,0}) \quad \text{et} \quad u(t, x_2) = u_0(x_{2,0}),
$$

which implies that $u_0(x_{1,0}) = u_0(x_{2,0})$, and lead to a contradiction. Defining a solution for all time in this situation imposes thus necessarily that it is discontinuous. Integrating (1.4.3), one can find the speed at which these discontinuities propagates and, for a discontinuity localized in $x_s(t)$,

$$
\dot{x}_s(t) = \frac{(u(x^+_{s}(t)))^2 - (u(x^-_{s}(t)))^2}{2(u(x^+_{s}(t)) - u(x^-_{s}(t)))}.
$$

This brings a new problem: discontinuous solutions are not unique. This means not only that the system

![Figure 1.1: Example of initial condition $u_0$ leading to a discontinuity: the left-hand side of the solution of (1.4.3) is positive and move with positive speed, while the right-hand side is negative and move with negative speed.](image)

is no longer well-posed but also that some of the solutions considered are not physical. To solve this, Lax introduces an additional condition [132] : to be acceptable the two characteristics arriving at a discontinuity,
if they were positively extended in time, should cross the curve of discontinuity which moves at speed given by \(1.4.8\). In other words in our case, 
\[
u(x^+ < \dot{x} < u(x^-).\]

(1.4.9)

This condition ensures the unicity of the solution and thus a causality principle, i.e. at each time \(t > 0\) all the information of \(u(t, \cdot)\) was already contained in the initial condition \(u_0\), which is, after all, quite physical. Endowed with this condition, the system is again well posed, and the solution is said to be entropic. In more general cases, we can also define discontinuous and entropic solutions. The study of these solutions, and among others for stability studies, is often a complicated problem.

### 1.4.2 Definition of stability

We couple now the system \((1.4.1)\) with the boundary control, so far formal, 
\[
B(Y(t, 0), Y(t, L)) = u(t). \tag{1.4.10}
\]

where \(u(t)\) is the control which depends on the state \(Y\). As stated previously, \(u(t)\) is called a feedback law. Stabilization is about finding feedback laws \(u(t)\) such that, whatever the initial condition, any solution of \((1.4.1), (1.4.10)\) converges to a fixed steady state \(Y^*\), which plays the role of target to reach. We can give a more precise mathematical definition : for a Banach space \(X\) with a norm \(||\cdot||_X\), which will be called the \(X\) norm in the following, we define the exponential stability as follows :

**Définition 1.4.1.** The steady-state \(Y^*\) of the system \((1.4.1), (1.4.10)\) is exponentially stable for the \(X\) norm if there exists \(\gamma > 0, \eta > 0,\) and \(C > 0\) such that for any initial condition \(Y^0 \in X\) compatible with \((1.4.10)\) and such that \(||Y^0 - Y^*||_X \leq \eta\), the Cauchy problem \((1.4.1), (1.4.10), (Y(0, x) = Y^0)\) has a unique solution in \(C^0([0, +\infty[, X)\) and 
\[
| |Y(t, \cdot) - Y^*||_X \leq Ce^{-\gamma t}| |Y^0 - Y^*||_X, \quad \forall t \in [0, +\infty[.
\]

\(1.4.11\)

We can make three remarks :

- This definition depends on the norm \(X\) considered. A question that one could legitimately ask is: is it really necessary to specify the norm \(X\), at least as long as the system is well posed ? The answer is yes. In general, for nonlinear hyperbolic systems, the stabilities in the different norms are not equivalent \[61\]. We will come back to this in part \(4\) (see Section \(1.6\).

- We see that this definition requires that the system be well-posed, which can sometimes be a problem. Indeed, as we have already slightly mentioned it, even the well-posedness of the usual systems must be re-studied when the control is a feedback law. For example, in Chapter \(6\) we have to show this well-posedness for the Burgers’ equation with our feedback law before studying stability as such. A less demanding definition would be to simply require that solutions exist without asking for uniqueness and that they all satisfy the stability estimate \((1.4.11)\). In most cases, however, it is not a given that this greatly simplifies the problem.

- The exponential stability we are talking about is local. Indeed, when stabilizing nonlinear hyperbolic systems by boundary controls, it is generally impossible to obtain a global stabilization, that is to say for disturbances arbitrarily wide. This is due to the finite propagation speeds of the system, which implies that if the state of the system begins close enough to a shock inside the domain, then the shock will form before the influence of the controls at the boundaries can reach it.

Finally, if we consider the change of coordinates \(u(t, x) = S(x)(Y(t, x) - Y^*(x))\), the steady state we aim at stabilizing is now \(u^* = 0\). Of course the exponential stability of \(Y^*\) for the original system \((1.4.1), (1.4.10)\) is equivalent to the exponential stability of the steady state \(u^* = 0\) for the new system \((1.4.2)\) with the

\(^3\)A more precise definition is given in Chapter \(2\).
corresponding boundary control. For this reason in the following we will consider most of the time this last system.

A renown boundary control: the proportional control Stabilization already existed well before the first definition of controllability by Kalman. In antiquity, people were already trying to stabilize systems, and proportional control is perhaps the first control that has been designed on finite dimensional systems. Without really mathematizing it, Ctesibius used it already in the 3rd century B.C. to stabilize the flow of a clepsydra. Much later, in 1788 it is the Watt regulator, used to stabilize the steam engines, that took up the same principle. As its name indicates, this control consists in acting on the system proportional to what one measures. If we take, for example, the equation $\dot{x} = x + u(t)$ where $u(t)$ is the control, we see that choosing $u(t) = -kr$ is a simple way to stabilize exponentially the system if $k > 1$. The solution then being $x(t) = x_0 e^{-kt}$. For infinite dimensional systems the principle is the same and we assign to our boundary control $u(t)$ a value proportional to the values of $Y$ at the boundaries of the domain, that is to say in $0$ and in $L$. The boundary control (1.4.10) then becomes

$$\left(\frac{S(0)(Y(t,0) - Y^*(0))}{S(L)(Y(t,L) - Y^*(L))}\right) = K\left(\frac{(S(L)(Y(t,L) - Y^*(L))}{(S(0)(Y(t,0) - Y^*(0))}\right),$$

(1.4.12)

where $K \in \mathcal{M}_n(\mathbb{R})$ is a matrix that can be chosen, $S$ diagonalizes $F(Y^*)$ so that $S(Y - Y^*)$ represents locally the components that propagate with speeds given by the eigenvalues of $F(Y^*)$. The notation $S(Y - Y^*)^+$ denotes the components that propagate with a positive speed and $S(Y - Y^*)^-$ those which propagate with a negative speed. This boundary control therefore simply means that the incoming information in the system is a function of the outgoing information, which is the simplest notion of feedback law. By extension we can consider a nonlinear control:

$$\left(\frac{S(0)(Y(t,0) - Y^*(0))}{S(L)(Y(t,L) - Y^*(L))}\right) = G\left(\frac{(S(L)(Y(t,L) - Y^*(L))}{(S(0)(Y(t,0) - Y^*(0))}\right),$$

(1.4.13)

where $G$ is now a function $C^1$ (and no longer a matrix) such that $G(0) = 0$. Since $G$ is $C^1$, the expression (1.4.13) linearizes locally as (1.4.12) with $K = G'(0)$. For this reason, these controls are the natural generalization of proportional controls, and we will call them output feedback laws or output controls in the following. These controls are the most famous and probably the most studied in mathematics (see Section 1.6.1). These are the ones we will study in the first two parts. Nevertheless, in the industry, a variant allowing to gain in robustness exists: the proportional-integral control, or PI control. We will give a more precise definition in the part III (see Section 1.8) dedicated to a more specific study of these controls.

A natural question then arises: how to find a control that stabilizes our system? In other words, how to choose $G$?

1.4.3 An almighty tool for linear systems: the Spectral Mapping Theorem

For linear 1D systems, there exists an almighty tool: the Spectral Mapping Theorem which can be stated as follows:

**Théorème 1.4.1.** Let a linear hyperbolic system of the form (1.4.2), where $A$ does not depend on $u$ and $B = M(x)u$, endowed with boundary conditions of the form (1.4.12). We define $\mathcal{L}$ by

$$\mathcal{L}(u) = -A\partial_x u - Mu$$

(1.4.14)

\footnote{Even if the name of Watt remained, this regulator was probably already invented before. Paternity remains difficult to establish but Mead used a similar regulator in 1787 and a scheme of di Giorgio Martini from the 15th century seems to represent a similar regulator.}

\footnote{Technically the result stated in [141] only deals with local boundary conditions in the sense that the control in 0 only depends on values measured in 0 and the control in $L$ only depends on values measured in $L$. Nevertheless, one can always reduce the problem to this case through an appropriate doubling of variable as presented in [70].}
on the domain $D(\mathcal{L}) := \{ (u) \in W^{1,2}(0,L), \mathcal{C}^0 ) | (u(0))_+, (u(L))_-^T = K((u(L))_+,(u(0))_-^T \}$. Then
\[
\sigma(\mathcal{C}^0) \setminus \{ 0 \} = \overline{\sigma(\mathcal{L}) \setminus \{ 0 \}}, \text{ pour } t \geq 0,
\]
where $\sigma(C)$ refers to the spectrum of the operator $C$, and $\overline{\sigma(\mathcal{L})}$ to the closure of $\sigma(\mathcal{L})^*$.

This theorem, if it may seem arid at first sight, means in summary that it is enough to know the eigenvalues of the operator to know the limits of the stability of the linear system. It thus reduces to a relatively simple problem the resolution of a complicated question like the stability of an infinite dimensional system.

Unfortunately, as soon as the system is nonlinear, this theorem no longer applies, even if it is only very weakly nonlinear or only studied locally. To be convinced of this, we can look at the following system taken from \cite{61}:
\[
\begin{align*}
\partial_t u + \begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{2+10^{-5}u_2}} & 1 \end{pmatrix} \partial_x u &= 0, \\
u(t,0) &= a \begin{pmatrix} 1 & \xi \\ -1 & \eta \end{pmatrix} \nu(t,1).
\end{align*}
\]
In \cite{61}, the authors show that we can find $a \in \mathbb{Q}$, $\xi > 1$ and $\eta > 1$, such that the linearized system associated with (1.4.16) is exponentially stable, while the system (1.4.16) itself is not exponentially stable for the $C^1$ norm, even when $\|\nu\|_{C^1}$ is as close to 0 as we want. Adding even a very weak nonlinearity can thus radically change the stability. Thus, other methods must be found to stabilize nonlinear systems.

1.4.4 A new way to formulate the problem : the Lyapunov approach

Many strategies have been established to stabilize nonlinear systems, among them:

- The method of the characteristics. This is the historical and most natural method. For 1D hyperbolic systems, it is possible to trace the characteristics of the system to obtain explicit estimates. Nevertheless this method quickly becomes complex and hard to use when the system gets complicated, especially when it has source terms, that is, $B \neq 0$.

- The Grammian method. Introduced first in finite dimension, then for linear systems of infinite dimension by \cite{177} and \cite{122,123}, it consists in using the properties of the semi-groups to define a feedback law and deduce the stability. This semi-group approach can be generalized to nonlinear systems in some cases.

- The Backstepping method. This very powerful method, introduced in \cite{39,121,193} for finite dimensional systems, has been adapted in \cite{51} and modified in \cite{9,32} (see also \cite{128,179}) for infinite dimensional systems. For infinite dimensional systems this modified method consists in transforming, via an invertible linear transformation, the system into a system easier to stabilize, then to perform the inverse transformation once a feedback law has been found for this simpler system. Unfortunately, these successive transformations result in general in complicated controls depending on the whole state of the system, and no longer only on the state at the boundaries.

- The Lyapunov approach, that we detail here.

The spirit of the Lyapunov approach is to find a functional $V$ that decreases along the trajectories $u$, tends to 0, and such that $V(u) = 0$ implies $u = 0$. Intuitively, $V$ can be seen as an energy of the perturbations $u$ that we want to bring back to 0. The idea behind this is that we have more flexibility when studying $V(u)$ than $u$. Indeed, even if the norm of $u$ tends to 0, it is possibly hard to show if it does not decrease all the time, for instance if it makes damped oscillations. On the other hand, if it tends to 0, it is possible to make an envelope of the solution $V(u)$ which decreases all the time. The convergence of $V(u)$ is then easier.
Figure 1.2: Example of Lyapunov function $V(u(t))$, decreasing with time $t$ while the solution $u(t)$ does not decrease with $t$ and has damped oscillations. Here $u(t)$ corresponds to the angle of a pendulum, as described later on in Section 1.4.4 and $V(u(t))$ to the associated mechanical energy.

to show since it we “only” need to compute its derivative with respect to time. This is illustrated in the following figure (Fig. 1.2).

This method has first been introduced for finite dimensional systems by Lyapunov in 1892 [148, 149].

**Théorème 1.4.2** (Lyapunov). Let a system $\dot{x} = f(x)$ with $f$ locally Lipschitz and $f(0) = 0$. If there exists a neighborhood $U$ of the origin and a function $V \in C^1(U, \mathbb{R})$ such that

\[
V(x) > 0, \text{ for all } x \in U \setminus \{0\}, \quad V(0) = 0,
\]
\[
\nabla V(x) \cdot f(x) < 0 \text{ for all } x \in U \setminus \{0\},
\]

(1.4.17)

then we say that $V$ is a Lyapunov function, and the system is (locally) asymptotically stable.

In some cases, it is difficult to get a strict inequality in (1.4.17). This theorem can then be supplemented by LaSalle’s invariance principle: 7

**Théorème 1.4.3** (LaSalle). Let the same system as previously, if there exists a neighborhood $U$ of the origin and a function $V \in C^1(U, \mathbb{R})$ such that

\[
V(x) > 0, \text{ for all } x \in U \setminus \{0\}, \quad V(0) = 0,
\]
\[
\nabla V(x) \cdot f(x) \leq 0 \text{ for all } x \in U \setminus \{0\},
\]

(1.4.18)

and such that the set $\{x \in U | \nabla V(x) \cdot f(x) = 0\}$ does not contain any trajectory except from the null trajectory, then the system is (locally) asymptotically stable.

In infinite dimension this method still applies, with some precisions. Indeed, to be applied, LaSalle’s invariance principle requires to show the pre-compactness of the trajectories of the infinite dimensional system [40], which is sometimes technical and complicated. In the following, we are interested in the exponential stabilization (see the Definition 1.4.1), which is why we define the Lyapunov functions in the following way, slightly more restrictive than previously.

6 The definition of weakly nonlinear does not seem to be clearly established but we understand the idea: the Spectral Mapping Theorem cannot apply stricto sensu to a system of the form $u_t + (1 + 0.00001u)u_x = 0$.

7 In reality LaSalle’s principle is a little more general than [143] and the theorem we give is its direct application to the stability of a point, see [188, Theorem 13.3.3] for more details.
Définition 1.4.2. Let a hyperbolic quasilinear system of the form \((1.4.2)\) with boundary conditions such that the system is well-posed in \(X\). We call Lyapunov function for the \(X\) norm a functional \(V \in C^0(X, \mathbb{R})\) such that there exists \(\gamma > 0, c_1 > 0, c_2 > 0\) and a neighborhood of the origin \(W \subset X\) such that
\[
c_1 \|U\|_X < V(U) < c_2 \|U\|_X, \quad \text{pour tout } U \in W,
\]
and such that for any \(T > 0\) and any solution of the system \(u\) on \([0, T]\) with a trajectory included in \(W\),
\[
\frac{dV(u(t, \cdot))}{dt} \leq -\gamma V(u(t, \cdot)),
\]
in a distribution sense on \([0, T]\).

We can notice that we do not ask anymore the function \(V\) to be \(C^1\), which results in an inequality \((1.4.20)\), in a distribution sense. This will be useful in the following when we define Lyapunov functions equivalent to the norm \(L^\infty\) or \(W^{l,\infty}\) and not necessarily differentiable. Moreover, this definition is a little stricter than the one previously used because we require the decay of the Lyapunov function to be exponential and no longer only asymptotic. This slight constraint allows unsurprisingly to obtain the exponential stability of the system:

Proposition 1.4.4. Let a hyperbolic quasilinear system of the form \((1.4.2)\) with boundary conditions such that the system is well-posed in \(X\). If there is a Lyapunov function for the \(X\) norm, then the system is exponentially stable for the \(X\) norm.

This method is very useful and very general, but the price to pay is that it is also very abstract and does not give much guidance on how to build the function \(V\), especially in infinite dimension when \(V\) belongs to \(C^0(X, \mathbb{R}_+)\), a very large space.

An important case : the basic Lyapunov functions.

Among the possible Lyapunov functions, there is a natural and important class : the basic Lyapunov functions.

Définition 1.4.3. For a Sobolev space \(W^{l,p}([0, L])\), where \((l, p) \in \mathbb{N} \times \mathbb{N}^* \cup \{+\infty\}\), we call a basic Lyapunov function for the norm \(W^{l,p}\) a function \(V \in C^0(W^{l,p}([0, L]), \mathbb{R})\) defined by:
\[
V(U) = \sum_{n=0}^{l} \|F(\cdot)E(U, \cdot)D_n U\|_{L^p([0, L])}, \quad \forall U \in W^{l,p}([0, L]),
\]
where \(E(U, x)\) is a matrix diagonalizing \(A(U, x)\), \(D_n\) is the operator defined iteratively by \(D_1U = -A(U, x)\partial_x U + B(U, x)\), and for \(n \geq 2, D_n U = \partial_x (D_{n-1} U) D_1 U\), with \(\partial_x\) the differentiation operator with respect to \(x\), and \(F = (f_1, ..., f_n)\) are \(C^1\) and positive functions on \([0, L]\), such that \(V\) is a Lyapunov function for the \(W^{l,p}\) norm.

Defined this way, these functions might not seem that natural. In reality, applied to a solution of \((1.4.2)\), the expression \((1.4.21)\) becomes
\[
V(u) = \sum_{n=0}^{l} \|F(\cdot)E(u(t, \cdot), \cdot)\partial_x^n u(t, \cdot)\|_{L^p},
\]
which justifies the form \((1.4.21)\). These Lyapunov functions are therefore a kind of weighted norms for Sobolev spaces \(W^{l,p}\), considering temporal derivatives instead of spatial derivatives. If these functions have been used for a long time for the \(L^2\) norm, because of their link with the dissipative entropies that we will see in the next paragraph, the name basic Lyapunov function appears in \([11]\), while we introduce the \(C^1\) version in \([106]\), even if similar functions were already used in \([48]\).

\*Precisely, it is required that for every \(T > 0\) the system is well posed from \([0, T]\) to \(X\), more precise and less abstract definitions are given at the beginning of each relevant chapter.

\*For a precise definition of an inequality in a distribution sense, see the Remark \([2.3.3]\). In the present case \((1.4.20)\) is equivalent to \(V(u(t, \cdot)) \leq V(u(t', \cdot))e^{-\gamma(t-t')}\) for all \(0 < t \leq t' < T\), which explains that we talk about exponential decay.

\*The problem does not really arise in finite dimension : all norms being equivalent, one can always choose a norm more regular than the infinite norm.
Link dissipative entropies - Lyapunov functions

Physical systems have physical quantities, such as energy or entropy, that can help study their stability. The stability of a pendulum subject to friction, for example, is easily studied using the mechanical energy. The dynamics of the pendulum is given by

\[ \ddot{\theta} = -\frac{g}{mR} \sin(\theta) - \frac{f}{m} \dot{\theta}, \]

where \( R \) is the pendulum length, \( m \) its mass, \( \theta \) its angle with the vertical axis, \( f > 0 \) the friction coefficient and \( g \) the gravity acceleration on Earth’s surface. Introducing the mechanical energy \( V(\theta) = (1/2)m(R^2(\dot{\theta})^2 + 2gR(1 - \cos(\theta))) \) and using (1.4.23) we obtain the energy equation along trajectories

\[ \frac{dV(\theta)}{dt} = \frac{1}{2}(-2gR \sin(\theta)\dot{\theta}) - 2R^2 f \dot{\theta}^2 + 2gR \sin(\theta)\dot{\theta} = -\frac{R^2 f}{m} \dot{\theta}^2 \leq 0. \]

Theorem 1.4.3 is then applied. Indeed \( \dot{V}(\theta) = 0 \) implies \( \dot{\theta} = 0 \) and, according to (1.4.23), the only path contained in a neighborhood of the origin such as \( \dot{\theta} = 0 \) for any time is the path identically zero \( \theta = 0 \). The system is therefore asymptotically stable and, incidentally, \( V \) is a weak Lyapunov function, that is, it satisfies (1.4.18). In the same way, if we consider an isolated solid with an inhomogeneous temperature at \( t = 0 \), the second principle of thermodynamics imposes that the physical entropy increases, which imposes the convergence to a uniform equilibrium for the temperature.

All this motivated the definition of a mathematical notion, called entropy, which, by abuse of language, encompasses the previous physical quantities but also other quantities to which it is sometimes difficult to give a physical meaning, as we will see it later (see Chapter 6 and 7).

**Définition 1.4.4.** Let a system of the form (1.4.2), we call entropy a function \( \eta \in C^\infty(W \times [0, L]; \mathbb{R}) \), where \( W \subset \mathbb{R}^n \) is a neighborhood of the origin, such that for all \( U \in W \) and all \( x \in [0, L] \),

\[ \partial_\eta(U, x)A(U, x) = A(U, x)^T \partial_\eta(U, x). \]

Clearly, if the dimension \( n \) is too big, this condition imposes more than one equation while \( \eta \) has only one component. This problem does not necessarily admit of solution. Nevertheless, what is admirable is that physical systems often have a special structure that gives them an entropy, as we will see with the example of density-velocity systems in the part II. If an entropy exists, we can define the flow associated with the entropy by

\[ \partial_\eta q = \partial_\eta A \]

Obviously, \( \eta \) is only defined up to an affine function, and \( q \) is only defined up to a constant. Therefore, for a given convex entropy \( \eta \), we can define a new entropy by

\[ \eta_r(U) = \eta(U) - \eta(U^*) - \partial_\eta_U(U^*)(U - U^*). \]

such that \( \eta_r(U^*) = \partial_\eta(U^*) = 0 \) and such that \( \eta_r \) is locally quadratic around \( U^* \). This entropy is called entropy relative to \( U^* \). Finally, these entropies are said to be dissipative if they satisfy the following entropy production condition:

\[ \partial_\eta(U, x)(B(U, x)) - \partial_x q(U, x) \geq 0 \]

The existence of a convex dissipative entropy ensures the “internal” stability of the system. In other words if the system is studied on the whole \( \mathbb{R} \) (and not on a bounded domain) and that there is a convex dissipative entropy, then the system will be asymptotically stable for the \( L^2 \) norm. In our stabilization problem, we can choose the conditions at the boundaries, thus they do not pose a problem. That is why, when there is such an entropy, there is always a boundary control such that the system is stable. Moreover, the form of \( \eta \) makes it possible to find explicit conditions on this control.

The dissipative entropies and Lyapunov functions are closely related. It can be noted that the integrand of a basic Lyapunov function for the \( L^2 \) norm is directly a convex dissipative entropy, locally at least. This is also the case for the basic Lyapunov functions for the \( H^0 \) norm by seeing the \( H^0 \) norm as the \( L^2 \) norm of
the augmented system that takes as variable \((u, \partial_t u, \ldots, \partial^p_t u)\) and is obtained by differentiating successively \(p\) times the system (1.4.2) with respect to time (see Appendix 2.8.4 for example). Indeed, at the first order, \(\partial^2_t u \eta\) is diagonal just like \(A\), which solves the problem of the condition (1.4.25), up to a higher order rest, while the condition (1.4.28) corresponds exactly to the internal condition found in [13, Theorem 6.10] (see Section 1.6.1) for basic Lyapunov functions for the \(H^p\) norm. On the other hand, one can note that (1.4.25) is exactly the condition requested by the authors in [16] to define a basic Lyapunov function for the norm \(L^2\) for a linearized density-velocity system. Finally, the dissipative convex entropies are good Lyapunov functions and take locally the quadratic form \(\eta(U) = \frac{1}{2} U^T \partial^2_t U \eta(0, x) + O(|U|^2)\), close to 0, which is close to the integrand of a basic Lyapunov function.

1.4.5 Main difficulties

The stabilization of 1D and homogeneous hyperbolic systems, i.e. of the form (1.4.2) with \(B \equiv 0\), is a well-known problem and quite well circumscribed when it is studied in regular spaces and using proportional boundary controls or output feedback laws (see for example [138] and [61]). The stabilization of general systems, on the other hand, is still relatively unknown because it is more complicated. Two phenomena make it intrinsically difficult to study the stability of general systems:

- **The source terms (or inhomogeneities).** These source terms induces several problems, among others, an intrinsic coupling between the different equations of the system that cannot be solved by simply using a change of variables (see Section 1.6 for more details). This will be the subject of Chapters 2 and 3 and partially of Chapters 4, 5 and 9.

- **The shocks.** The occurrence of discontinuities complicates the stabilization methods, especially when they change the sign of one of the propagation velocities. For this reason, studies are generally restricted to regular solutions. The subject of Chapters 6 and 7 will be to extend these results to the stabilization of shock steady-states.

Moreover, an additional difficulty can arise in the choice of the control. This is the case when we want to use a control that is no longer just proportional, but proportional-integral. As will be seen in part III, PI controls are still poorly understood mathematically when they are applied to nonlinear systems of infinite dimension. An example illustrates this well: the example of the transport equation treated in Chapter 8. If finding optimal conditions to stabilize this system with proportional controls or output feedback laws is easy and can be done in one paragraph, doing it with a PI control requires a new method, which is introduced in Chapter 8 and takes about twenty pages.

1.5 Objectives of the thesis

Hyperbolic systems have been studied for many years for their broad mathematical and practical interest. In view of the multitude of their applications, the question of their stability and stabilization is fundamental and has nourished many researches over the last forty years. This thesis is placed in their following and is articulated around three parts.

In the first part, we look for general results to guarantee the stability in \(C^1\) norm of inhomogeneous systems. The second part focuses on the equations of fluid mechanics and shows in particular the existence of a local dissipative entropy for the 1-D “density-velocity” systems, which include the Saint-Venant equations and the isentropic Euler equations in their most general forms. This entropy trivializes the study of their stabilities whatever the framework. Finally we introduce a method to deal with shocks and stabilize a shock steady-state.

In the third part we are interested in proportional-integral (PI) controls, which are widely used in industry for their robustness but poorly understood mathematically when they are applied to nonlinear systems of 11In reality we can show that when \(A\) is diagonal, since it has distinct eigenvalues, \(\partial^2_t \eta(0, x)\) is diagonal, and \(E(U, x) = Id\), which corresponds exactly to the integrand of a basic Lyapunov function.
infinite dimension. We introduce a so-called "extraction method" to find optimal stability conditions for a scalar equation. We then study the Saint-Venant equations stabilized by a PI control.

Summary of the thesis:

1.6 Part 1: Stability of inhomogeneous quasilinear systems for the $C^1$ norm.

As mentioned in Section 1.4.2, stability estimates of the type (1.4.11) for different norms are not equivalent when the system is nonlinear. For a first-order non-linear differential system, the classical natural solutions are the solutions $C^1$. This is why the general study of the stability of these systems with boundary controls has historically been conducted for this standard and for homogeneous systems. These homogeneous systems, that is to say without source term, are written

$$\partial_t Y + F(Y)\partial_x Y = 0,$$

(1.6.1)

where $F(Y)$ is diagonalizable around a steady state $Y^*$ with distinct and real eigenvalues, as in Section 1.4.1. Inhomogeneous systems are written

$$\partial_t Y + F(Y)\partial_x Y + G(Y, x) = 0.$$  

(1.6.2)

These systems induce many difficulties compared to the homogeneous case:

- **Nonuniform steady-states.** When there is a source term the stationary states are generally non-uniform with variations that can be large, as can be seen by looking for a stationary solution to (1.6.2). The stabilization of one of these stationary states can then always be reduced to stabilization of the constant steady state equal to 0 via a change of variables as seen in Section 1.4.1. But there is a price to pay: the system takes the form (1.4.2) and the transport matrix $A$ and the source term $B$ then explicitly depend on $x$, whether the original matrices depend on it or not.

- **Internal coupling between equations.** When the system is homogeneous and we use the change of variables $u = S(Y - Y^*)$ to put it in the form (1.4.2), $B \equiv 0$ and the matrix $A$ is diagonal for $u = 0$. The system thus takes a locally diagonal form around $u^* = 0$. The components of $u$ are therefore locally weakly coupled to each other due to nonlinearity. When the system is inhomogeneous, on the contrary, $B \neq 0$ and the components are intrinsically coupled together by the source term, even locally.

These differences make the study of the stability of inhomogeneous systems significantly more difficult than that of homogeneous systems. To my knowledge, the only general results are in [13, Théorème 6.10] and [114] which study the stability in $H^2$ norm to overcome these difficulties. In [13, Théorème 6.10], Bastin and Coron use output boundary feedback laws, thus of the form (1.4.13). In [114] the authors have a completely different approach and use the backstepping method, already mentioned in Section 1.4.4, which gives a result that works all the time but also gives rise to boundary controls much more complicated and difficult to implement in practice.

Nevertheless, in both cases one thus loses the natural norm $C^1$ in favor of a norm more regular and therefore more restrictive: the norm $H^2$ (we recall that, thanks to the injection of Sobolev, a function $H^2$ is automatically also $C^1$ in 1D). In the following we will consider only output feedback laws and proportional controls of the form (1.4.13) which are at the same time among the most studied, but also among the most practical because they only need to know the state of the system at the boundaries, as stated previously. The purpose of this part is to obtain the stability in the $C^1$ norm using an approach similar to that of Bastin and Coron but we will see that it brings significantly different conditions. Finally, we will study in more detail the systems of two equations and we will see that we can establish links between the stability for the $H^2$ norm and stability for the $C^1$ norm for these systems.
1.6.1 Existing results

**Homogeneous systems** The first result for hyperbolic quasilinear systems is probably the result of Li and Greenberg in 1984 \[93\]. In this article, Li and Greenberg study a homogeneous system of two equations, so \(B \equiv 0\) in its diagonalized form \[1.4.2\], with local boundary controls of the form:

\[
\begin{align*}
\mathbf{u}_+(0) &= f(\mathbf{u}_-(0)), \\
\mathbf{u}_-(L) &= g(\mathbf{u}_+(L)),
\end{align*}
\] (1.6.3)

and show by studying precisely the characteristics that the system is exponentially stable for the \(C^1\) norm if

\[
|f'(0)g'(0)| < 1.
\] (1.6.4)

Note that the controls (1.6.3) are a special case of the output feedback laws (1.4.13). Indeed, when the system is in diagonal form with \(u = S(Y - \mathbf{Y}^*)\), the controls (1.4.13) are written

\[
\begin{pmatrix}
\mathbf{u}_+(t, L) \\
\mathbf{u}_-(t, L)
\end{pmatrix} = G \begin{pmatrix}
\mathbf{u}_+(t, 0) \\
\mathbf{u}_-(t, 0)
\end{pmatrix},
\] (1.6.5)

which covers the case of (1.6.3). This result was then generalized to systems of dimension \(n \in \mathbb{N}^*\) by Qin \[170\] then Zhao \[202\], then with general boundary conditions of the form (1.6.5) by Halleux et al. \[71\] then recently by \[48\]. The proof of this last article uses Lyapunov functions in a way from which we will use while modifying them to be able to manage the new difficulty related to the source term. With general conditions at the boundaries of the form (1.6.5), the system is then exponentially stable in the \(C^1\) norm if

\[
\rho_{\infty}(G'(0)) < 1
\] (1.6.6)

where \(\rho_{\infty}\) is defined by

\[
\rho_{\infty}(M) = \inf(\|\Delta M \Delta^{-1}\|_{\infty}, \Delta \in D_n^+),
\] (1.6.7)

with \(D_n^+\) the set of diagonal matrices with positive coefficients.

Later on, Coron and Bastin showed that the stability is easier to get in the more regular (and thus smaller) space \(H^2\). The condition in this space becomes:

\[
\rho_2(G'(0)) < 1
\] (1.6.8)

where \(\rho_2\) is defined by

\[
\rho_2(M) = \inf(\|\Delta M \Delta^{-1}\|_2, \Delta \in D_n^+),
\] (1.6.9)

and one can check that \(\rho_2 \leq \rho_{\infty}\) \[50\]. Given the similarity of the conditions (1.6.7) and (1.6.8) one may wonder if it is possible to find stability conditions for other Sobolev spaces using \(\rho_p\) defined by

\[
\rho_p(M) = \inf(\|\Delta M \Delta^{-1}\|_p, \Delta \in D_n^+).
\] (1.6.10)

Coron and Nguyen showed that it was indeed the case \[61\], more precisely the homogeneous system of the form (1.4.2) with \(B \equiv 0\) endowed with the control (1.6.5) is exponentially stable for the \(W^{2,p}\) norm if

\[
\rho_p(G'(0)) < 1.
\] (1.6.11)

It remains to be seen which of these conditions is the least restrictive. If we can show that \(\rho_{\infty} \geq \rho_p\) for all \(p \in \mathbb{N}^*\), and therefore that the stability in norm \(C^1\) is more difficult to obtain than the stability in norm \(W^{2,p}\) for all \(p \in \mathbb{N}^*\), the comparison of \(\rho_p\) between them is not clear. Since the less the solution is regular, the more general it is, one would expect that the stability in the spaces of solutions less regular (and therefore wider) is more difficult to obtain and thus that \(\rho_{p_1} \geq \rho_{p_2}\) for \(p_1 > p_2\). This intuition is in fact misleading, as \(p_1 = \rho_{\infty}\) and therefore \(p_1 \leq \rho_p\) for all \(p \in \mathbb{N}^*\ \cup \{\infty\}\) \[56\] Remark 1.4]. But in all these cases, we are moving away from the least restrictive and most natural norm: the \(C^1\) (or \(W^{1,\infty}\)) norm.
Inhomogeneous systems If particular inhomogeneous hyperbolic systems have interested mathematicians for years, the study of the stability of inhomogeneous hyperbolic systems in general has only recently begun. We now consider a system of the form (1.4.4) and we want to stabilize a stationary state $\mathbf{Y}^*$, potentially non-uniform, with controls at the boundaries of the form (1.4.13). To simplify the study, we use again the change of variables $\mathbf{u} = \mathbf{S}(\mathbf{Y} - \mathbf{Y}^*)$ presented in Section 1.4.1 so that the system becomes

$$
\partial_t \mathbf{u} + A(\mathbf{u}, x) \partial_x \mathbf{u} + B(\mathbf{u}, x) = 0
$$

(1.6.12)

where $A(0, \cdot) = \Lambda$, and $B(0, x) = 0$, as described in introduction, and of which we try to stabilize the steady-state 0. The boundary controls (1.4.13) have again the form (1.6.5), and we introduce in addition the notation $M(0, x) = \partial_\mathbf{u} \mathbf{g}(0, x)$.

Again, the first really general result for these systems was for $2 \times 2$ systems, that is, composed of two potentially coupled equations, with propagation speeds of different signs [11]. Rigorously, the result of [11] deals with linear systems but it uses an approach that can be extended almost immediately to nonlinear systems for the norm $H^2$ and is thus:

**Théorème 1.6.1** ([11]). Let a quasilinear hyperbolic system of dimension 2 be of the form (1.4.2) where $A$, $B$ and $G$ are of class $C^2$, $\Lambda_1 > 0$ and $\Lambda_2 < 0$. There exists a control of the form (1.6.5) such that there exists a basic Lyapunov function for the $H^2$ norm for this system if and only if there exists a function $\eta$ defined on $[0, L]$ and solution of :

$$
\eta'(x) = \begin{vmatrix} M_{21}(0, \cdot) \eta^2 + \varphi \frac{M_{12}(0, \cdot)}{\Lambda_1} \end{vmatrix},
$$

(1.6.13)

$$
\eta(0) = 0,
$$

where $\varphi$ is a notation referring to :

$$
\varphi = \exp \left( \int_0^x \frac{M_{11}(0, s)}{\Lambda_1} + \frac{M_{22}(0, s)}{\Lambda_2} ds \right).
$$

(1.6.14)

Besides, for any $\sigma > 0$ such that

$$
\eta_\sigma \geq \begin{vmatrix} M_{21}(0, \cdot) \eta^2 + \varphi \frac{M_{12}(0, \cdot)}{\Lambda_1} \end{vmatrix},
$$

(1.6.15)

has a solution $\eta_\sigma$ on $[0, L]$ if

$$
G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix} \text{ with } l_1^2 < \eta_\sigma^2(0) \text{ and } l_2^2 < \frac{1}{\eta_\sigma^2(L)},
$$

(1.6.16)

there exists a basic Lyapunov function for the $H^2$ norm and the system (1.4.2), (1.6.16) is exponentially stable for the $H^2$ norm.

An interesting phenomenon appears which did not occur before : as before there is a condition on the boundary control, but there is also now an internal condition (1.6.15), independent of control and therefore intrinsic to the system. Since the differential equation (1.6.15) is non-linear, its solutions can explode and cease to exist if $L$ is too big, this is for example the case if $M_{11}(0, \cdot) = M_{22}(0, \cdot) = 0$ and $M_{12}(0, \cdot) = M_{21}(0, \cdot) = 1$. In this case $\eta$ is simply the tangent function and explodes in $L = \pi/2$. This condition means that we can not guarantee stability over a certain distance The $L_{\text{max}}$. It is an inherent condition of the system that links the forms and sizes of the source terms with distance.

This result has then been generalized in [13 Chapitre 6] to the general $n \times n$ case and the authors have shown the following result :

**Théorème 1.6.2** ([13]). Let a hyperbolic quasilinear system (1.4.2) with boundary control (1.6.5) such that $A$, $B$ and $G$ are of class $C^2$. The system is exponentially stable for the $H^2$ norm if
\begin{itemize}
  \item (internal condition) the matrix 
  \[ -(QA)'(x) + Q(x)M(0,x) + M(0,x)^TQ(x)^T \] 
  is positive definite for any \( x \in [0,L] \),
  \item (Boundary condition) the matrix 
  \[ \begin{pmatrix} \Lambda_+(L)Q_+(L) & 0 \\ 0 & -\Lambda_-(0)Q_-(0) \end{pmatrix} - K^T \begin{pmatrix} \Lambda_+(0)Q_+(0) & 0 \\ 0 & -\Lambda_-(L)Q_-(L) \end{pmatrix} K \] 
  is semi-positive definite.
\end{itemize}

We find again the two conditions: condition \( \text{(1.6.18)} \) on the boundary control and condition \( \text{(1.6.17)} \) intrinsic to the system. Once again, this results uses basic Lyapunov functions for the \( H^2 \) norm.

### 1.6.2 Nouveaux résultats

#### General case

In Chapter 2 we are interested in the \( C^1 \) norm and we show the following result:

**Theorem 1.6.3.** Let a quasilinear hyperbolic system of the form \( (1.4.2), \text{(1.6.5)} \) with \( A \) and \( B \) of class \( C^1 \).

If the two following conditions are satisfied,

1. (internal condition) the system 
   \[ \Lambda_1 f'_1 \leq -2 \left( -M_{1i}(0,x) f_i + \sum_{k=1,k\neq i}^n |M_{ik}(0,x)| \left| f_i \right|^3 \right) / \sqrt{f_k} \]  
   has a solution \( (f_1, ..., f_n) \) on \( [0,L] \) such that for any \( i \in [1,n] \), \( f_i > 0 \),

2. (boundary conditions) there exists a diagonal matrix \( \Delta \) with positive coefficients such that 
   \[ \| \Delta G'(0) \Delta^{-1} \|_\infty < \frac{\inf_k \left( \frac{f_k(d_k)}{\Delta^2_k} \right)}{\sup_k \left( \frac{f_k(L-d_k)}{\Delta^2_k} \right)} \]  

then, there exists a basic Lyapunov function for the \( C^1 \) norm and the system \( (1.4.2), \text{(1.6.5)} \) is exponentially stable for the \( C^1 \) norm.

We obtain again an inner condition and a boundary condition. Let us note that the existence of a solution \( (f_1, ..., f_n) \) on \( [0,L] \) to the system 

\[ f'_i = -2 \left( -M_{1i}(0,x) f_i + \sum_{k=1,k\neq i}^n |M_{ik}(0,x)| \left| f_i \right|^3 \right) / \sqrt{f_k} \]  

with \( f_i > 0 \) for all \( i \in \{1, ..., n\} \), is also a sufficient but a stricter internal condition. On the other hand, when \( M \equiv 0 \), we find the result of Li, Greenberg, Qin, Zhao and Halleux et al. for the homogeneous systems: the inner condition is satisfied by any constant functions \( (f_1, ..., f_n) \) and choosing \( f_i = \Delta_i^2 \), the edge condition comes back the existence of \( \Delta \in D^n_+ \) such that \( \| \Delta G'(0) \Delta^{-1} \|_\infty < 1 \), which is equivalent to \( \rho_\infty(G'(0)) < 1 \).

To show this result we introduce the basic Lyapunov functions for the norm \( C^1 \), which is the analog of that for the norm \( H^2 \) used in \( [11, 13] \), and we approach them by functions equivalent to the norm \( W^{1,p} \) where \( p \) is then made to infinity, in the same fashion as what is done in \( [13] \). The main difference comes from the fact that, when one differentiates the functions equivalent to the norm \( W^{1,p} \) with respect to time, appears because of the inhomogeneity a polynomial that one would like to be positive. In \( [13] \) which deals with the case of the norm \( H^2 \) this polynomial is a quadratic form. Here it is a polynomial of degree \( p \) and guaranteeing its sign therefore requires a little more work (see \( (2.5.31) - (2.5.38) \) and Lemma \( 2.5.1 \) for more details). Besides, we go a little further by showing the following result which illustrates the sharpness of the internal condition of Theorem \( 1.6.3 \):
Théorème 1.6.4. Let a quasilinear hyperbolic system be of the form (1.4.2) with $A$ and $B$ of class $C^3$. There exists a boundary control of the form (1.6.5) such that there exists a basic Lyapunov function for the $C^1$ norm if and only if

$$\Lambda_i f_i' \leq -2 \left(-M_{ii}(0, x)f_i + \sum_{k=1, k\neq i}^n |M_{ik}(0,x)| f^{3/2}_j \right),$$

(1.6.22)

admits a solution $(f_1, ..., f_n)$ on $[0, L]$ such that for all $i \in [1, n]$, $f_i > 0$.

This result is proved by contradiction, assuming that there exists a basic Lyapunov function for the $C^1$ norm when this condition is not satisfied, and showing that one can find an initial disturbance that makes it increase at least over a short time interval.

Remarque 1.6.1. The regularity of the coefficients is not exactly the same in Theorem 1.6.3 and 1.6.4. The explanation is in the perturbation that is constructed to prove the Theorem 1.6.4: we try to make it more regular than just $C^1$ to be able to estimate the derivative of the Lyapunov function in a classical sense. It is very probable that the regularity can be lowered to that of Theorem 1.6.3. Nevertheless, in the physical cases $A$ and $B$ are often not only $C^{inf}$ but even analytic. Notice that we impose the regularity of $A$ and $B$ as a functional and not the regularity of $A(u(t, \cdot), \cdot)$ which could be lower if $u$ is less regular.

These results can be generalized in fact to the stabilization in norm $C^q$ for all $q \geq 1$, under the same conditions. In other words if $A$, $B$ and $G$ are of class $C^q$ (resp. $C^{q+2}$) then Theorem 1.6.3 (resp. Theorem 1.6.4) still applies when replacing the stability in norm $C^1$ by the stability in norm $C^q$. Nevertheless, we seek to stabilize the system in the least regular possible norm to cover the largest possible solution space. That is why it is the $C^1$ norm that interests us most in general. Finally, in the case where the system is semi-linear, that is when $A$ does not depend on $u$, these results can be generalized to the $C^0$ norm.

Case $2 \times 2$ and comparison with the $H^2$ norm In the Chapter 3 we are interested in the case where the system is composed of only two equations. This case is interesting, not only because it is the simplest case of systems having a coupling between two equations, but also because it corresponds to numerous physical systems (signal transmission along an electric line, river dynamics, behavior of isentropic gas, road traffic, etc.) and that it exhibits particular behaviors. In the following we consider the case $n = 2$ and we introduce the notations:

$$\varphi_1 = \exp \left( \int_0^x \frac{M_{11}(0, s)}{\Lambda_1} ds \right),$$

$$\varphi_2 = \exp \left( \int_0^x \frac{M_{22}(0, s)}{\Lambda_2} ds \right),$$

$$\varphi = \frac{\varphi_1}{\varphi_2},$$

$$a = \varphi M_{12}(0, \cdot),$$

$$b = \varphi^{-1} M_{21}(0, \cdot).$$

(1.6.23)

Before continuing, let us take a moment to explain these notations. First of all we can notice that $\varphi$ corresponds exactly to the $\varphi$ introduced in the theorem 1.6.1 of Bastin and Coron. In fact, the functions $\varphi_1$ and $\varphi_2$ represent the influence of the diagonal terms of $M(0, \cdot)$ that would lead to an exponential variation of the amplitude if there was no coupling between $u_1$ and $u_2$. The functions $a$ and $b$ represent this coupling induced by $M(0, \cdot)$ after a change of variable to remove the diagonal terms of $M$ (see (3.4.1) and (3.4.4) for more details). This change of variable to delete the diagonal terms of $M$, obtained by setting $z = \varphi \cdot u$, is apparently innocuous but it makes it possible for the linearized operator $L(U) = -A(0, x) \partial_x U - M(0, x) U$ and the linearized homogeneous operator $L^*(U) = -A(0, x) \partial_x U$ to be equipped respectively with semigroups $S$ and $S^*$ whose difference is compact. This difference is not compact in general when $M$ has diagonal non-zero terms. This remark, formulated by Russell in 1978 [175] and proved by Hu and Olive in [113], can prove
useful to obtain properties of controllability (see for example [113]), although this not explicitly used here.

We then show the following results :

- **Case where the propagation speeds have same sign :**

  **Théorème 1.6.5.** Let a quasilinear hyperbolic system be of the form (1.4.2), with \( n = 2 \), where \( A, B \) and \( G \) are of class \( C^2 \) and such that \( \Lambda_1 \Lambda_2 > 0 \). If

  \[
  G'(0) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix},
  \]

  \[
  k_1^2 < \exp \left( \int_0^L 2 \frac{M_{11}(0,s)}{|\Lambda_1|} - 2 \max \left( \left| \frac{a(s)}{\Lambda_1} \right|, \left| \frac{b(s)}{\Lambda_2} \right| \right) ds \right), \tag{1.6.25}
  \]

  \[
  k_2^2 < \exp \left( \int_0^L 2 \frac{M_{22}(0,s)}{|\Lambda_2|} - 2 \max \left( \left| \frac{a(s)}{\Lambda_1} \right|, \left| \frac{b(s)}{\Lambda_2} \right| \right) ds \right),
  \]

  then there exists a basic Lyapunov function for the \( C^1 \) norm and a basic Lyapunov function for the \( H^2 \) norm. In particular the system is exponentially stable for the \( C^1 \) norm and the \( H^2 \) norm.

- **Case where the propagation speeds have different signs :**

  **Théorème 1.6.6.** Let a quasilinear hyperbolic system be of the form (1.4.2) with \( A \) and \( B \) of class \( C^3 \) with \( \Lambda_1 > 0 \) and \( \Lambda_2 < 0 \). There exists a control of the form (1.6.5) such that there exists a basic Lyapunov function for the \( C^1 \) norm if and only if

  \[
  d'_1 = \left| \frac{a(x)}{\Lambda_1} \right| d_2,
  \]

  \[
  d'_2 = -\left| \frac{b(x)}{\Lambda_2} \right| d_1, \tag{1.6.26}
  \]

  admits a positive solution \( d_1, d_2 \) sur \([0,L]\), or equivalently

  \[
  \eta' = \left| \frac{a}{\Lambda_1} \right| + \left| \frac{b}{\Lambda_2} \right| \eta^2, \tag{1.6.27}
  \]

  \[
  \eta(0) = 0,
  \]

  admits a solution on \([0,L]\), where \( a \) et \( b \) are defined by (1.6.24).

  Besides, if one of the previous condition is satisfied and if

  \[
  G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_1^2 < \varphi(L)^2 \left( \frac{d_2(L)}{d_1(L)} \right)^2 \text{ et } k_2^2 < \left( \frac{d_1(0)}{d_2(0)} \right)^2, \tag{1.6.29}
  \]

  where \( d_1 \) and \( d_2 \) form a solution of (1.6.26)–(1.6.27), then the system (1.4.2), (1.6.5) is exponentially quickly for the \( C^1 \) norm.

  **Remarque 1.6.2.** An interesting remark can be made : in the Théorème 1.6.3 the necessary and sufficient internal condition concerns the existence of solutions to a system of differential inequalities. This theorem shows with (1.6.26)–(1.6.27) that, in the case of systems \( 2 \times 2 \), it is equivalent to the existence of a solution to a system of differential equations, moreover linear, which is practical. This result is false in general in dimension \( n \geq 3 \).

  It is not by mistake that we call again \( \eta \) the solution of the ODE (1.6.28) : these conditions are different from those obtained previously for the norm \( H^2 \), and recalled in Théorème 1.6.1, but have the same form. It is therefore natural to want to compare them, and this is what is done in the following corollary:
Corollaire 1. Let a quasilinear hyperbolic system be of the form (1.4.2), (1.6.5) with \( n = 2 \) and \( A \) and \( B \) of class \( C^2 \), such that \( \lambda_1 > 0 \) and \( \lambda_2 < 0 \).

1. If there exists a basic Lyapunov function for the \( C^1 \) norm, then there exists a boundary control of the form (1.6.5) such that there exists a basic Lyapunov function for the \( H^2 \) norm.

Moreover, if \( M_{12}(0, \cdot)M_{21}(0, \cdot) \geq 0 \), then the converse is true.

2. In particular if the system (1.4.2), (1.6.5) admits a basic Lyapunov function for the \( C^1 \) norm and

\[
G'(0) = \begin{pmatrix} 0 & k_2 \\ k_2 & 0 \end{pmatrix} \quad \text{avec} \quad k_2^2 < \varphi(L)^2 \left( \frac{d_2(L)}{d_1(L)} \right)^2 \quad \text{et} \quad k_1^2 < \left( \frac{d_1(0)}{d_2(0)} \right)^2,
\]

where \( d_1 \) and \( d_2 \) are positive solutions of (1.6.26)–(1.6.27), then with the same boundary control there exists a basic Lyapunov function for the \( H^2 \) norm.

Conversely, if the system admits a basic Lyapunov function for the \( H^2 \) norm and if \( M_{12}(0, \cdot)M_{21}(0, \cdot) \geq 0 \) and

\[
G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \quad \text{avec} \quad k_2^2 < \left( \frac{\varphi(L)}{\eta(L)} \right)^2 \quad \text{et} \quad k_1^2 < \eta(0)^2,
\]

where \( \eta \) is a positive solution of

\[
\eta' = \left| \frac{a}{\lambda_1} + \frac{b}{|\lambda_2|^2} \eta \right|^2,
\]

then there exists a basic Lyapunov function for the \( C^1 \) norm and the system is exponentially stable for the \( C^1 \) norm.

Remarque 1.6.3. The existence of a solution to (1.6.32) when there exists a basic Lyapunov function for the \( H^2 \) norm is given by Theorem 1.6.7.

To my knowledge, the only such link that existed so far dealt with the trivial case \( M \equiv 0 \) where there always exists both a basic Lyapunov function for the \( H^2 \) norm and a basic Lyapunov function for the \( C^1 \) norm. This link can in fact be extended to the stability in \( H^p \) and \( C^q \) norms for \( p \geq 2 \) and \( q \geq 1 \) with the same conditions.

Since Theorem 1.6.1 and Theorem 1.6.6 both use basic Lyapunov functions for the \( H^2 \) and \( C^1 \) norm respectively, we may want to build one according to the other when we know, thanks to Corollary 1 that both exist. This is what we do in this theorem :

Théorème 1.6.7. If there is an boundary control of the form (1.6.5) such that there exists a basic Lyapunov function for the \( C^1 \) norm with the coefficient \( g_1 \) and \( g_2 \), then for every \( 0 < \varepsilon < \min \{0, \varepsilon\} (|\varphi_2/\varphi_1|\sqrt{|g_1/g_2|}/L) \)

there exists a boundary control of the form (1.6.5) such that

\[
\frac{1}{\lambda_1} \left( \sqrt{\frac{g_1}{g_2}} \varphi_1 \varphi_2 - \varphi_1^2 \varepsilon I_d \right) \quad \text{et} \quad \frac{1}{|\lambda_2|} \sqrt{\frac{g_2}{g_1}} \varphi_1 \varphi_2
\]

are the coefficients of a basic Lyapunov function for the \( H^2 \) norm, where \( I_d \) denotes the identity function.

If there exists a basic Lyapunov function for the \( H^2 \) norm with, for coefficients, \((q_1, q_2)\) and if \( M_{12}(0, \cdot)M_{21}(0, \cdot) \geq 0 \), then for any \( A \geq 0 \) and \( \varepsilon > 0 \) there exists a boundary control of the form (1.6.5) such that \( g_1 \) and \( g_2 \) given by :

\[
g_1(x) = A \exp \left( 2 \int_0^x \frac{M_{12}(0, \cdot)}{\lambda_1} - \frac{|M_{21}(0, \cdot)|}{\lambda_2} \right) \sqrt{\frac{|A_1|}{|A_2|}} \ v_g \ ds - \varepsilon x \right),
\]

\[
g_2 = \frac{|A_2|}{\lambda_1} q_2 \ v_g,
\]

are the coefficients of a basic Lyapunov function for the \( C^1 \) norm.
Corollary 1.6.8 gives a sufficient condition for the existence of a basic Lyapunov function for the $H^2$ norm to be equivalent to the existence of a basic Lyapunov function for the $C^1$ norm. Looking at the necessary conditions (1.6.28) and (1.6.15), we quickly see that there are counter-examples when the sufficient condition of Corollary 1 is not verified. One might think that these counter-examples are artificial, but in fact, and somewhat surprisingly, there are even counter-examples that correspond to well-known physical systems. This is for example the case of the Saint-Venant equations used to model the atmosphere and the waterways (more details are given in Section 1.7.3). We will see in the part II a general form of the Saint-Venant equations, but here, to simplify, we introduce a special case in which the slope is constant, the rectangular section uniform, and the friction model well determined. The equations of Saint-Venant are then written

\[
\begin{align*}
\partial_t H + \partial_x (HV) &= 0, \\
\partial_t V + \partial_x \left( \frac{V^2}{2} + gH \right) + \left( \frac{kV^2}{H} - C \right) &= 0,
\end{align*}
\]  

(1.6.36)

where $C$ is the slope influence, $k$ the friction coefficient, $g$ the gravity acceleration on Earth’s surface and the variables $H$ and $V$ denote the height of water and its horizontal averaged velocity, for example in a river. We denote $(H^*, V^*)$ the steady state we are trying to stabilize and we introduce the boundary controls

\[
\begin{align*}
(H(t, 0) - H^*(0)) &= b_1(V(t, 0) - V^*(0)), \\
(H(t, L) - H^*(L)) &= b_2(V(t, L) - V^*(L)).
\end{align*}
\]

(1.6.37)

Since the system is not diagonal, $H$ and $V$ are not the quantities that propagate. Nevertheless, we can show that this boundary controls has still the form (1.6.5) (see Section 3.3 in particular (3.3.13), (3.5.13) - (3.5.17)). When the influence of friction is greater than the influence of the slope, we can always find $b_1$ and $b_2$ such that there exists a basic Lyapunov function for the $C^1$ norm, more precisely

**Proposition 1.6.8.** Let the nonlinear Saint-Venant equations (1.6.36) with the boundary control (1.6.37).

If $kV^2(0)/H^*(0) > C$ and

\[
 b_1 \in \left[ \frac{H^*(0)}{V^*(0)}, \frac{V^*(0)}{g} \right] \quad \text{et} \quad b_2 \in \mathbb{R} \setminus \left[ \frac{H^*(L)}{V^*(L)}, \frac{V^*(0)}{g} \right],
\]

(1.6.38)

then the system admits a basic Lyapunov function for the $C^1$ norm.

When the influence of the slope is stronger than the influence of the friction, on the other hand, there is not always a basic Lyapunov function for the $C^1$ norm. This result is all the more interesting as there is always a basic Lyapunov function for the $H^2$ norm (see part II Theorem 1.7.1).

**Théorème 1.6.9.** Let the nonlinear Saint-Venant equations (1.6.36) on the domain $[0, L]$, with a steady state $(H^*, V^*)$. If $kV^2(0)/H^*(0) < C$ then:

1. There exists $L_1 > 0$ such that if $L < L_1$, there exists a boundary control of the form (1.6.37) such that the system admits a basic Lyapunov function for the $C^1$ norm.

2. There exists $L_2 > 0$ independent of the boundary control chosen such that, if $L > L_2$, the system does not admit any basic Lyapunov function for the $C^1$ norm.

**Generalization to $H^p$ and $C^q$ norm.** As before, all these results can be generalized to the $H^p$ and $C^q$ norms for all integers $p \geq 2$ and $q \geq 1$, with the same conditions, provided that $A$, $B$, and $G$ are sufficiently regular. Moreover, when the system is semi-linear, these results can also be generalized to the $H^1$ and $C^0$ norms.

### 1.7 Part 2 : Physical equations and density-velocity systems.

In this part we study equations coming from fluid mechanics, and in particular three famous examples : the Burgers’ equation, the Euler equations, and the Saint-Venant equations.
1.7.1 The Burgers’ equation

Although seemingly the simplest of the equations in this part, the Burgers’ equation is actually the most recent one. It is written as:

\[ \partial_t y + \partial_x \left( \frac{y^2}{2} \right) = 0. \]  \hspace{2cm} (1.7.1)

Given its simplicity, it is difficult to date the first work dealing with it. Forsyth already mentioned it in 1906 in [SS] with an additional viscosity term, just like Bateman in 1915 [23] who deduced it from a physical problem and wondered what would be the behavior of the solutions if the viscosity tended towards 0. In 1939, it is Burgers who becomes interested in this equation and removes the viscosity term to keep only the form (1.7.1) who already features the behaviors that interested him. His many works on the subject until 1948, summarized in [35] and [36], popularize it and give it its current name. Behind this approach, Burgers was actually interested with the turbulence, a phenomenon he had been studying since 1923 [37], and saw in the Burgers’ equation a simple model for studying it. Unfortunately, the works of Hopf in 1950 [111] and Cole in 1951 [45] showed that it is possible to solve this equation explicitly, at least when it is studied on the whole \( \mathbb{R} \).

Nevertheless, if the Burgers’ equation is so famous, it is because it features generic behaviors found in the equations of fluid mechanics. Among these behaviors, the most famous is perhaps the phenomenon of shocks: even starting from a regular solution to \( t = 0 \) the system can spontaneously “break” the solution and evolve towards a solution that has discontinuities (See Section 1.4.1 for more details on shocks).

Its simplicity makes this equation a good way to study the main lines of these behaviors, and a good testbed for testing new mathematical techniques for later use on more complicated equations, such as the Euler equations that we will see right after. The first study of shocks, for instance, formulated by Hopf in 1950 in [111] (see also [131, 132]), begins with the Burgers’ equation. The Burgers’ equation is also sometimes used directly for the modeling of physical systems, either because it already represents a good approximation of the phenomenon, or because we are trying to model a network that couples many identical equations with each other and it is then useful to have a simple enough basis equation to successfully draw conclusions about the entire network. This is the case, for example, with the road network [103 Chapter 3] whose traffic jams are relatively well modeled by shocks [28]. Far from being an exhausted subject, these models still have a lot of room for progress and give rise to interesting open issues.

There is much to be said about this equation and one could probably devote several thesis to the work that has been done on the Burgers’ equation. We will stop here, but for a description a little more detailed one can refer to [151, 1.2.1].

1.7.2 The Euler equations

The history of Euler’s equations is directly related to the history of partial differential equations. Most of the first equation is in fact proposed by D’Alembert in 1749 in an essay for a competition of the Berlin Academy of Sciences, published in 1752 [69]. His essay is now recognized as a cornerstone of fluid mechanics and introduces several new notions [184] : the partial derivatives, the notion of internal pressure of a fluid, the field of velocities, and the first equation describing fluids as a function of several variables. On the decision of the jury, which includes Euler, the contest will be shifted to 1752 and won by Adami, a protégé of Euler. D’Alembert sees Euler as directly responsible and it will take more than ten years for them to reconcile. Euler nevertheless remains very interested in D’Alembert’s works and completes them in 1751 [77] and then 1755 [87] by adding a fundamental element : the pressure gradient. His equations are written then:

\[ \partial_t \rho + \text{div}(\rho \mathbf{V}) = 0 \]
\[ \partial_t (\rho \mathbf{V}) + \text{div}(\rho \mathbf{V} \mathbf{V}) = -\nabla p - \rho g. \]  \hspace{2cm} (1.7.2)

\[ ^{12}\text{It should be recalled however that on this date Alembert is a foreign member of the Berlin Academy and won the competition for the year 1746, in which Euler was already a member of the jury. The relations between d’Alembert and Euler are complex and are described in more detail in [183,184].} \]

\[ ^{13}\text{At the same time, in another domain and quite another context comes out the first edition of a certain Encyclopédie ou Dictionnaire raisonné des sciences, des arts et des métiers, directed by Diderot and d’Alembert, but this is another story.} \]
where \( \rho \) is the fluid density, \( \mathbf{V} \) its velocity vector, \( p \) the pressure and \( \mathbf{g} \) the acceleration of gravity, thus completing what will be the main ingredient of fluid mechanics to the present day.

These equations will in turn be supplemented later by the works of Navier in 1823 and Stokes in 1845 (with contributions from Cauchy and Poisson in 1829 and Saint-Venant in 1843) to account for the viscosity and to give the famous Navier-Stokes equations. Finally, if \( \rho \) is a constant, these equations then become the incompressible Euler equations.

The Euler equations are linked to many complicated problems. In 3D, even in the incompressible case, their well-posedness, that is, the existence and uniqueness of a solution, is still one of the most difficult mathematical problems nowadays which, according to Villani\(^\text{14}\) is even more difficult to solve than the well-posedness of the Navier-Stokes equation, yet one of the millennium problems.

In 1D their well-posedness is much better controlled. Looking at (1.7.2), we see that we need an additional equation for the pressure to describe the system. A simplification consists in considering that the pressure is a function (increasing) of the density of the fluid \( \rho \), which gives the isentropic Euler equations which we will study in the following:

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho \mathbf{V}) &= 0, \\
\partial_t \mathbf{V} + \mathbf{V} \partial_x \mathbf{V} &= -\frac{1}{\rho} \partial_x (P(\rho)) - \frac{1}{2} \mathbf{V} |\mathbf{V}| + g \sin \alpha(x),
\end{align*}
\]

where \( \alpha \) is the angle of the slope, \( P \) the pressure, and \( \theta \) is a friction term which describes the friction between the fluid and the surroundings and which appears from the boundary conditions when we integrate the 3D equation to go to the 1D equation. Finally, other simplification exists:

- Polytropic gas approximation: \( P(\rho) = C \rho^\gamma \) where \( C \) is a constant and \( \gamma > 1 \).
- Isothermal approximation: \( P(\rho) = C \rho \) where \( C \) is a constant.

The first approximation is often directly considered with the isentropic Euler equations [], to make them explicit, even if they are not rigorously related to it. The second approximation gives rise to the isothermal Euler equations. We will not use either of these two simplifications in the following.

### 1.7.3 The Saint-Venant equations

Discovered heuristically by Saint-Venant in 1871 to describe the flow in a one-dimensional channel [10], Saint-Venant’s equations model the motion of fluids when the depth is small compared to the length of the system and the system is an open channel flow. They are therefore very suitable for describing rivers in 1D, while their natural generalization in 2D is used to model the atmosphere. They are written as follows:

\[
\begin{align*}
\partial_t A + \partial_x (AV) &= 0, \\
\partial_t V + V \partial_x V + g \partial_x H - S_b(x) + S_f(A, V, x) &= 0
\end{align*}
\]

where \( A \) is the submerged section, \( V \) the velocity of the fluid, \( H \) the height of the fluid, \( S_b = -\partial_x Z \) the influence of the slope with \( Z \) the bathymetry, and \( S_f \) the influence of friction. It can be noticed that the higher the water height, the higher the immersed section, at least as long as one does not reach the top of the wall. In other words, there is a function \( F \) depending on the river profile such that \( A = F(H, x) \) and \( \partial_H F(H, x) > 0 \). Surprisingly, the equations given by Saint-Venant in 1871 are already the most general in 1D and have not been considerably supplemented thereafter, even if variants with viscosity and many other models less famous but more adapted to certain situations have been and are still developed (e.g. [29][31][153]). As we will see later, these equations were even simplified initially in the stability studies, the general equations being too complicated.

\(^{14}\)in Théorème Vivant
If Saint-Venant obtained them heuristically, these equations are consistent with the already existing equations in fluid mechanics and can in fact be deduced from the Euler equations under the hypothesis of shallow depth. A more recent variant including some viscosity was introduced in 2001 by Gerbeau and Perthame as a consequence of the Navier-Stokes equations under this same hypothesis of shallow depth.

Their relative precision and apparent simplicity have made these equations both a highly studied mathematical object, analytically and numerically, and a technical tool widely used in engineering, particularly in the regulation of waterways. This last application is far from being negligible when we know that there are nearly 700,000 km of navigable waterways in the world. Also, hydroelectricity accounts for 70% of renewable energy produced globally, and river hydroelectricity is the one with the easiest local development and the least environmental damage.

For the Saint-Venant equations, there is a famous shock phenomenon: the hydraulic jump. In order to understand what a hydraulic jump is, we should ask ourselves: what is the difference between a river and a torrent? The first answer that comes to mind is the speed of flow, and that is about right. In a torrential flow, \( V > \sqrt{gH} \) and it can be shown that both velocities of (1.7.4), given by \( V + \sqrt{gH} \) and \( V - \sqrt{gH} \) (see Chapter 4), are strictly positive. In a fluvial flow, on the other hand, \( V < \sqrt{gH} \) and we can show that the propagation velocities have opposite signs. The transition from the first regime to the second is accompanied by an entropic shock, called a hydraulic jump. These hydraulic jumps frequently appear in the rivers. They are at the origin of the tidal bores, for example in the Gironde. They also appear sometimes in the atmosphere and are at the origin of the Morning Glory Clouds, clouds that can be seen from time to time in Australia. They are also suspected of causing some glider crashes. The importance of these hydraulic jumps is not only their spontaneous appearances but also their utility in engineering where there are sometimes provoked voluntarily in hydraulic facilities to dissipate energy and protect the natural surroundings or the facilities.

1.7.4 A more general framework: the density-velocity systems

Let us take a step back and introduce a more general framework that will be useful for the future. As their name suggests, "density-velocity" systems are the systems described by the density of a certain quantity (matter, probability, etc.) and the velocity of the same quantity for which the flux is preserved. They consist of a continuity equation (or conservation of a flow) coupled with an energy equation or a moment equation, and are written in the form

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho V) &= 0, \\
\partial_t V + V \partial_x V + \partial_x (P(\rho, x)) + S(\rho, V, x) &= 0,
\end{align*}
\]  

where \( \rho \) is the density, \( V \) the speed, \( P \) is an increasing function of \( \rho \) and represents for example the influence of the forces of pressure or of forces deriving from a potential and \( S \) is a source term. This framework covers many fluid mechanics systems, in particular the isentropic Euler equations and the Saint-Venant equations described just before, but also other examples like the equations describing the movement of a fluid in a rigid pipe, which one can find in [13, Chapter 1],

\[
\begin{align*}
\partial_t \left( \exp \left( \frac{gP}{c^2} \right) \right) + \partial_x \left( V \exp \left( \frac{gP}{c^2} \right) \right) &= 0, \\
\partial_t V + V \partial_x V + \partial_x (gP) + S_f(V, x) - gC(x) &= 0
\end{align*}
\]  

where \( P \) is the piezometric head, \( V > 0 \) the velocity of the fluid, \( c \) the velocity of sound in the fluid, \( g \) the acceleration of gravity on the surface of the Earth, \( S_f \) the term of friction and \( C \) the influence of slope. Another example are the equations governing the osmosis phenomenon as modeled in [152] and which consist of the isentropic Euler equations with an additional barrier of potential acting on the solute:

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho V) &= 0, \\
\partial_t V + V \partial_x V + \partial_x (P(\rho)) + \frac{1}{2} \frac{\theta V |V| + g \sin \alpha(x) - c(x)}{\rho} \partial_x \mathcal{U} &= 0,
\end{align*}
\]  

15 A point of view defended in [161]
where \( \mathcal{H} \) is the profil of the barrier of potential, compactly supported, and \( c \) is the solute concentration.

In this part, we try to guarantee the stability in \( H^2 \) norm of these systems by output feedback laws. For this, we must solve the two problems mentioned in Section 1.4.5: the source terms and the shocks.

Before going any further we could ask ourselves the following question: why consider the \( H^2 \) norm, even though we have spent the Section 1.6 explaining that the \( C^1 \) norm is more natural mathematically? The answer is to look in the physics side. In this part we try to guarantee the stability by using physical quantities hidden in the structure of the system, like energies or entropies. But these quantities are often locally integrals of quadratic forms and therefore have the form of a \( H^p \) norm, with \( p \geq 0 \), as mentioned in Section 1.4.4. But, in general, the minimal \( H^p \) regularity for which we know how to study the stability of quasilinear systems is the norm \( H^2 \). This is the reason why this norm is considered in [13, Chapter 6].

1.7.5 A new entropy for inhomogeneous density-velocity systems.

The study of the stabilization of density-velocity systems by boundary controls is not new. For the same reasons as in the Section 1.6 we will only talk here about the results with feedback laws that only need to measure the state of the system at the boundaries. Other results using the backstepping method and feedback laws requiring knowledge of the state of the system over the entire domain are described in the introduction to Chapter 4. For feedbacks that only require knowledge of the system at the boundaries, the three approaches mentioned in Section 1.4.4 have been used in the past with a peculiarity: the Lyapunov functions used are often entropies or physical energies for the system, such as the mechanical energy.

For the Saint-Venant equations, the first results on the nonlinear systems was in [52], in 1999, where the authors study homogeneous nonlinear equations using a Lyapunov function. This Lyapunov function is not strict and requires the use of LaSalle invariance principle to conclude the asymptotic stability. Later, in [53] is introduced a strict Lyapunov function, working with all homogeneous systems that take the form of Riemann invariants, and thus covering all density-velocity systems in the absence of a source term. Later again, in [83, 169] the authors show, using the method of the characteristics, that one can stabilize the Saint-Venant equations with a source term when the source term is sufficiently small in \( C^1 \) norm, with a bound that depends on the length \( L \) of the domain. This bound can also be seen as a bound on the length \( L \) for a fixed source term. In [47] Section 13.4] and [50] the authors show the same type of result, but using a Lyapunov function. Then, in [1], the authors study the inhomogeneous linear equations with an arbitrary large source term, and show that the Lyapunov function used in [53] can also be used to stabilize this system but in the very particular case where the steady state is uniform. Finally, recently in [10] the authors manage to stabilize inhomogeneous nonlinear equations for the norm \( H^2 \) without any bound on the source term or the length of the domain, but in the absence of slope, that is to say in the case where the source term is dissipative and does not involve anything else than friction. This last result is, to my knowledge, the most accomplished in this matter before the works presented in this part. Unfortunately, in practice, the most frequent steady states are precisely the steady states where the influence of the slope is stronger than the influence of the friction, the others cease to exist after a finite distance, sometimes short, [110], Chapter 5]. This study, however, provides a general framework and a general condition for the stabilization of density-velocity systems in the \( H^2 \) norm.

Concerning the isentropic Euler equations, several results have been obtained by adding the isothermal hypothesis presented before: the authors of [52] show that it is possible to stabilize the system by output feedbacks for the \( H^1 \) norm using a Lyapunov function. Note that the stability in \( H^1 \) norm, potentially better than the stability in the \( H^2 \) norm, is then made possible by the isothermal hypothesis. Stability with uncertainty about the conditions at the boundaries and without source term has been studied in [101]. Still with the same hypothesis, the case of a network has been studied in [101]. In [85] the authors both exponentially stabilize the isothermal Euler equations for the \( H^2 \) norm and give an estimate of the decay rate, but with a limit on the length of the domain. Finally in [16], without using the isothermal hypothesis but only the assumption, less restrictive, of polytropic gas, the authors manage to show the stability if the nonlinear equations for the \( H^2 \) norm without any bound on the size of the domain, but still in the absence
of slope.

These studies therefore deal each time with particular cases, either because the length of the domain or the source terms must respect a certain limit, or because that they have a particular form. So far no result exists in the general case, which presents at least three important differences: the absence of bound on the source term and the length of the domain, the presence of a slope which can make the source term non-dissipative, that is to say that provides energy to the system rather than dissipates energy, and the presence of a variable section.

Before going any further, one can notice that for a regular steady state \((\rho^*, V^*)\) of (1.7.5), three cases are possible:

- **Torrential regime**: \(\partial_H P(\rho^*(x), x) \rho < V'^2(x)\) for all \(x \in [0, L]\) and the propagation speeds of the system are positive. This case is treated by Theorem 1.6.5 of part I.

- **Critical regime**: \(\partial_H P(\rho^*(x), x) \rho = V'^2(x)\) for all \(x \in [0, L]\), from (1.7.5) this case is very special and does not always exist. Moreover, one of the propagation velocities vanishes.

- **Fluvial regime**: \(\partial_H P(\rho^*(x), x) \rho > V'^2(x)\) for all \(x \in [0, L]\) and the propagation speeds have opposite signs. These are the complicated cases and they are the ones we will look at.

From now on, we will assume that we are in fluvial regime. As mentioned in the Section 1.6, an inhomogeneous system stabilized by output feedback has *a priori* an intrinsic length \(L_{\text{max}}\) beyond which we fail to obtain stability regardless of the choice of our boundary control. And this even in \(H^2\) norm. In Chapter 4 we show that in fact this phenomenon of maximum length \(L_{\text{max}}\) never occurs for the Saint-Venant equations with rectangular section. This result is true even when the source term is not dissipative, that is to say that it can provide energy to the system. This last case occurs when the influence of the slope is stronger than the influence of the friction, which corresponds to a large part of the practical cases. More precisely, we show the following result:

**Theorem 1.7.1.** Consider the nonlinear Saint-Venant equations (1.7.4) with \(A = H\) and \(S(A, V, \cdot) = kV^2/H - gC\), où \(C \in C^2([0, L])\), with a boundary control of the form

\[
V(t, 0) = B(H(t, 0)), \quad V(t, L) = B(H(t, L)).
\]  

(1.7.8)

If the following conditions are satisfied

\[
B'(H^*(0)) \in \left( -\frac{g}{V'^*(0)}, \frac{V'^*(0)}{H'^*(0)} \right),
\]

\[
\text{et} \quad B'(H^*(L)) \in \mathbb{R} \setminus \left[ -\frac{g}{V'^*(L)}, \frac{V'^*(L)}{H'^*(L)} \right],
\]  

(1.7.9)

then the system is exponentially stable for the \(H^2\) norm.

This result means that the Saint-Venant equations can always be stabilized using output feedback, regardless of source term and domain length. In fact, this result is stronger than that: very surprisingly, the control law do not depend directly on either the friction or the slope\(^{16}\). We even show in Chapter 5 that this result can be generalized to the general Saint-Venant equations and that the control law do not depend either on the profile of the section or the model used for friction. So we can stabilize the system by knowing only the height and the velocity at the boundaries and those of the target steady state, ignoring all the rest including a part of the model as well as all the physical data of the river, which are sometimes difficult to observe (bathymetry, coefficients of friction, profile of the section along the river).

\(^{16}\)Of course they depend on it indirectly through the steady states. But when we want to stabilize a water level and a water velocity, the minimum is to know how high and how fast we want to stabilize them.
A local dissipative entropy This result comes from the existence of a local dissipative entropy for the inhomogeneous Saint-Venant equations. For the homogeneous Saint-Venant equations, the natural entropy is well known: it is the mechanical energy \( gH^2/2 + HV^2/2 \), widely used. For the inhomogeneous Saint-Venant equations, on the other hand, this mechanical energy does not correspond to a dissipative entropy, even if the inhomogeneous term contains only friction and therefore would be purely dissipative [16, Remark 1]. In Chapter 4, we show that we can still locally find a dissipative entropy around any stationary state \((H^*, V^*)\), and this, whatever the source term. This entropy is written

\[
\mathcal{E} = \exp \left( \int_0^x \frac{-3gC}{gH^* - V^2} ds \right) \frac{H^{*2}/2}{gH^* - V^2} 
\times (g(H - H^*)^2 + 2V^*(H - H^*)(V - V^*) + H^*(V - V^*)^2).
\] (1.7.10)

It corresponds to a limit case as the entropy production inequality (1.4.28) introduced in Section 1.4.4, and which gives the dissipative behavior, becomes in fact an equality for an entire class of perturbations: if \( HV = H^*V^* \),

\[
\partial_{(H,V)} \mathcal{E}(H,V,x).P(H,V,x) = 0,
\] (1.7.11)

where \( P(H,V,x) = \left( \begin{array}{c} 0 \\ S(H,V,x) \end{array} \right) \) is the source term of the Saint-Venant equations.

We have seen in Section 1.4.4 the link between dissipative entropy and Lyapunov functions. To understand where this entropy comes from, we can look at the Lyapunov point of view. According to the result of Bastin and Coron recalled in Theorem 1.6.1, there exists a basic Lyapunov function for the norm \( H^2 \) if and only if \( \eta^* \)

\[
\eta' = \left| \frac{a}{\Lambda_1} + \frac{b}{\Lambda_2} \eta^* \right|^2,
\] (1.7.12)

admits a solution \( \eta \) on \([0, L]\) such that \( \eta(0) > 0 \). Here \( \Lambda_1 > 0, \Lambda_2 < 0 \) are the propagation speeds and \( a \) and \( b \) are the quantities of the system defined in (1.6.24) in the case of the Saint-Venant equations. An explicit expression is given in Chapter 4 (see (4.3.17), (4.3.20), (4.3.21), (4.3.23), (4.3.26)). In general, finding explicit solutions to (1.7.12) is unsuccessful when the functions \( a/\Lambda_1 \) and \( b/\Lambda_2 \) are not trivial. Therefore, a method consists instead in finding an explicit function \( \eta \) defined on \([0, L]\), such that \( \eta(0) > 0 \) and

\[
\eta' > \left| \frac{a}{\Lambda_1} + \frac{b}{\Lambda_2} \eta^* \right|^2,
\] (1.7.13)

and trying to get as close as possible from the equality case. This is what is used for instance in [10] for the Saint-Venant equations. We show that the equations of Saint-Venant have in fact the following remarkable property:

**Lemma 1.7.1.** The function

\[
\eta = \frac{|\Lambda_2|}{\Lambda_1} \varphi \quad (1.7.14)
\]

is a solution of equation (1.7.12).

The function \( \eta \) that we find here is thus optimal in the sense that we are able to reach the equality case. On the other hand, the higher the value of \( \eta(0) \), the less restrictive the control condition in 0 is (more details in Chapter 4, especially (4.3.40) and (4.3.43), or [11]). It is therefore legitimate to ask whether there is another function \( \eta_2 \) even better, which would also be a solution of (1.7.12) for any amplitude of the source term and any length \( L > 0 \) such that \( \eta_2(0) > |\Lambda_2|\phi(0)/\Lambda_1(0) \). The answer, negative, is given by the following proposition

**Proposition 1.7.2.** Let a steady state \((H^*, V^*)\) such that \( S(H^*, V^*, x) < 0 \). For any \( \eta_{2,0} > |\Lambda_2(0)|/\Lambda_1(0) \), there exists \( L > 0 \) such that \((H^*, V^*) \in C^1([0, L]; \mathbb{R}^2) \) and the solution \( \eta_2 \) of (1.7.12) with \( \eta_2(0) = \eta_{2,0} \) satisfies

\[
\lim_{x \to -L} \eta_2(x) = +\infty. \quad (1.7.15)
\]

\[\text{This form is slightly different from the one given in Section 1.6.2, but perfectly equivalent as shown in [10], and in Chapter 3, and has the advantage of relaxing the initial condition.}\]
This function $|\Lambda_2|\varphi/\Lambda_1$ and the Lemma \ref{lem:stabilization1} are the keystone of Theorem \ref{thm:stabilization1}.

In Chapter \ref{sec:stabilization1} we show that this local dissipative entropy and this result are in fact generalizable not only to the general Saint-Venant equations but also to all density-velocity systems, of which we gave a few examples above. More precisely if we consider the system
\[ \partial_t \varrho + \partial_x (\varrho V) = 0, \]
\[ \partial_t V + V \partial_x V + \partial_x (P(\varrho, x)) + S(\varrho, V, x) = 0, \]
introduced in \ref{sec:examples1} où $S \in C^2([0, +\infty[^3 ; \mathbb{R})$ et $P \in C^2([0, +\infty[^2 ; \mathbb{R})$ with the boundary control
\[ V(t, 0) = B(A(t, 0)), \]
\[ V(t, L) = B(A(t, L)), \]
we show:

**Theorem 1.7.3.** Let $(\varrho^*, V^*)$ be a steady state of the system \ref{eq:system1}--\ref{eq:system2} such that
\[ \frac{\partial_t S(\varrho^*, V^*(x), x)}{V^*(x)} - \frac{\partial_\varrho S(\varrho^*, V^*(x), x)}{\partial_\varrho (\varrho^*, x)} \geq 0, \quad \forall \ x \in [0, L]. \] (1.7.18)

If the following conditions are satisfied
\[ B'(\varrho^*(0)) \in \left\{ \frac{\partial_\varrho P(\varrho^*(0), 0)}{V^*(0)}, -\frac{V^*(0)}{\varrho^*(0)} \right\}, \]
\[ B'(\varrho^*(L)) \in \mathbb{R} \setminus \left\{ \frac{\partial_\varrho P(\varrho^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{\varrho^*(L)} \right\}, \] (1.7.19)
then the steady-state $(\varrho^*, V^*)$ is exponentially stable for the $H^2$ norm.

**Remark 1.7.1.** Condition \ref{eq:condition1} which appears in this theorem is actually a very physical condition. Note that it is satisfied for all source terms from outside forces that do not depend on $A$ or $V$. For the Saint-Venant equations it is obviously satisfied since the slope does not depend on either $A$ nor $V$ while the friction increases with the velocity and decreases (in a broad sense) with the section, therefore $\partial_\varrho S(\varrho^*, V^*, \cdot) \leq 0$ and $\partial_A S(\varrho^*, H^*, \cdot) \geq 0$. It is also satisfied for all the examples given in Section \ref{sec:examples1}.

### 1.7.6 Stabilization of shock steady states

As mentioned in the introduction, the nonlinear hyperbolic partial differential equations have the particularity that they can make naturally appear discontinuities. These discontinuities, called shocks, lead to technical difficulties in the study of these equations, so that the great majority of stability studies focus on regular steady state and solutions, as we have done so far. Nevertheless the shocks correspond to concrete physical phenomena, such as the hydraulic jump for the equations of Saint-Venant, and their study is far from being devoid of interest.

In the past several works have considered this problem. In \cite{225, 150}, for example, are considered regular solutions and regular steady states but whose profiles are close to a shock. Other works have addressed systems that may have discontinuous solutions. In \cite{33}, for example, the controllability of a hyperbolic homogeneous system is studied for BV class solutions, which is usually the most general class of functions with shocks for hyperbolic systems.\footnote{Surprisingly this article shows with an example that exact controllability can not be obtained in general, whereas this phenomenon does not occur when the solutions are regular.} The stabilization of a scalar equation is treated in \cite{221, 162} while the stabilization of a homogeneous hyperbolic system is studied in \cite{33, 57}. But in all these articles the steady state to stabilize is regular (and even constant), ideally one would like to be able to stabilize a steady state presenting a shock, such as for example a hydraulic jump. This is the purpose of Chapters \ref{sec:stabilization2} and \ref{sec:stabilization3} which deal respectively with scalar conservation equations and the Saint-Venant equations. More precisely in Chapter

\[34\]
we consider the following system consisting of the Burgers’ equation on a bounded domain \([0, L]\) with two controls at the boundaries \(u_0(t)\) and \(u_L(t)\):

\[
\partial_t y + \partial_x \left( \frac{y^2}{2} \right) = 0
\]

\[
y(t, 0^+) = u_0(t) \tag{1.7.20}
\]

\[
y(t, L^-) = u_L(t).
\]

We are interested in states \(y\) which have a shock localized in \(x_s\in[0, L]\) and are regular elsewhere. The dynamics of this shock can be obtained from (1.7.20) and Rankine-Hugoniot conditions which impose

\[
\dot{x}_s(t) = \frac{y(t, x_s(t)^+ + y(t, x_s(t)^-)}{2} \tag{1.7.21}
\]

We want to stabilize the following steady-states:

\[
y^*(x) = \begin{cases} 
1, & x \in [0, x_0], \\
-1, & x \in [x_0, L],
\end{cases}
\]

\[
x^*_s = x_0.
\]

where \(x_0 \in [0, L]\). One can easily check that this steady-state is entropic, and in fact any stationary entropic state of (1.7.20) with a single shock can be reduced to it without loss of generality, using a scaling. It is clear that if we did not try to stabilize the shock location in \(x_0\) but only the system state \(y\), we could choose constant controls \(u_0(t) = -u_L(t) = 1\) and be in open loop, that is to say with a control without feedback. Our goal, and this is the main difficulty, is to completely stabilize the steady state: that is to say both the amplitude \(y\) before and after the shock but also the shock location \(x_s\). We will therefore choose a feedback of the form:

\[
u_0(t) = 1 + k_1(y(t, x_s(t)^-) - 1) + b_1(x_0 - x_s(t)),
\]

\[
u_L(t) = -1 + k_2(y(t, x_s(t)^+) + 1) + b_2(x_0 - x_s(t)). \tag{1.7.23}
\]

These feedbacks are probably the simplest one can take in the sense that they require the minimum information possible: Let us assume that the state of the system is close to the steady state, then according to (1.7.20) the propagation speed is positive before the shock and negative after the shock, so no information passes from one side of the shock to the other. This means that it is necessary to measure the state of the system at least at one point on each side, and this is what is used by the first feedback term of (1.7.23). On the other hand, since we want to stabilize the location of the shock, the minimum is to know this location \(x_s\), and that is what is used by the second feedback term of (1.7.23).

We show first that this system is well-posed:

\textbf{Théorème 1.7.4.} For any \(T < 0\), there exists \(\delta(T) > 0\) such that, for all \(x_0 \in [0, L]\) and \(y_0 \in H^2([0, x_0]; \mathbb{R}) \cap H^2([x_0, L]; \mathbb{R})\) satisfying the first-order compatibility conditions associated to (1.7.23) and

\[
|y_0 - 1|_{H^2([0, x_0]; \mathbb{R})} + |y_0 + 1|_{H^2([x_0, L]; \mathbb{R})} \leq \delta(T),
\]

\[
|x_0 - x_0| \leq \delta(T), \tag{1.7.24}
\]

the system (1.7.20), (1.7.21), \(y(0, \cdot) = y_0, x_s(0) = x_0\), (1.7.23) has a unique entropic and piecewise \(C^1\) solution \(y \in C^2([0, T]; H^2([0, x_0(t)]; \mathbb{R}) \cap H^2([x_0(t), L]; \mathbb{R}))\) with \(x_s \in C^1([0, T]; [0, L])\) its only shock. Besides, there exists \(C(T)\) such that the following estimate is satisfied for all \(t \in [0, T]\)

\[
|y(t, \cdot) - 1|_{H^2([0, x_s(t)]; \mathbb{R})} + |y(t, \cdot) + 1|_{H^2([x_s(t), L]; \mathbb{R})} + |x_s(t) - x_0| \\
\leq C(T) \left( |y_0 - 1|_{H^2([0, x_0]; \mathbb{R})} + |y_0 + 1|_{H^2([x_0, L]; \mathbb{R})} + |x_0 - x_0| \right). \tag{1.7.25}
\]

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In fact we can even prove in a very similar way to what is done in [132, Proof of Theorem 3.4] that this solution is the only entropic solution for this system and these initial conditions. In summary, this result tells us that if the state starts with a single shock then it keeps a single shock during its evolution. More details on the definitions, in particular on the compatibility conditions and the definition of an entropic piecewise $C^1$ solution, are to be found in Chapter [6].

Then, since the solution is not regular and we try to stabilize also the shock location, we must slightly change the definition of the exponential stability as follows :

**Définition 1.7.1.** The steady-state $(y^*, x_0) \in H^2([0, x_0]; \mathbb{R}) \cap H^2([x_0, L]; \mathbb{R}) \times [0, L]$ of the system (1.7.20), (1.7.21), $y(0, \cdot) = y_0$, $x_0(0) = x_0_0$, (1.7.23) is exponentially stable for the $H^2$ norm with a decay rate $\gamma$, if there exists $\delta > 0$ and $C > 0$ such that if $y_0 \in H^2([0, x_0]; \mathbb{R}) \cap H^2([x_0, L]; \mathbb{R})$ and $x_0_0 \in [0, L]$ are compatible with (1.7.23) and satisfy

\[
\begin{align*}
|y_0 - y_1^*(0, \cdot)|_{H^2([0, x_0]; \mathbb{R})} + |y_0 - y_2^*(0, \cdot)|_{H^2([x_0, L]; \mathbb{R})} &\leq \delta^* , \\
|x_0 - x_0_0| &\leq \delta^*
\end{align*}
\]

the system has a unique solution $(y, x_0) \in C^0([0, +\infty[, H^2([0, x_0(t)]; \mathbb{R}) \cap H^2([x_0(t), L]; \mathbb{R})) \times C^1([0, +\infty[, \mathbb{R})$ and

\[
\begin{align*}
|y(t, \cdot) - y_1^*(t, \cdot)|_{H^2([0, x_0(t)]; \mathbb{R})} + |y(t, \cdot) - y_2^*(t, \cdot)|_{H^2([x_0(t), L]; \mathbb{R})} + |x_0(t) - x_0_0| &\leq Ce^{-\gamma t} \left(|y_0 - y_1^*(0, \cdot)|_{H^2([0, x_0]; \mathbb{R})} + |y_0 - y_2^*(0, \cdot)|_{H^2([x_0, L]; \mathbb{R})} + |x_0 - x_0_0|\right), \quad \forall t \in [0, +\infty].
\end{align*}
\]

We show then the following result of stability :

**Théorème 1.7.5.** Let $\gamma > 0$. If the following conditions are satisfied :

\[
\begin{align*}
b_1 &\in \left[\gamma e^{-\gamma x_0}, \frac{\gamma e^{-\gamma x_0}}{1 - e^{-\gamma x_0}}\right], \quad b_2 \in \left[\gamma e^{-\gamma(L-x_0)}, \frac{\gamma e^{-\gamma(L-x_0)}}{1 - e^{-\gamma(L-x_0)}}\right],
\end{align*}
\]

\[
\begin{align*}
k_1^2 &< e^{-\gamma x_0} \left(1 - \frac{b_1}{\gamma} \left(\frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0} + b_2} - \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right),
\end{align*}
\]

\[
\begin{align*}
k_2^2 &< e^{-\gamma(L-x_0)} \left(1 - \frac{b_2}{\gamma} \left(\frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0} + b_2} - \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}}\right)\right),
\end{align*}
\]

then the steady state $(y^*, x_0)$ of the system (1.7.20), (1.7.21), $y(0, \cdot) = y_0$, $x_0(0) = x_0_0$, (1.7.23) is exponentially stable for the $H^2$ norm with decay rate $\gamma/4$.

**Remarque 1.7.2.** We also show that for all $\gamma > 0$ there are indeed parameters $b_1$, $b_2$, $k_1$ and $k_2$ satisfying (1.7.28) . This result is therefore a stabilization result called “rapid”, since the decay rate $\gamma$ can be chosen as large as desired.

To show this result, the difficulty is twofold :

- One the one hand the solutions are not regular.
- On the other hand we have a direct control on the amplitude of the solution on both sides of the shock via the boundary conditions (1.7.20), but we have no direct control on the location of shock whose dynamics is given by Rankine-Hugoniot conditions (1.7.21).

We overcome the first difficulty by using Theorem [1.7.3] which says that if the solution starts with a single shock it remains with a single shock. We can thus divide the problem in two : before and after the shock. This gives two regular solutions $z_1$ and $z_2$ defined on a variable domain but one can straighten the domain
with a change of coordinates to obtain a fixed domain. This is equivalent to setting the following variable \( z \) (see [6.3.1] - [6.3.3] for more details).

\[
\begin{align*}
    z(t, x) &= \left( z_1(t, x), z_2(t, x) \right) = \left( y(t, x \frac{x(t)}{x_0} - 1), y(t, \frac{x_0 - x}{x_0} + 1) \right), \quad x \in [0, x_0],
\end{align*}
\]  

(1.7.29)

where \( z_1 \) represents the state of the system before the shock and \( z_2 \) represents the state of the system after the shock.

The second difficulty means in essence that one can only succeed by using the coupling between the state of the system and the shock location given by \([1.7.21]\). For this we define a new form of Lyapunov function \( V(z, s) \) (see \([6.3.1] - [6.3.3]\) for more details).

We make an interesting remark on this feedback law: if we take \( k_1 = k_2 = b_1 = b_2 = 0 \), then, according to \([1.7.23]\), \( u_0(t) \equiv 1 \) and \( u_L(t) \equiv -1 \) and the system is not exponentially stable because one can not stabilize the location of the shock. Now let us look at the conditions \([1.7.20] - [1.7.21]\), with \( z \) defined just before by \([1.7.29]\), by:

\[
V(z, s) = V_1(z) + V_2(z, s) + V_3(z, s) + V_4(z, s) + V_5(z, s) + V_6(z, s)
\]

(1.7.30)

with

\[
\begin{align*}
    V_1(z) &= \int_0^{x_0} p_1 e^{-\frac{\mu x}{x_0}} z_1^2 + p_2 e^{-\frac{\mu x}{x_0}} z_2^2 dx, \\
    V_2(z, s) &= \int_0^{x_0} p_1 e^{-\frac{\mu x}{x_0}} z_1^2 + p_2 e^{-\frac{\mu x}{x_0}} z_2^2 dx, \\
    V_3(z, s) &= \int_0^{x_0} p_1 e^{-\frac{\mu x}{x_0}} z_1^2 + p_2 e^{-\frac{\mu x}{x_0}} z_2^2 dx, \\
    V_4(z, s) &= \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{x_0}} z_1(x_0 - x) dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{x_0}} z_2(x_0 - x) dx + \kappa(x - x_0)^2, \\
    V_5(z, s) &= \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{x_0}} z_1(x_0 - x) dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{x_0}} z_2(x_0 - x) dx + \kappa(x - x_0)^2, \\
    V_6(z, s) &= \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu x}{x_0}} z_1(x_0 - x) dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu x}{x_0}} z_2(x_0 - x) dx + \kappa(x - x_0)^2.
\end{align*}
\]

(1.7.31) - (1.7.36)

where, \( \mu, p_1, p_2, \bar{p}_1, \bar{p}_2 \) are positive constants, \( \kappa > 1 \) and

\[
\eta_1 = 1, \quad \eta_2 = \frac{x_0}{L - x_0}.
\]

We go then a little further and show that Theorem \([1.7.3]\) can extend to any scalar conservation law of the form

\[
\partial_t y + \partial_x f(y) = 0
\]

(1.7.38)

with \( f \) of class \( C^3 \), convex, such that \( f(1) = f(-1) \) and satisfying:

\[
\min(f'(1), |f'(-1)|) \geq 1.
\]

(1.7.39)

**Remarque 1.7.3.** We can make an interesting remark on this feedback law: if we take \( k_1 = k_2 = b_1 = b_2 = 0 \), then, according to \([1.7.23]\), \( u_0(t) \equiv 1 \) and \( u_L(t) \equiv -1 \) and the system is not exponentially stable because one can not stabilize the location of the shock. Now let us look at the conditions \([1.7.28]\). If it seems logical that the larger \( \gamma \) is, the smaller the gains \( k_1 \) and \( k_2 \) should be, it may seem counterintuitive that \( b_1 \) and \( b_2 \) should tend to 0 when \( \gamma \) goes to \( +\infty \), since if we put \( b_1 = 0 \) and \( b_2 = 0 \), we can not stabilize the location of the shock. In other words, for all \( \gamma > 0 \) the feedback law we define works, while the limit feedback law we would get by making \( \gamma \to +\infty \) can not even ensure the asymptotic stability of the system. The explanation behind this apparent paradox is that if \( \gamma \) goes to infinity, the Lyapunov function used is no longer equivalent to the norm of the solution and can no longer guarantee its exponential decay. More details are given in Chapter \([6] \) (especially Remark \([6.2.6]\).)

\(^{19}\)We can in fact make this hypothesis without loss of generality, the two important hypotheses are the convexity and \([1.7.39]\).
It is worth mentioning that a preprint [163], released on Hal shortly after the submission of our work, also aims at stabilizing a steady state with a shock using boundary controls for the \( L^1 \) norm in the space of solutions of class \( BV \). The norm is thus more general, the stabilization on the other hand is simply exponential with a decay rate which can not be as great as desired contrary to our study. The method is very interesting and totally different.

**Stabilization of a hydraulic jump** In Chapter 7 it is shown that this method can be extended to the Saint-Venant equations and to the phenomenon of hydraulic jump, with an additional difficulty however, as we will see.

We are interested here in the homogeneous Saint-Venant equations in the particular case where the section is rectangular,

\[
\begin{align*}
\partial_t H + \partial_x Q &= 0, \\
\partial_t Q + \partial_x \left( \frac{gH^2}{2} + \frac{Q^2}{H} \right) &= 0.
\end{align*}
\] (1.7.40)

The interest of considering such a special case is to study the phenomenon of hydraulic jump while keeping the framework as simple as possible. However, given Section 1.7.5, it is likely that the result can be extended to the general Saint-Venant equations. Moreover, we consider here the formulation of the Saint-Venant equations with variables \( H \) and \( Q = HV \) and a second equation representing a balance of momentum, rather than the variables \( H \) and \( V \) and a second equation that represents a balance of energy, as we did previously with (1.6.36) and (1.7.4). In fact, when the solutions are regular these two formulations are equivalent. When solutions have discontinuities, however, this equivalence is lost and this can be seen through the steady state: the formulation (1.7.40) imposes a continuity of the flow \( Q^* \) and applied forces \( (gH^* + Q^*/2)/H^* \), which is compatible with a discontinuity in the water level for the stationary state. On the contrary, a formulation like (1.7.4) would imply a continuity of the flow \( H^*V^* \) and the energy \( (gH^* + V^*/2) \) which would directly imply the continuity of \( H^* \) and \( V^* \) and thus prevent any jump (see Chapter 7 Section 7.1, paragraph Physical remarks). This is actually quite logical: when a hydraulic jump occurs there is a local loss of energy, dissipated in the jump, which prevents the conservation of energy.

Since we are trying to study a system involving a hydraulic jump, whose location is denoted by \( x^* \), the steady states \((H^*, Q^*)\) of (1.7.40) must be in a torrential regime before the jump and in fluvial regime after the jump. It can be shown that this means that the steady states are the functions satisfying the following conditions:

1. \( Q^* \) is a positive constant, \( x^* \in [0, L] \) and

\[
H^* = \begin{cases} 
H_1^* > 0, & x \in [0, x^*], \\
H_2^* > 0, & x \in [x^*, L],
\end{cases}
\] (1.7.41)

2. Condition for having a torrential regime before the jump:

\[
\lambda_1 = \frac{Q^*}{H_1^*} - \sqrt{gH_1^*} > 0, \quad \lambda_2 = \frac{Q^*}{H_1^*} + \sqrt{gH_1^*} > 0, \quad \text{for} \ x \in [0, x^*),
\] (1.7.42)

Condition for having a fluvial regime after the jump:

\[
-\lambda_3 = \frac{Q^*}{H_2^*} - \sqrt{gH_2^*} < 0, \quad \lambda_4 = \frac{Q^*}{H_2^*} + \sqrt{gH_2^*} > 0, \quad \text{for} \ x \in [x^*, L].
\] (1.7.43)

Which implies in particular that \( H_1^* < H_2^* \).

3. Condition at the interface

\[
\frac{H_2^*}{H_1^*} = \frac{-1 + \sqrt{1 + 8 \frac{(Q^*)^2}{g(H_1^*)^2}}}{2}.
\] (1.7.44)
The last condition is called Bélanger equation \[42\] and can be deduced from (1.7.40) and Rankine-Hugoniot conditions.

Compared to the Burgers’ equation, an additional difficulty appears: For the Burgers’ equation there is only one propagation speed, and it goes from a positive sign before the shock to a negative sign after the shock. This makes it possible to have an incoming information both in \(x = 0\) and in \(x = L\) and thus to have a control on each of the two halves of the solution (cf Fig [1.3]).

For the Saint-Venant equation in contrast, there are two components and two propagation speeds. Both are discontinuous but only one sign changes. Thus, by dividing the problem into two between before the shock and after the shock, we have in total four quantities (two before, two after), and we would like to have the corresponding number of controls at the boundaries. Before the shock, the two propagation speeds are positive, so we can have two controls at the boundary in \(x = 0\). After the shock, on the other hand, one of the propagation speeds has changed sign and allows to have an incoming information in \(x = L\) and thus a boundary control, but the other has not changed sign, the corresponding information in \(x = L\) is always outgoing and we cannot have any control on it (see Fig [1.3]). So we have four quantities with only three boundary controls and we must exploit the coupling not only to stabilize the shock location but also to stabilize this fourth component.

\[
\begin{pmatrix}
H(t,0) - H^*_1 \\
Q(t,0) - Q^* \\
Q(t,L) - Q^* \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
G_4(H(t,L) - H^*_2) \\
\end{pmatrix}
= 
G
\begin{pmatrix}
Q(t,x^-) - Q^* \\
Q(t,x^+) - Q^* \\
H(t,x^-) - H^*_1 \\
x^- - x^* \\
\end{pmatrix}
\]  
(1.7.45)

where \(G = (G_1, G_2, G_3)^T : \mathbb{R}^4 \to \mathbb{R}^3\) and \(G_4 : \mathbb{R} \to \mathbb{R}\) are of class \(C^2\) and satisfy

\[G(0) = 0, \quad G_4(0) = 0, \quad G_4'(0) = -\lambda_4.\]  
(1.7.46)

Figure 1.3: Change of variables to bring the problem back to a regular system. a) Burgers’ equation. b) Saint-Venant equations, in red the component corresponding to the propagation speed which does not change sign and on which one can not apply a boundary control.

This difficulty imposes to be a little more precise in the estimates than for a scalar equation, but the method still works by choosing controls similar to the ones we previously used:
And, using the matrices $D$, $\tilde{D}$, $K$ et les $b_i$ defined by

$$
D(x, \gamma) = \text{diag} \left( \frac{s_i(1 - s_i \frac{\lambda_i}{\lambda_4})}{b_i} e^{\frac{s_i x_i}{\lambda_i}}(x_i^* - x), i \in \{1, 2, 3\} \right),
$$

$$
\tilde{D}(\gamma) = \text{diag} \left( \sum_{j=1}^{3} e^{\frac{s_j x_j}{\lambda_j}} \gamma \frac{s_j x_j}{\lambda_j}, 1 - s_j \frac{\lambda_j}{\lambda_4}, i \in \{1, 2, 3\} \right),
$$

$$
K = \begin{pmatrix}
\frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} - \frac{\lambda_3}{\lambda_3 - \lambda_2} & 0 & 0 \\
\frac{\lambda_3}{\lambda_3 - \lambda_2} - \frac{\lambda_2}{\lambda_2 - \lambda_1} & 0 & \frac{\lambda_1}{\lambda_4} \\
0 & \frac{\lambda_1}{\lambda_4} & 0
\end{pmatrix} G(0) \begin{pmatrix} 1 & 1 & 0 \\
\frac{\lambda_2}{\lambda_4} & \frac{\lambda_2}{\lambda_4} & 1 + \frac{\lambda_3}{\lambda_4} \\
\frac{1}{\lambda_4} & \frac{1}{\lambda_4} & 0
\end{pmatrix},
$$

(1.7.47)

$$
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix} = \begin{pmatrix}
\frac{\lambda_2 \lambda_1}{\lambda_2 - \lambda_1} - \frac{\lambda_3}{\lambda_3 - \lambda_2} & 0 & 0 \\
\frac{\lambda_3}{\lambda_3 - \lambda_2} - \frac{\lambda_2}{\lambda_2 - \lambda_1} & 0 & \frac{\lambda_1}{\lambda_4} \\
0 & \frac{\lambda_1}{\lambda_4} & 0
\end{pmatrix} G(0) \begin{pmatrix} 0 \\
0 \\
1
\end{pmatrix},
$$

and denoting $s_1 = s_2 = 1, s_3 = -1, x_1 = x_2 = 1, x_3 = x_3^*/(L - x_3^*)$ and $x_4 = x_4^*/(x_4^* - L)$, we obtain the following result :

**Théorème 1.7.6.** For any steady-state $((H^*, Q^*)^T, x_3^*)$ of the system $\text{(1.7.40)}$ satisfying $\text{(1.7.41)} - \text{(1.7.44)}$, for any $\gamma > 0$, if for $i = 1, 2, 3$

$$
b_i \in \begin{cases}
-\gamma e^{-\frac{s_i x_i}{\lambda_i}}(H^*_1 - H^*_2) & \text{if } s_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right) < 0, \\
-\gamma x_i e^{-\frac{s_i x_i}{\lambda_i}}(H^*_1 - H^*_2) & \text{if } s_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right) > 0,
\end{cases}
$$

(1.7.48)

and if the matrix

$$
D(x_3^*, \gamma) - K^T D(0, \gamma) K = \sum_{k=1}^{3} \frac{\lambda_k}{\gamma} (H^*_1 - H^*_2)^2 b_k s_k (1 - s_k \frac{\lambda_k}{\lambda_4}) (e^{\frac{s_i x_i}{\lambda_i}} - 1) \tilde{D}(\gamma),
$$

(1.7.49)

is positive definite, with $(b_1, b_2, b_3)^T$, $D$, $\tilde{D}$ and $K$ defined by $\text{(1.7.47)}$, then the steady state $((H^*, Q^*)^T, x_3^*)$ is exponentially stable for the $H^2$ norm with decay rate $\gamma/4$.

**Remarque 1.7.4.**
- The control function $G$ is actually very similar to the controls used for the Burgers’ equation: its linearized is written analogously to $u_0(t)$ and $u_l(t)$ defined by $\text{(1.7.23)}$, where $K$ plays the role of $k_i$ and where $b_i$ play the same role as before. We used here a nonlinear $G$ to show, as it could be expected, that the linearity of the control does not play a role in this method.
- Here again, for all $\gamma > 0$ we can find $G$ such that $K$ and $(b_1, b_2, b_3)^T$ defined by $\text{(1.7.47)}$ satisfies $\text{(1.7.48)} - \text{(1.7.49)}$. This result is therefore again a rapid stabilization result.

### 1.8 Part 3 : PI control

Backstepping aside, most stabilization studies use output feedback, thus of the form $\text{(1.4.13)}$. Since the goal is to apply these controls to practical systems, it is natural to wonder what happens when one makes small
mistakes, for example a small constant error on the control applied. What happens if, in Theorem 1.7.1, instead of applying \( B(H(t, L)) \), we apply \( B(H(t, L)) + \varepsilon \)? The answer is simple: nothing works anymore\(^{20}\). These control laws are therefore not robust to so-called "off-set" errors \([6, \text{ Chapter 11.3}]\). Fortunately there is a simple solution, known for a long time, to this problem: adding an integral term that also uses the information of previous errors and absorbs this measurement error \([155]\). The control then becomes:

\[
V(t, L) = k_p(H(t, L) - H^*(L)) - k_I Z
\]  

(1.8.1)

where \( Z \) is the integral term \( \dot{Z} = H(t, L) \). This solution has been known for a long time, its first appearance goes back to the work of the Périer brothers towards the end of the XVIIIth century who used an integral control to stabilize their "fire pump" \([72, \text{ Pages 50-51 et figure 231, Plate 26}]\, [22, \text{ Chapitre 2}]\). Then the regulator of Jenkins, which works on the same principle, was then studied by Maxwell who mentions this type of regulator in his famous article: "On governors" presented to the Royal Society of London in 1868. Obviously these regulators were not yet called "PI controls" and their theory was not yet mathematized, but they worked on the same principle. It was Minorsky at the beginning of the 20th century who began to theorize this class of controls for finite dimensional systems. At the time Minorsky studied these controls to apply them to the automatic pilots of the ships of the US Navy. Since then, motivated by the very large number of applications \([74]\), many studies have been published in the case of finite dimensional systems \([4,6]\). Then, analyzing the eigenvalues and using the Spectral Mapping Theorem described in Section 1.4.3, other works have studied linear systems of infinite dimension\(^2\), in particular \([21, 84, 130, 167, 192, 197, 198]\). Nonlinear systems of infinite dimension, and in particular non-linear hyperbolic systems, have remained hardly ever studied since they are very difficult to handle with these controls. The additional difficulty added by the infinite dimension is no longer compensated by the possible use of the Spectral Mapping Theorem and only few techniques are known. The purpose of this part is to contribute to their study.

### 1.8.1 Transport equation

To our knowledge, only one general result exists for a nonlinear system of infinite dimension: the case of a transport equation with integral control:

\[
\begin{align*}
\partial_t z + \lambda(z)\partial_x z &= 0, \\
z(0, t) &= -k_I I(t), \\
\dot{I} &= z(L, t),
\end{align*}
\]  

(1.8.2)

(1.8.3)

(1.8.4)

where \( \lambda(0) > 0 \) and \( k_I \) is the control coefficient to be designed. The transport equation \((1.8.2)\) is the simplest nonlinear hyperbolic system but is already quite rich. It is studied either as a toy model, as a preparatory step to the study of more complicated systems, or as a basis model of transport to model for example the dynamics of a population \(\text{Cite PerthameCours}^\text{, road traffic, or the transport of gas in first approximation.}^\text{\text{\footnote{41}}}

For this system, the associated linearized system is easy to study using the Spectral Mapping Theorem, and the eigenvalues \( \varrho \) satisfy the following equation:

\[
k_I + \varrho e^{\lambda_0 L} = 0.
\]  

(1.8.5)

where \( \lambda_0 := \lambda(0) \). We can show the following properties:

\footnote{Note that in this case it is sometimes possible to reach a close steady state: this is the case for instance for systems stabilized by a centrifugal governor. More information on this governor can be found in \([155, \text{ Section X}]\). Depending on the applications, the proportional controls and the output feedbacks are still interesting in themselves, in addition to being a natural preparatory step for PI controls.}

\footnote{His paper was in fact ignored for several years, because apparently considered hard to understand \([156]\).}

\footnote{The Spectral Mapping Theorem as defined in Section 1.4.3 is only valid for output feedback. Nevertheless other versions exist with a PI control by considering the integrator \( I \) as an additional variable.}
If $\varphi$ is a solution of (1.8.5), then $\bar{\varphi}$ is also a solution.

If $k_I \leq 0$, there always exists a real solution $\varphi$ nonnegative to (1.8.5).

If $k_I \in \left[0, \frac{2k\lambda_0}{L} + \frac{\pi\lambda_0}{2L}\right]$, then there exists exactly $k$ pairs of conjugate solutions $\varphi \in \mathbb{C}$ to (1.8.5) with positive real parts [190].

One can then obtain the following result: the linearized system is exponentially stable if and only if

$$k_I \in \left[0, \frac{\pi\lambda_0}{2L}\right].$$

The nonlinear system was studied by Trinh, Andrieu and Xu [192] who showed the following result using a Lyapunov approach:

**Théorème 1.8.1** ([192]). There exists $k_I > 0$ such that the system (1.8.2)–(1.8.4) is exponentially stable for the $H^2$ norm.

In reality, their proof is a little stronger and shows that the system is exponentially stable for $k_I \in \left[0, \lambda(0)\Pi(2 - \sqrt{2})/2L\right]$, where $\Pi(x) = \sqrt{x(2-x)}e^{-x/2}$ and thus $\Pi(2 - \sqrt{2})/2 \approx 0.34$. This result is a sufficient condition and therefore probably too strict. Naturally, we would like to know what would be the optimal condition on $k_I$, i.e. its maximum limit, and we would like to know if it is $\pi\lambda(0)/2L$ just like in the linear case, or not. This is the problem we are studying in Chapter 8.

Our approach starts with finding a Lyapunov function a bit different from the one used by Trinh, Andrieu and Xu, exploiting the coupling between the integrator and the state of the system, which allows us to obtain the following proposition:

**Proposition 1.8.2.** The nonlinear system (1.8.2)–(1.8.4) is exponentially stable for the $H^2$ norm if

$$k_I \in \left[0, \frac{\lambda(0)}{L}\right].$$

Unfortunately this condition is still too strict, and all our attempts to find a relatively simple Lyapunov function to go beyond have failed. This problem is part of a more general problem already mentioned briefly: Lyapunov functions are a very good tool to study the stability of systems, but unless one can find a perfect function, they often lead to too strict conditions when the systems become complicated. This is the reason why we developed a new method to find optimal conditions: the extraction method.

**The extraction method** The idea is the following: if we look at the linearized system, there are two eigenvalues which limit the stability that we denote by $\varphi_1$ and $\bar{\varphi}_1$ and which correspond to the eigenvalues whose real part is positive when $\frac{\pi\lambda_0}{2L} < k_I < \frac{2\pi\lambda_0}{L} + \frac{\pi\lambda_0}{2L}$. If these eigenvalues did not exist then, according to the properties of (1.8.5) that we have stated, the system would also be stable for larger $k_I$ and actually for $0 < k_I < 5\pi\lambda_0/2L$. So if we could extract from the solution the part corresponding to these two eigenvalues using a suitable projector, we would end up with two parts:

- A part $(\varphi_1, X_1)$ corresponding to the eigenvalues $\varphi_1$ and $\bar{\varphi}_1$, which belongs to a two dimension space and which is the limiting factor for the stability (no longer exponentially stable from $k_I = \frac{\pi\lambda_0}{2L}$ because then $\varphi_1$ has a nonnegative real part).

- A part $(\varphi_2, X_2)$ which is the “rest” and belongs to a space of infinite dimension but which is stable up to larger values of $k_I$ (at least $5\pi\lambda_0/2L$).

\[^{23}\text{We see that this method already assumes that the eigenvalues which limit the stability are continuous with the parameters of the system, which motivates the Conjecture 1.42.}\]
Figure 1.4: Illustration of the strategy of the extraction method. In red the stability domain that we can obtain directly with a Lyapunov function and in blue the total stability domain. The values are normalized so that $\lambda_0/L = 1$.

Our goal is then to succeed in proving the stability using Lyapunov functions in order to transpose the reasoning to non-linear. Let us study the two parts separately: since $(\varphi_1, X_1)$ belongs to a relatively simple finite space finding an optimal Lyapunov function that proves the stability of $(\varphi_1, X_1)$ up to $k_I = \pi \lambda_0/2L$ is relatively easy. The second part $(\varphi_2, X_2)$ belongs to a space of infinite dimension, but is stable for $k_I$ which can go much further than $\pi \lambda_0/2L$, so if we use our Lyapunov function for this part, even if it is too strict in the absolute, there is hope that the $k_I$ limit is beyond $\pi \lambda_0/2L$, and so the sum of the two functions would have the total stability limit $\pi \lambda_0/2L$. All this is done so far for the linear system, but since we use Lyapunov functions that have a simple form we can hope that it works with the nonlinear system. This is actually the case, with a main difference: for the linearized system the equations on $(\varphi_1, X_1)$ and $(\varphi_2, X_2)$ are not coupled and therefore $(\varphi_1, X_1)$ and $(\varphi_2, X_2)$ can be studied separately, while for the nonlinear system $\varphi_1$ depends of $(\varphi_2, X_2)$, and vice versa, which requires us to study them together.

The main difficulties are the following:

- Finding the projector, and proving that it is a projector. Finding the potential shape of the projector is actually not very complicated, but, to our surprise, succeeding in showing that it is a projector is more difficult. To overcome this problem, one has to use precisely the relation (1.8.5).

- As we saw above with the Proposition 1.8.2, use directly the Lyapunov function on the total solution i.e. $(\varphi_1, X_1) + (\varphi_2, X_2)$ is not satisfactory. In this method we use our Lyapunov function on $(\varphi_2, X_2)$ instead and we want to use the extra information we have on $(\varphi_2, X_2)$ to improve the condition we get.

The problem is that this information proves quite difficult to exploit. To do so, we use the projector to find a function $\theta$ such that $\int_0^L \varphi_2(t, x) \theta(x) dx = 0$, we can then add $\kappa \int_0^L \varphi_2(t, x) \theta(x) dx = 0$ to the expression of the Lyapunov function differentiated with respect to time along the trajectories and optimize $\kappa$ (see (8.5.14)–(8.5.21) for more details).

In summary, here are schematically the different steps:

1. Finding a good Lyapunov function candidate $V$.
2. Identifying the eigenvalues limiting the stability, if needed through an implicit relation.
3. Finding a projector on the space generated by the corresponding eigenvectors.
4. Finding a function $\theta$ such that $\int_0^L \varphi_2(t, x) \theta(x) dx = 0$ using the projector.
5. Finding an optimal Lyapunov function for the finite dimensional part $(\varphi_1, X_1)$.
6. Using $\theta$ to improve the time derivative of $V$ taken along the trajectories $(\varphi_2, X_2)$ to reach the limit given by the eigenvalues limiting the stability.
If the condition we get on $(\varphi_2, X_2)$ is unsatisfactory, we can iterate the extraction to remove as many eigenvalues as necessary. Obviously the last step becomes more precise as we extract eigenvalues but all the more complicated to achieve because we have more information to exploit and the eigenspaces corresponding to these eigenvalues are not necessarily orthogonal to each other.

Using this method, we then show the final result:

**Theorem 1.8.3.** The nonlinear system \[(1.8.2) - (1.8.4)\] is exponentially stable for the $H^2$ norm if

\[ k_I \in \left[ 0, \frac{\pi \lambda(0)}{2L} \right]. \] (1.8.8)

So we find the same condition as for the linearized system. Optimality is illustrated by the following proposition:

**Proposition 1.8.4.** There exists $k_1 > \frac{\pi \lambda(0)}{2L}$, such that for all $\frac{\pi \lambda(0)}{2L} < k_1 < k_I$ the nonlinear system \[(1.8.2) - (1.8.4)\] is unstable for the $H^2$ norm.

Figure 1.5: Example of numerical simulation of $z(t, 0)$ as a function of time $t$ varying between 0 and 10 for different values of $k_I$ between $0.1k_{I,c}$ and $2k_{I,c}$, where $k_{I,c} = \frac{\pi \lambda_0}{2L}$ is the critical value of Theorem 1.8.3 and Proposition 1.8.4. The black line represents the trajectory for $k_I = k_{I,c}$. On the left $k_I$ is large and the (nonlinear) system is unstable. On the right $k_I$ is smaller and the system is stable. The system parameters are chosen such that $\lambda(x) = 1 + x$, $\lambda_0 = L = 1$, and $\varphi_0(x) = 0.1$ on $[0, L/2]$ and $\varphi_0(L) = 0$ so that $\varphi_0$ is compatible for all $k_I \in [0.1k_{I,c}, 2k_{I,c}]$.

**Remark 1.8.1.** Note that this method aims to show that the nonlinear system is exponentially stable with the same limit as the linearized system. But we saw in the introduction that it is wrong for some systems. It would be interesting to know when is this the case and when it is not. This is the subject of the conjecture that will be given later in Section 1.9.

In fact, this result is a little more general than it seems: one can reduce the stabilization of any steady steady state of a hyperbolic quasilinear equation to this problem. More precisely, if we consider the system

\[
\begin{align*}
\partial_t y + \lambda(y, x) \partial_x y + g(y, x) &= 0, \\
y(t, 0) &= y^*(0) - k_I Z, \\
\dot{Z} &= (y(t, L) - y^*(L)),
\end{align*}
\] (1.8.9)
where \( y^* \) is the regular steady state we aim at stabilizing with \( \lambda(y^*,\cdot) > 0 \), then we show the following result:

**Theorem 1.8.5.** The steady state \( y^* \) of the system (1.8.9) is exponentially stable for the \( H^2 \) norm if

\[
K_I \in \left[ 0, \frac{\pi \exp \left( \int_0^L (\partial_y g(y^*,s) + y^*_s \lambda(y^*,s) / \lambda(y^*,s)) + y^* x_s \lambda(y^*,s) \lambda(y^*,s) ds \right) / 2 l^{-1}(L) \right],
\]

where \( l \) is the primitive of \( \lambda(y^*,\cdot) \) vanishing in 0.

And the proposition,

**Proposition 1.8.6.** There exists \( k_1 > k_c := \pi \exp \left( \int_0^L (\partial_y g(y^*,s) + y^*_s \lambda(y^*,s)) / \lambda(y^*,s) ds \right) / 2 l^{-1}(L) \), such that for all \( k_I \in [k_c, k_1[ \) the nonlinear system (1.8.9) is unstable for the \( H^2 \) norm.

### 1.8.2 The Saint-Venant equations

In practice, the waterways, which are modeled by the Saint-Venant equations, are regulated by PI [146] controllers, [13, Chapter 8]. To do this, the control laws are obtained either by linearizing the system [22, 144, 145, 198, 199, 143, Chapter 1] and using the Spectral Mapping Theorem described in Section 1.4.3 or by approximating the model with a fixed finite dimensional system [146]. These two approaches have a major disadvantage: nothing guarantees a priori that the “true” non-linear system will be stable if these approximate systems are stable. The purpose of this sub-part is to remedy this by treating the nonlinear case.

We consider for simplicity the Saint-Venant equations with rectangular section and defined friction model,

\[
\partial_t H + \partial_x (HV) = 0,
\]

\[
\partial_t V + \partial_x \left( \frac{V^2}{2} + gH \right) + \left( \frac{kV^2}{H} - C(x) \right) = 0,
\]

(1.8.11)

where \( k \) is once again the friction coefficient and \( C \) the influence of the slope. As in Section 1.7, our results can be generalized to the general Saint-Venant equations or the density-velocity systems. For a steady state \((H^*, V^*)\), the PI controller is given by

\[
H(t,L)V(t,L) - H(L)V(L)^* = k_p (H(t,L) - H^*(L)) - k_I Z,
\]

\[
\dot{Z} = H(t,L) - H^*(L),
\]

(1.8.12)

while the upstream flow is an unknown constant \( Q_0 \),

\[
H(0)V(0) = Q_0.
\]

(1.8.13)

We can notice three things compared to the controls we have previously used:

- On the one hand we use only one downstream control, the upstream condition is imposed.

- On the other hand the control PI given by (1.8.12) acts on the flow and not directly on the speed. This does not change the analysis, but it makes more sense from a practical point of view: most of the time the hydraulic installations actually control the flow.

- Finally, since the PI control is set up to a constant thanks to the integrator \( Z \), it is not necessary to know \( V^*(L) = Q_0/H^*(L) \), by simply to know the value of \( H^*(L) \) we want to reach. We can then re-write the control in the simpler form,

\[
H(t,L)V(t,L) = k_p (H(t,L) - H^*(L)) - k_I Z,
\]

\[
\dot{Z} = H(t,L) - H^*(L).
\]

(1.8.14)
For this system, many linear studies based on the Spectral Mapping Theorem exist. In [198, 199], for instance, the authors give sufficient conditions to ensure the stability of the linearized inhomogeneous system, whereas in [13] Sections 2.2.4.1, 3.4.4, a necessary and sufficient condition of stability can be found for the linearized homogeneous system. For the nonlinear system, some stability studies have been done, in [13] Section 2.2.4.2 the authors find a necessary and sufficient condition for the homogeneous system and a sufficient condition for the inhomogeneous system in [13] Sections 5.4.4.5.5 but in the particular case where the stationary states are constant. The most advanced result, to my knowledge, is the following, taken from [17], which deals with the case where there is no slope:

\[\text{Théorème 1.8.7 (17). If } C \equiv 0 \text{ then let a steady state } (H^*, V^*) \in H^2([0, L]; \mathbb{R}^2) \text{ of } (1.8.11)-(1.8.13), \text{ if the following conditions are satisfied}
\]

\[k_p > 1, \quad k_t > 0, \quad (1.8.15)\]

then the system is exponentially stable for the $H^2$ norm.

Unfortunately, as mentioned in the second part, the case without slope corresponds to particular steady states which exist only on generally small distances and are not the most frequent cases in practice. Our goal is to remove this hypothesis on the slope, as in the second part. In particular we show:

\[\text{Théorème 1.8.8. Let a steady-state } (H^*, V^*) \in H^2([0, L]; \mathbb{R}^2) \text{ of } (1.8.11)-(1.8.13), \text{ if the following conditions are satisfied}
\]

\[k_p > 0 \text{ and } k_t > 0,
\]

\[\text{or } k_p < -\frac{gH^*(L) - V^{*2}(L)}{V^*(L)} \text{ and } k_t < 0, \quad (1.8.16)\]

then the system is exponentially stable for the $H^2$ norm.

Note here that the conditions on $k_t$ do not depend on the system parameters. More surprisingly, if we restrict ourselves to the first condition, they do not depend either on the steady state considered. They are also less restrictive than the conditions obtained in the previous result (Theorem 1.8.7). It would be interesting to know if these conditions are optimal or not. Unfortunately, unlike the transport equation, even in the linearized case we do not know the optimal stability conditions. It can nevertheless be noted that when the system is homogeneous, the conditions (1.8.16) are necessary and sufficient [22].

In fact the result we obtain in Chapter [9] is even a little more precise than Theorem 1.8.8 because we make this study in the case, more general, where one seeks to stabilize not only steady states but also states that may vary slightly with time while keeping a fixed height $H_c$ downstream, which often corresponds to the industrial goal. This choice obeys a certain logic: in practice the flow entering upstream of the watercourse may vary slightly with time while keeping a fixed height $H$. In this study in the case, more general, where one seeks to stabilize not only steady states but also states that exist only on generally small distances and are not the most frequent cases in practice. Our goal is to remove this hypothesis on the slope, as in the second part. In particular we show:

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\]

\[k_p > 1, \quad k_t > 0, \quad (1.8.15)\]

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\]

\[k_p > 0 \text{ and } k_t > 0,
\]

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\[\partial_t H_1 + \partial_x (H_1 V_1) = 0,
\]

\[\partial_t V_1 + V_1 \partial_x V_1 + g \partial_x H_1 + \left(\frac{kV_1^2}{H_1} - C(x)\right) = 0, \quad (1.8.17)\]

with, as initial conditions, $H_1(0, \cdot) = H^*$, $V_1(0, \cdot) = V^*$, where $(H^*, V^*)$ would be the steady state if $Q_0$ was constant equal to $Q_0(0)$. To prove the existence of these target states we will define a family of intermediate functions $(H_0, V_0)$. For each $t^* \in [0, +\infty]$ fixed, we define $(H^*_{t^*}, V^*_{t^*})$ the unique solution of

\[\partial_x (HV) = 0,
\]

\[V \partial_x V + g \partial_x H + \left(\frac{kV^2}{H} - C(x)\right) = 0,
\]

\[H(L) = H_c, \quad (1.8.18)\]

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with initial condition \(H_\ast^*(0)V_\ast^*(0) = Q_0(t^*)\). For \(H_c\) fixed we can find a bound \(Q_{\text{infty}}\) such that if \(Q_0(t^*) \leq Q_\infty\), the solution exists on \([0, L]\) and the problem is well posed (see [105] or Chapter 5 for more details). This bound is not necessarily weak and corresponds to the fact that we must keep \(\sqrt{gH_\ast^2} > V_\ast^*\) on \([0, L]\) for the solution to be in the fluvial regime. The function \((H_\ast^*, V_\ast^*)\) then corresponds to what would be the steady state of the problem if \(Q_0\) was constant equal to \(Q_0(t^*)\). If \(\|Q_0\|_{L^\infty} \leq Q_\infty\), we can define \((H_0, V_0) : (t, x) \rightarrow (H_\ast^*(t, x), V_\ast^*(t, x))\) which can be seen as a family of functions indexed by \(t\) or as a function of two variables. We can show this function of two variables is as regular as \(Q_0\) ([105], Chapter 5, Th 3.1], see Chapter 9 for more details). We then show the following proposition:

**Proposition 1.8.9.** The exist constants \(C > 0, \nu > 0, \mu > 0\) and \(\delta > 0\) such that if \(\|\partial_t Q_0\|_{C^3([0, +\infty])} \leq \delta\), then there exists a unique solution \((H_1, V_1) \in C^0([0, +\infty], H^2([0, L]))\) and

\[
\|H(t, \cdot) - H_0(t, \cdot)\|_{H^2([0, L])} + \|V_1(t, \cdot) - V_0(t, \cdot)\|_{H^2([0, L])} \\
\leq C \left( \int_0^t |\partial_t Q_0(s) + \partial_t^2 Q_0(s) + \partial_{ttt} Q_0(s)| e^{\frac{\nu}{2}} ds \right) e^{-\frac{\mu}{2} t},
\]  

(1.8.19)

We now seek to stabilize the target state \((H_1, V_1)\). We quickly see that we can no longer use the PI control given by (1.8.14), simply because \(V_1(t, L)\) is no longer a constant and \((H_1, V_1)\) no longer satisfies the conditions (1.8.14). So we define the new natural control:

\[
H(t, L)V(t, L) - H_1V_1(t, L) = k_p(H(t, L) - H_c) + k_1Z, \\
\dot{Z} = H(t, L) - H_c.
\]  

(1.8.20)

We show then the expected result:

**Théorème 1.8.10.** There exists \(\delta > 0\) such that if \(\|\partial_t Q_0\|_{C^3([0, +\infty])} \leq \delta\), and if

\[
k_p > 0 \text{ and } k_1 > 0,
\]

(1.8.21)

\[k_p < -\frac{gH_1(t, L) - V_1^2(t, L)}{V_1(t, L)} \text{ and } k_1 < 0,
\]

(1.8.21)

then the target state \((H_1, V_1)\) of the system \((1.8.11), (1.8.13), (1.8.20)\) is exponentially stable\(^{24}\) for the \(H^2\) norm.

Given the control law (1.8.20), one might wonder what happens when one acts in the unknown and does not know the upstream flow \(Q_0\). In this case, it is impossible to know \(H_1\) and \(V_1\) and therefore to impose the control law (1.8.20). Since we do not know the target, it is impossible to obtain the exponential stability if only we do not know how to put a boundary condition compatible with the target. Nevertheless, one can still give a bound on the error that one commits by using the control law (1.8.14) with \(H^\ast(t, L) = H_c\), and we prove the following input-to-state stability (ISS) result:

**Théorème 1.8.11.** There exist constants \(\nu > 0, \delta > 0, \gamma > 0\) and \(C\), such that if \(\|\partial_t Q_0\|_{C^3([0, +\infty])} \leq \delta\), then for all \(T > 0\) and \((H^0, V^0) \in (H^2([0, L]))^2\) such that

\[
\|H^0 - H^\ast\|_{H^2([0, L])} + \|V^0 - V^\ast\|_{H^2([0, L])} \leq \nu,
\]

the system \((1.8.11), (1.8.13), (1.8.14)\) with initial condition \((H^0, V^0)\) has a unique solution \((H, V) \in C^0([0, T], H^2([0, L]))\) which satisfies the following ISS inequality

\[
\|H(t, \cdot) - H_0(t, \cdot)\|_{H^2([0, L])} + \|V(t, \cdot) - V_0(t, \cdot)\|_{H^2([0, L])} \\
\leq C e^{-\gamma t} \left( \|H^0 - H^\ast, V^0 - V^\ast\|_{H^2([0, L])} + \int_0^t |\partial_t Q_0(s) + \partial_t^2 Q_0(s) + \partial_{ttt} Q_0(s)| e^{\gamma s} ds \right).
\]  

(1.8.22)

\(^{24}\)When we consider a time-dependent target trajectory, the exponential stability definition must be slightly changed to ensure that it does not depend on the initial time. For a rigorous definition, see the Definition 10.2.1.47
1.9 Interesting questions about 1D hyperbolic systems

Before to complete this introduction, we can return to a question to which we are still far from giving a perfect answer:

If the linearized system is exponentially stable, is the nonlinear system exponentially stable locally? That is, if the initial condition is arbitrarily close to the stationary state to be stabilized, is the knowledge of the stability of the associated linearized system sufficient? The answer is no, we saw it in Section 1.4.3. The more in-depth question would then be to ask: under what condition does the exponential stability of the linearized system guarantee the stability of the nonlinear system? If we do not have the answer here, we can try to formulate the following conjecture.

Conjecture 1. Let a hyperbolic system be of the form (1.4.2), with boundary conditions (1.4.13). We define \( \mathcal{L} \) the operator of the associated linearized system. If

- the associated linearized system is exponentially stable,
- all the eigenvalues of \( \mathcal{L} \) are continuous with \( A(0, \cdot) \) and \( B(0, \cdot) \),

then the system is exponentially stable for the \( C^1 \) and \( H^2 \) norms.

In this conjecture, the additional critical condition is therefore the continuity of the eigenvalues of the linearized system with respect to the system parameters.

1.10 Perspectives

These results and these works bring several perspectives:

- In the first part we use basic Lyapunov functions to find an inner condition that guarantees that a hyperbolic system can be stabilized for the \( C^1 \) norm by output feedbacks. This condition is more restrictive than the one found for the \( H^2 \) norm by the same method. It would be interesting to extend the method to the \( W^{2,p} \) norm to find out how the internal conditions would behave. Would they be more restrictive than for the \( H^2 \) norm and less restrictive than for the \( C^1 \) norm? If not, what would be the most and the least restrictive in terms of \( p \)? Will the links we obtain in the case \( 2 \times 2 \) still be valid with these norms?

- In the second part it is shown that one can always stabilize the steady states of density-velocity systems by boundary controls whatever the source term. It would be interesting to know if we can use the same approach to stabilize systems with more equations, for example the Euler equations without the isentropic hypothesis.

- This work was done in 1D in space. Is it possible to extend the results to a higher dimension? This is probably one of the main perspectives. Far from being a mere technical generalization, it is to be expected that new and very different behaviors may appear. This very interesting question is all the more relevant as the stability analysis using the Spectral Mapping Theorem is no longer valid even for linear systems for a space dimension \( n \geq 2 \).

- When stabilizing a steady state with a shock, we consider discontinuous solutions that have only one shock. Would it be possible to consider a larger space of solution as the space of the entropic solutions of class \( BV \)? As long as there is only one shock that changes one of the propagation speeds, the same method can probably be used to deal with a solution that has a finite number of shocks. The case of solutions of class \( BV \) is undoubtedly much more ambitious.
• Is the exponential stabilization of hydraulic jumps always possible when there is a source term? Given the part [1], it is likely that the method still works by changing the Lyapunov function to account for the source terms. Nevertheless, it is also probable that the result does not allow rapid stabilization. On the contrary, it is also likely that the problem of stabilizing the shock location is simplified when the steady states are no longer uniform.

• In the third part, we stabilize the Saint-Venant equations with a PI control and we obtain conditions that are optimal when there is no source term or when the steady states stabilized are constant, but we do not know anything else. Are they optimal in general? If not, can we make them optimal? The answer to this question would be very interesting because it would close the series of studies of stabilization of the Saint-Venant equations with a PI control.

• In practice, many finite dimensional stabilization systems use a PID regulator that is derived from the PI by adding a differentiation term. Very surprisingly, this regulator does not make it possible to obtain stability in the case of an infinite dimension equation [64]. Is it possible, by applying a filtering on the derived term, to regain the exponential stability and to have a faster return to equilibrium than with a PI control?

• Finally, it would be interesting to try to use the extraction method on systems $2 \times 2$ or even $n \times n$. And, if it has been shown that this method works well with a PI control, is it possible to successfully extend it to other types of control?
Part I

Stability of quasilinear inhomogeneous systems for the $C^1$-norm.
Chapter 2

Exponential stability of general 1-D quasilinear systems with source terms for the $C^1$ norm under boundary conditions

This chapter is taken from the following article (also referred to as [106]):


Abstract. We address the question of the exponential stability for the $C^1$ norm of general 1-D quasilinear systems with source terms under boundary conditions. To reach this aim, we introduce the notion of basic $C^1$ Lyapunov functions, a generic kind of exponentially decreasing function whose existence ensures the exponential stability of the system for the $C^1$ norm. We show that the existence of a basic $C^1$ Lyapunov function is subject to two conditions: an interior condition, intrinsic to the system, and a condition on the boundary controls. We give explicit sufficient interior and boundary conditions such that the system is exponentially stable for the $C^1$ norm and we show that the interior condition is also necessary to the existence of a basic $C^1$ Lyapunov function. Finally, we show that the results conducted in this chapter are also true under the same conditions for the exponential stability in the $C^p$ norm, for any $p \geq 1$. 


2.1 Introduction

Hyperbolic systems have been studied for several centuries, as their importance in representing physical phenomena is undeniable. From gas dynamics to population evolution through wave equations and fluid dynamics they are found in many areas. As they represent the propagation phenomena of numerous physical or industrial systems \[1\] [94] [135], the issue of their controllability and stability is a major concern, with both theoretical and practical interest. If the question of controllability has been well-studied [138], the problem of stabilization under boundary control, however, is only well known in the particular case of an absence of source term. However, in many case neglecting the source term is a crude approximation and reduces greatly the analysis, in particular because it implies that the system can be reduced to decoupled equations or slightly coupled equations (see [71] for instance). For most physical equations the source term cannot therefore be neglected and the steady-states we aim at stabilizing can be non-uniform with potentially large variations of amplitude (e.g. Saint-Venant equations, see [42] Chapter 5 or [110], Euler equations, see [81] or [100], Telegrapher equations, etc.). Taking into account these nonuniform steady-states and stabilizing them is impossible if not taking the source term into account, although it is an important issue in many applications. In presence of a source term some results exist for the \(H^2\) norm (and actually \(H^p, p \geq 2\)), however, few results exist for the more natural \(C^1\) norm (and consequent \(C^p\) norms, \(p \geq 1\)). It has to be underlined that for nonlinear systems the stability in these two main topologies are not equivalent as shown in [61]. In this article we deal with the stability in \(C^1\) norm of such hyperbolic systems of quasilinear partial differential equations with source term under boundary conditions.

Several methods are usually used to study the stability of systems. The Lyapunov approach, one of the most famous, is the one we opted for in this article. This approach has the advantage, among others, of guaranteeing some robustness and of being convenient to deal with non-linear problems \[47\] [119]. We first introduce the basic \(C^1\) Lyapunov functions, a kind of natural Lyapunov functions for the \(C^1\) norm and we then find a sufficient condition such that the system admits a basic \(C^1\) Lyapunov function. We show that this sufficient condition is twofold: a first intrinsic condition on the system and a second condition on the boundary controls. We show then that this sufficient condition on the system is in fact necessary in the general case for the existence of a basic \(C^1\) Lyapunov function.

The organisation of this chapter is as follows: In Section 1, we recall some preliminary properties about 1-D quasilinear hyperbolic system. Section 2 presents an overview of the context and previous results. Section 3 states the main results, which are proven in Section 4. Section 5 presents several remarks and further detail to the results.

2.2 Preliminary properties of 1-D quasilinear hyperbolic systems

A general quasilinear hyperbolic system can be written as:

\[
\begin{align*}
\mathbf{Y}_t + F(\mathbf{Y})\mathbf{Y}_x + D(\mathbf{Y}) &= 0, \quad (2.2.1) \\
\mathcal{B}(\mathbf{Y}(t,0), \mathbf{Y}(t,L)) &= 0, \quad (2.2.2)
\end{align*}
\]

with \(\mathbf{Y} : [0, +\infty) \times [0, L] \rightarrow \mathbb{R}^n\) and \(F : U \rightarrow \mathcal{M}_n(\mathbb{R})\) and \(D : U \rightarrow \mathbb{R}^n\) where \(U\) is a non empty connected open set of \(\mathbb{R}^n\) and \(F\) is strictly hyperbolic, i.e. for all \(\mathbf{Y} \in U, F(\mathbf{Y})\) has real, distinct eigenvalues. We suppose in addition that these eigenvalues are non-vanishing. \(\mathcal{B}\) is a map from \(U \times U\) to \(\mathbb{R}\) whose form will be precised later on, such that the system (2.2.1)–(2.2.2) is well-posed.

We call \(\mathbf{Y}^*\) a steady-state of the previous system that we aim at stabilizing. Note that, due to the source term, \(\mathbf{Y}^*\) is not necessarily uniform and the problem cannot be directly treated as a null stabilization. We therefore use the following transformation:

\[
u(x, t) = N(x)(\mathbf{Y}(x, t) - \mathbf{Y}^*(x)), \quad (2.2.3)\]
where $N$ is such that:
\[ NF(Y^*)N^{-1} = \Lambda, \]  
(2.2.4)
where $\Lambda$ is diagonal and corresponds to the eigenvalues of $F(Y^*)$. Note that such $N$ exists as the system is strictly hyperbolic. Therefore, the system (2.2.1)–(2.2.2) is equivalent to
\[ u_t + A(u, x)u_x + B(u, x) = 0, \quad B(N(0)^{-1}u(0, t) + Y^*(0), N(L)^{-1}u(L, t) + Y^*(L)) = 0, \]
(2.2.5)  
(2.2.6)
with
\[ A(u, x) = N(x)F(Y)N^{-1}(x) = N(x)F(N^{-1}(x)u + Y^*(x))N^{-1}(x), \]
\[ B(u, x) = N(F(Y)(Y^*_x + (N^{-1})^T)u + D(Y)). \]
(2.2.7)  
(2.2.8)

The difficulty when there is a source term is twofold, and its first aspect can be seen in (2.2.7): we cannot assume that the steady state $Y^*$ we aim at stabilizing is uniform. Therefore $A$ depends not only on $u$ but also directly on $x$, and having $A(u(t, x), x)$ is different from having $A(u(t, x), x)$ especially when $u$ is a perturbation: if $u$ can still be seen as a perturbation, the dependency on $x$ can no longer be seen itself as a perturbation.

Its second aspect is that the source term creates a coupling between the two quantities which is a zero order term that can disturb the Lyapunov function and we will see in Section 2, 3 and 4 that this implies that there does not always exist a simple quadratic Lyapunov function ensuring exponential stability even when the boundary conditions can be chosen arbitrarily, while this phenomenon cannot appear in the absence of source term.

From the strict hyperbolicity we can denote by $m$ the integer such that
\[ \Lambda_i > 0, \forall i \leq m, \text{ and } \Lambda_i < 0, \forall i \in [m + 1, n]. \]
(2.2.9)

We now denote by $u_+$ the vector of components associated to positive eigenvalues $(u_1, \ldots, u_m)^T$ and similarly $u_-$ refers to $(u_{m+1}, \ldots, u_n)^T$. In the special cases where $m = 0$ or $m = n$ $u$ is equal to $u_-$ or $u_+$ respectively.

From now on we will focus on boundary conditions of the form
\[ \left( \begin{array}{c} u_+(t, 0) \\ u_-(t, L) \end{array} \right) = G \left( \begin{array}{c} u_+(t, L) \\ u_-(t, 0) \end{array} \right), \]
(2.2.10)

Note that with these boundary conditions the incoming signal is a function of the outgoing signal, which is what is typically expected from a feedback control law and enables the well-posedness of the system (see Theorem 2.2.1 later on). However the method presented in this article could also be applied to any other boundary conditions of the form (2.2.2) that also ensure well-posedness.

We also introduce the consequent first order compatibility conditions for an initial condition $u^0$:
\[ \left( \begin{array}{c} u^0_+(0) \\ u^0_-(0) \end{array} \right) = G \left( \begin{array}{c} u^0_+(L) \\ u^0_-(0) \end{array} \right), \]
(2.2.11)
\[ \left( \begin{array}{c} (A(u^0(0), 0)\partial_x u^0(0) + B(u^0(0), 0))^+ \\ (A(u^0(L), L)\partial_x u^0(L) + B(u^0(L), L))^+ \\ (A(u^0(0), 0)\partial_x u^0(0) + B(u^0(0), 0))^+ \end{array} \right) = G' \left( \begin{array}{c} u^0_+(L) \\ u^0_+(0) \end{array} \right) \left( \begin{array}{c} (A(u^0(L), L)\partial_x u^0(L) + B(u^0(L), L))^+ \\ (A(u^0(0), 0)\partial_x u^0(0) + B(u^0(0), 0))^+ \end{array} \right). \]
(2.2.12)

Well-posedness of the system (2.2.5), (2.2.10) for any initial condition $u^0$ that satisfies the compatibility conditions (2.2.11), (2.2.12) is given by Li [140] (see also [170]), one has the following theorem:
Theorem 2.2.1. For all $T > 0$ there exist $C_1(T) > 0$ and $\eta(T) > 0$ such that, for every $u_0 \in C^1([0, L], \mathbb{R}^n)$ satisfying the compatibility conditions (2.2.10), (2.2.11), and such that $|u_0|_1 \leq \eta(T)$, the system (2.2.5)-(2.2.10), with $A$ and $B$ of class $C^1$, has a unique solution on $[0, T] \times [0, L]$ with initial condition $u_0$. Moreover, one has:

$$|u(t, \cdot)|_1 \leq C_1(T)|u(0, \cdot)|_1, \forall t \in [0, T].$$

(2.2.13)

2.3 Context and previous results

General hyperbolic system without source term The exponential stability of general strictly hyperbolic systems of the form (2.2.5) without source term, i.e. $B \equiv 0$, has been mainly studied in the linear or non-linear case (see for instance [21, 48, 50, 57, 83, 139, 178]) under various boundary conditions or boundary controls (e.g. Proportional-integral control, dead beat control, single boundary control, etc.). A large non-linear case (see for instance [21, 48, 50, 57, 93, 139, 178]) under various boundary conditions or boundary conditions of the form (2.2.10). For such boundary conditions in non-linear systems the exponential stability depends on the topology [61] and in particular that the stability in $H^2$ norm does not imply the stability in $C^1$ norm. In [61] the authors also gave a sufficient condition for stability in the $W^{2,p}$ norm for $p \in [1, +\infty]$:

$$\rho_p(G'(0)) < 1,$$

(3.1)

where $G$ is given in (2.2.10) and the definition of $\rho_p$ is

$$\rho_p(M) = \inf(\|\Delta M \Delta^{-1}\|_p, \Delta \in D_n^+), \ 1 \leq p \leq +\infty$$

(3.2)

where $\|\cdot\|_p$ is the usual $p$ norm for matrices and $D_n^+$ are the diagonal $n \times n$ matrices with positive eigenvalues.

The case of the $C^1$ norm for systems with no source term has also been treated in [48] by Jean-Michel Coron and Georges Bastin by a Lyapunov approach that inspired the first part of this chapter. There, they proved the following sufficient condition for exponential stability through a Lyapunov approach:

$$\rho_\infty(G'(0)) < 1.$$

(3.3)

However the general case with a non-zero source term changes several things. As mentioned previously it implies that the steady-states $Y^*$ are no longer necessarily uniform and as a direct consequence the matrix $A$ defined in (2.2.7) depends explicitly not only on $u$ but also on $x$. In addition, there are some cases where, for any $G$, no basic quadratic $H^2$ Lyapunov function can be found (see for instance [13] and in particular Proposition 5.12) or no basic $C^1$ Lyapunov function can be found, as shown later on.

General hyperbolic system with non-zero source term in the $H^p$ norm For general quasilinear hyperbolic systems with source term, also called inhomogeneous quasilinear hyperbolic systems, the analysis of the exponential stability is much less advanced and actual knowledge in the matter is still partial. To our knowledge the exponential stability of such systems with non zero and non negligible source term was only treated in the framework of the $H^p$ norm for $p \in \mathbb{N} \setminus \{0, 1\}$ and in [13] (in Chapter 6) the authors find a sufficient (but a priori non-necessary) condition: exponential stability of the system (2.2.5)-(2.2.13) for the $H^p$ norm where $p \geq 2$ is achieved if there exists $Q \in C^1([0, L], D_n^+)$ such that the two following conditions hold:

- (Interior condition) the matrix

$$-(Q\Lambda)'(x) + Q(x)M(0,x) + M(0,x)^TQ(x)^T$$

(3.4)

is positive definite for all $x \in [0, L]$.
• (Boundary conditions) the matrix
\[
\begin{pmatrix}
\lambda_+ (L)Q_+ (L) \\
0 \\
-\lambda_- (0)Q_- (0)
\end{pmatrix}
- K^T
\begin{pmatrix}
\lambda_+ (0)Q_+ (0) \\
0 \\
-\lambda_- (L)Q_- (L)
\end{pmatrix}
K
\] (2.3.5)
is positive semi-definite

where \( M(0, \cdot) = \frac{\partial B}{\partial u} (0, \cdot) \) and \( K = G'(0) \).

It has to be underlined that with a non-zero source term there does not always exist a simple quadratic Lyapunov function ensuring exponential stability for the \( H^p \) norm whatever the boundary conditions are. Thus appears not only a boundary condition (2.3.5) as in the previous paragraph but also an interior condition (2.3.4).

This phenomenon is not specific to non-linear systems but also appears in linear systems: In [11] for instance, the authors study a linear 2 \( \times \) 2 system and found a necessary and sufficient condition for the existence of \( Q \) such that (2.3.4) hold. In general for linear hyperbolic systems the condition (2.3.4) also appears although it is only sufficient when \( n > 2 \). This is the consequence of the non-uniformity of the steady-states combined with non-identically vanishing zero order term even close to the steady states. If this phenomenon is not new, we will see however that the interior condition that appears for the \( C^1 \) norm is different from the condition that typically appears when studying Lyapunov functions for \( H^p \) norms.

Our contribution in this article is to deal with the exponential stability for the \( C^1 \) norm of such general hyperbolic systems with source term. This article intends to give a necessary and sufficient interior condition to the existence of a simple quadratic Lyapunov function ensuring exponential stability in the \( C^1 \) norm of the system and a sufficient condition on the boundary conditions.

**Useful observations and notations** Before going any further let us note that by definition of \( B \) and as \( Y^* \) is a steady-state
\[
B(0, x) = N(0)(F(Y^*)(Y^*_x) + D(Y^*)) = 0. \tag{2.3.6}
\]

Thus if we assume that \( F \) and \( Y^* \) are \( C^3 \) functions, then, from (2.2.8), \( B \) is \( C^2 \) and there exists \( \eta_0 > 0 \) and \( M \in C^2(B_{\eta_0} \times [0, L], M_n(\mathbb{R})) \), where \( B_{\eta_0} \) is the ball of radius \( \eta_0 \) in the space of continuous function endowed with the \( L^\infty \) topology, such that,
\[
B(u, x) = M(u, x)u, \tag{2.3.7}
\]
and therefore \( \frac{\partial B}{\partial u}(0, x) = M(0, x) \).

Besides, \( A \) is also a \( C^2 \) function and \( \eta_0 > 0 \) can be chosen small enough such that there exists \( E \in C^2(B_{\eta_0} \times [0, L], M_n(\mathbb{R})) \), satisfying (see [13] in particular Lemma 6.7),
\[
E(u, x)A(u, x) = \lambda(u, x)E(u, x) \quad \forall (u, x) \in B_{\eta_0} \times [0, L], \tag{2.3.8}
\]
and \( E(0, x) = Id, \tag{2.3.9} \)

where \( \lambda \) is a diagonal matrix, whose diagonal entries are the eigenvalues of \( A(u, x) \).

Also we introduce the following notations:

**Definition 2.3.1.** For a \( C^0 \) function \( U = (U_1, ..., U_n)^T \) on \( [0, L] \) we define the \( C^0 \) norm \( |U|_0 \) by
\[
|U|_0 := \sup_i \left( \sup_{[0,L]} |U_i| \right). \tag{2.3.10}
\]

For a \( C^1 \) function \( U = (U_1, ..., U_n)^T \) on \( [0, L] \), we denote similarly the \( C^1 \) norm \( |U|_1 \) by
\[
|U|_1 := |U|_0 + |\partial_u U|_0. \tag{2.3.11}
\]
In the following for a $C^1$ function $u$ on $[0,T] \times [0,L]$, we will sometimes note for simplicity $|u|_0$ instead of $|u(t,\cdot)|_0$ and $|u|$ instead of $|u(t,\cdot)|_1$.

We recall the definition of the exponential stability for the $C^1$ norm:

**Definition 2.3.2.** The steady state $u^* = 0$ of the system $\{2.2.5\}, \{2.2.10\}$ is exponentially stable for the $C^1$ norm if there exist $\gamma > 0$, $\eta > 0$, and $C > 0$ such that for every $u^0 \in C^1([0,L])$ satisfying the compatibility conditions $\{2.2.11\}, \{2.2.12\}$ and $|u^0|_1 \leq \eta$, the Cauchy problem $\{2.2.5\}, \{2.2.10\}, (u(0,x) = u^0)$ has a unique $C^1$ solution and

$$|u(t,\cdot)|_1 \leq Ce^{-\gamma t}|u^0|_1, \forall t \in [0, +\infty].$$

(2.3.12)

**Remark 2.3.1.** Given our change of variable $Y \to u$, proving the exponential stability for the $C^1$ norm of the steady state $0$ of the system $\{2.2.5\}, \{2.2.10\}$ is equivalent to proving the exponential stability for the $C^1$ norm of the steady state $Y^*$ of the system $\{2.2.1\}$ and the associated boundary condition.

**Definition 2.3.3.** We call basic $C^1$ Lyapunov function a function $V$ defined by

$$V(U) = \left|(\sqrt{f_1}U_1, ... , \sqrt{f_n}U_n)^T\right|_0 + \left|(E(U,x)(A(U,x)U_x + B(U,x)))\sqrt{f_1}, ... , (E(U,x)(A(U,x)U_x + B(U,x)))\sqrt{f_n}\right|_0,$$

for some $(f_1, ... , f_n) \in C^1([0,L]; \mathbb{R}^n_+)$, such that there exist $\gamma > 0$ and $\eta > 0$ such that for any $T > 0$ and any solution $u$ of the system $\{2.2.5\} - \{2.2.10\}$ with $|u^0|_1 \leq \eta$,

$$V(t) \leq V(t)e^{-\gamma(t-t')}, \forall 0 \leq t' \leq t \leq T.$$  

(2.3.14)

Also, in that case, $(f_1, ... , f_n)$ are called coefficients inducing a basic $C^1$ Lyapunov function.

**Remark 2.3.2.** Note from $\{2.2.5\}$, that when $u$ is a solution of the system $\{2.2.5\}, \{2.2.10\}$, $V(u(t,\cdot), \cdot)$ becomes

$$V(u(t,\cdot)) = \left|(\sqrt{f_1}u_1, ... , \sqrt{f_n}u_n)^T\right|_0 + \left|\sum_{i=1}^{n}(Eu_i)\sqrt{f_i}, ... , (Eu_i)\sqrt{f_n}\right|_0,$$

(2.3.15)

where we denoted $E = E(u(t,x), x)$ to lighten the notations. The previous definition $\{2.3.13\}$ is used so that $V$ is actually defined as function on $C^1([0,L])$ only and to underline that therefore, the function $V(u) : \{2.2.5\}, \{2.2.10\}$ is actually defined as function on $C^1([0,L])$ only and to underline that therefore, the function $V(u) : t \to V(u(t,\cdot))$ does only depend on the state of the system at time $t$. Looking at $\{2.3.15\}$, one could wonder why we consider the components of $u$ while we consider the components of $Eu$ for the derivative. The interest of considering $Eu_i$ instead of $u_i$ is that $E$ diagonalizes $A$ and therefore when differentiating the Lyapunov function appears $2(Eu_i)\lambda_n(Eu_i)\lambda_n = -\lambda_n(u_i)(Eu_i)_n^2$ and first order derivative terms, and there is no crossed term of second order derivative which would be impossible to bound with the $C^1$ norm (the full computation is done in Appendix 2.8.4). Differentiating $u_n^2$, though, gives $-\lambda_n(u_n^2)_n - u_n(\lambda(A - \lambda)u_n)_n$ and zero order derivative terms, and the second term is a cubic perturbation that can be bounded by the cube of the $C^1$ norm. Nevertheless, the proof would work as well with $Eu$ instead of $u$, but we consider $u$ to keep the computations as simple as we can in the main proof (Section 2.3). Finally, we use in the definition $\{2.3.13\}$ the weights $\sqrt{f_i}$ instead of using directly the weights $f_i$ to be coherent with the existing definition of basic quadratic Lyapunov function for the $L^2$ norm introduced in $\Pi$ (see in particular (34)) for linear systems and to facilitate a potential comparison.

**Remark 2.3.3.** Note also that, in Definition $\{2.3.3\}$ the condition $\{2.3.14\}$ is actually equivalent to the condition

$$\frac{dV(u)}{dt} \leq -\gamma V(u),$$

(2.3.16)
Proof of proposition 2.3.1. From Theorem 2.2.1, let \( T > 0 \) and \( u_0 \in C^1([0,T],\mathbb{R}^n) \) satisfying the compatibility conditions (2.2.11) and such that \( |u_0| \leq \min(\eta(T), \gamma_0/C_1(T)) \), where \( \eta(T) \) and \( C_1(T) \) are given by Theorem 2.2.1 and \( \gamma_0 \) is given by (2.3.7)-(2.3.9). From Theorem 2.2.1 there exists a unique solution \( u \in C^4([0,T] \times [0,L]) \). Suppose that \( V \) is a basic \( C^1 \) Lyapunov function, induced by \( (f_1, \ldots, f_n) \) and \( \gamma \) and \( \eta_1 \) are the constants associated. From its definition \( V(u(t,\cdot)) \) is closely related to \( |u(t,\cdot)|_1 \), indeed, using that for all \( i \in \{1, n\} \), \( f_i \) are positive and bounded on \([0, L] \), it is easy to see that there exists a constant \( c_2 > 0 \) such that
\[
\frac{1}{c_2} |u(t,\cdot)|_0 + |E \partial_u u(t,\cdot)|_0 \leq V(u(t,\cdot)) \leq c_2 (|u(t,\cdot)|_0 + |E \partial_u u(t,\cdot)|_0).
\] (2.3.17)

But as, from (2.2.13) and the assumption on \(|u_0|_1, |u(t,\cdot)|_1 \leq \eta_0 \) for any \( t \in [0, T] \). Thus from (2.3.8)-(2.3.9) there exists a constant \( c_1 \) depending only on \( \eta_0 \) and the system such that
\[
\frac{1}{c_1} |\partial_t u(t,\cdot)|_0 \leq |E \partial_u u(t,\cdot)|_0 \leq c_1 |\partial_t u(t,\cdot)|_0,
\] (2.3.19)

thus, there exists \( c_0 > 0 \) such that
\[
\frac{1}{c_0} (|u(t,\cdot)|_0 + |\partial_t u(t,\cdot)|_0) \leq V(u(t,\cdot)) \leq c_0 (|u(t,\cdot)|_0 + |\partial_t u(t,\cdot)|_0).
\] (2.3.20)

But observe that, as \( u \) is a solution of (2.2.5), there exists \( \eta_a > 0 \) such that for \( |u(t,\cdot)|_0 < \eta_a \)
\[
|\partial_t u(t,\cdot)|_0 \leq 2 \sup_i (|\Lambda_i|_0) |\partial_x u(t,\cdot)|_0 + 2 \sup_{i,j} (|M_{ij}(0,\cdot)|_0) |u(t,\cdot)|_0,
\] (2.3.21)

and similarly
\[
|\partial_x u(t,\cdot)|_0 \leq 2 \inf_{i,x \in [0,L]} \left( |\partial_u u(t,\cdot)|_0 + \sup_{i,j} (|M_{ij}(0,\cdot)|_0) |u(t,\cdot)|_0 \right),
\] (2.3.22)

which implies that there exists \( c > 0 \) constant such that for \( |u(t,\cdot)|_0 < \eta_a \)
\[
\frac{1}{c} |u(t,\cdot)|_1 \leq V(u) \leq c |u(t,\cdot)|_1.
\] (2.3.23)

Let \( T \in \mathbb{R}^*_+, \) with \( T > 0 \) and \( T \) large enough such that \( c^2 e^{-\gamma T} < \frac{1}{2} \). From (2.3.14), for all solution \( u \) such that \( |u(0)|_1 < \min(\eta(T), \gamma_1, \eta_0/C(T)) \) where \( C(T) \) is defined in (2.2.13),
\[
V(u, T) \leq V(u, 0) e^{-\gamma T}.
\] (2.3.24)

Now, using (2.3.23) we get
\[
|u(T,\cdot)|_1 \leq |u(0,\cdot)|_1 e^2 e^{-\gamma T},
\] (2.3.25)

And from the hypothesis on \( T \)
\[
|u(T,\cdot)|_1 \leq \frac{1}{2} |u(0,\cdot)|_1,
\] (2.3.26)

and this imply that \( u \) is defined on \([0, +\infty) \) and that we can find \( C \) and \( \gamma_1 \) such that
\[
|u(t,0) - u^*|_1 \leq C e^{-\gamma_1 t} |u_0 - u^*|_1, \forall t \in [0, +\infty],
\] (2.3.27)

which gives the exponential stability and concludes the proof. \( \square \)
2.4 Main results

The aim of this article is to show the following results:

**Theorem 2.4.1.** Let a quasilinear hyperbolic system be of the form \((2.2.5), (2.2.10)\), with \(A\) and \(B\) of class \(C^1\), \(\Lambda\) defined as in \((2.2.4)\) and \(M\) as in \((2.3.7)\). Let assume that the two following properties hold

1. (Interior condition) the system
   \[ \Lambda_i f_i' \leq -2 \left( -M_{ii}(0,x)f_i + \sum_{k=1, k\neq i}^{n} |M_{ik}(0,x)| \frac{f_i^{3/2}}{\sqrt{F_k}} \right), \tag{2.4.1} \]
   admits a solution \((f_1, ..., f_n)\) on \([0, L]\) such that for all \(i \in [1, n]\), \(f_i > 0\),

2. (Boundary conditions) there exists a diagonal matrix \(\Delta\) with positive coefficients such that
   \[ \|\Delta G'(0)\Delta^{-1}\|_\infty < \inf_i \left( \frac{f_i(d_i)}{\Delta_i} \right) \sup_i \left( \frac{f_i(L-d_i)}{\Delta_i} \right), \tag{2.4.2} \]
   where \(d_i = L\) if \(\Lambda_i > 0\) and \(d_i = 0\) otherwise.

Then there exists a basic \(C^1\) Lyapunov function for the system \((2.2.5), (2.2.10)\).

**Remark 2.4.1.** Note that when \(M \equiv 0\) we recover the result found in [48] in the absence of source term: the interior condition is always verified by any positive constant functions \((f_1, ..., f_n)\) and when choosing \(f_i = \Delta_i^2\) the boundary condition reduces to the existence of \(\Delta \in D_n^+\) such that \(\|\Delta G'(0)\Delta^{-1}\|_\infty < 1\) which is equivalent to \(\rho_\infty(G'(0)) < 1\).

Note also that the existence of a solution \((f_1, ..., f_n)\) with \(f_i > 0\) on \([0, L]\) for all \(i \in \{1, ..., n\}\) for the system
\[ f_i' = -\frac{2}{\Lambda_i} \left( -M_{ii}(0,x)f_i + \sum_{k=1, k\neq i}^{n} |M_{ik}(0,x)| \frac{f_i^{3/2}}{\sqrt{F_k}} \right), \tag{2.4.3} \]
is also a sufficient interior condition as it obviously implies the existence of a solution with positive components for \((2.4.1)\).

Moreover, we show in the following Theorem that condition \((2.4.1)\) is also necessary in order to ensure the existence of a basic \(C^1\) Lyapunov function.

**Theorem 2.4.2.** Let a quasilinear hyperbolic system be of the form \((2.2.5)\) with \(A\) and \(B\) of class \(C^3\), there exists a control of the form \((2.2.10)\) such that there exists a basic \(C^1\) Lyapunov function for the system \((2.2.5), (2.2.10)\) if and only if
\[ \Lambda_i f_i' \leq -2 \left( \sum_{k=1, k\neq i}^{n} |M_{ik}(0,x)| \frac{f_i^{3/2}}{\sqrt{F_k}} - M_{ii}(0,x)f_i \right), \tag{2.4.4} \]
admits a solution \((f_1, ..., f_n)\) on \([0, L]\) such that for all \(i \in [1, n]\), \(f_i > 0\).

**Remark 2.4.2.** Note that Theorem \(2.4.2\) illustrates the sharpness of \((2.4.1)\) by showing that it is a necessary condition. This is not trivial as, to our knowledge, there is no similar condition for the \(H^p\) norm when \(n > 2\) yet. Note also that we have not imposed anything on the initial values of the \((f_1, ..., f_n)\) but we see from Theorem \(2.4.1\) and \(2.4.2\) that the more liberty we give them, the more restrictive the condition on the boundary \((2.4.3)\) might become.

The proof of these two results is given in the next section.
2.5 \( C^1 \) Lyapunov stability of \( n \times n \) quasilinear hyperbolic system

In this Section we shall prove Theorem 2.4.1 and Theorem 2.4.2. We will first start by proving the following Lemma which will be useful for finding the interior condition in the proof of Theorem 2.4.1 and for proving Theorem 2.4.2.

**Lemma 2.5.1.** Let \( (a_i, b_{ij})_{(i,j) \in [1,n]^2} \in C([0,L], \mathbb{R})^n \times C([0,L], \mathbb{R})^n \),

If

\[
(i) \exists p_1 \in \mathbb{N}^* : \sum_{i=1}^{n} \left( a_i(x) y_i^{2p} + \sum_{j=1}^{n} b_{ij}(x) y_i^{2p-1} y_j \right) > 0, \ \forall p > p_1, \forall y \in \mathbb{R}^n \setminus \{0\}, \forall x \in [0,L], \tag{2.5.1}
\]

then

\[
(ii) \ a_i(x) \geq \sum_{j=1, j \neq i}^{n} |b_{ij}(x)| - b_{ii}(x), \ \forall i \in [1,n], \forall x \in [0,L]. \tag{2.5.2}
\]

And if

\[
(iii) \ a_i(x) > \sum_{j=1, j \neq i}^{n} |b_{ij}(x)| - b_{ii}(x), \ \forall i \in [1,n], \forall x \in [0,L], \tag{2.5.3}
\]

then (i) holds.

**Proof of Lemma 2.5.1.** We start with (i) \( \Rightarrow \) (ii). Let \( x \in [0,L] \), let \( i_1 \in [1,n] \), assuming (i) is true for all \( y \in \mathbb{R}^n \setminus \{0\} \), we take \( m \in \mathbb{N}^* \), and define \( y_{i_1} := 1, y_j := -\text{sgn}(b_{i_1j})m/(m+1) \) for \( j \neq i_1 \). Then as (2.5.1) is true there exists \( p_1 \in \mathbb{N}^* \) such that

\[
\sum_{i=1, i \neq i_1}^{n} a_i(x) y_i^{2p} + \sum_{j=1}^{n} b_{ij}(x) y_i^{2p-1} y_j + a_{i_1}(x) + b_{i_1i_1} + \sum_{j=1, j \neq i_1}^{n} b_{i_1j}(x)y_j > 0, \ \forall p > p_1, \forall x \in [0,L]. \tag{2.5.4}
\]

Note that for any \( i \neq i_1 \), \( \lim_{p \to +\infty} |y_i|^{2p} = 0 \). Thus, by letting \( p \to +\infty \) one gets

\[
a_{i_1}(x) + b_{i_1i_1} \geq \frac{m}{m+1} \sum_{j=1, j \neq i_1}^{n} |b_{i_1j}(x)|, \ \forall x \in [0,L]. \tag{2.5.5}
\]

Hence, as it is true for all \( m \in \mathbb{N}^* \), letting \( m \to +\infty \)

\[
a_{i_1}(x) + b_{i_1i_1} \geq \sum_{j=1, j \neq i_1}^{n} |b_{i_1j}(x)|, \ \forall x \in [0,L]. \tag{2.5.6}
\]

This can be done for any \( i_1 \in [1,n] \), which concludes (i) \( \Rightarrow \) (ii).

Now let us prove that (iii) \( \Rightarrow \) (i). First of all observe that we can suppose without loss of generality that \( \forall i \in [1,n], b_{ii} := 0 \): one just has to redefine \( a_i := a_i + b_{ii} \). Then by (2.5.3), \( a_i > \sum_{j=1}^{n} |b_{ij}|, \ \forall i \in [1,n] \), then let us define:

\[
d_i(x) := a_i(x) - \sum_{k=1}^{n} |b_{ik}(x)|, \tag{2.5.7}
\]

then \( d_i \) is \( C^0 \) and positive on \([0,L]\). We denote by

\[
d_i^{(0)} := \inf_{[0,L]} (d_i) = \min_{[0,L]} (d_i) > 0. \tag{2.5.8}
\]
We introduce $y_i$ such that
$$|y_i| = \max_{t \in [1,n]} (|y_i|),$$
(2.5.9)
thus $y_i \neq 0$ and proving (2.5.1) is equivalent to proving that there exists $p_1 \in \mathbb{N}^*$ such that for all $p > p_1$,
$$\sum_{i=1}^{n} \left( a_i(x) \frac{|y_i|}{y_i} 2^p + \sum_{k=1}^{n} b_{ik}(x) \left( \frac{|y_i|}{y_i} \right)^{2p-1} \frac{y_k}{y_i} \right) > 0, \forall x \in [0,L].$$
(2.5.10)
Denoting $z_i = y_i/y_i$, (2.5.10) becomes
$$I := \sum_{i=1}^{n} \left( a_i z_i^{2p} + \sum_{k=1}^{n} b_{ik} z_i^{2p-1} z_k \right) > 0, \text{ on } [0,L].$$
(2.5.11)
Using (2.5.7) we know that
$$I = \sum_{i=1}^{n} d_i z_i^{2p} + \sum_{k=1}^{n} |b_{ik}| z_i^{2p} + \sum_{k=1}^{n} b_{ik} z_i^{2p-1} z_k.$$ 
(2.5.12)
By definition for $i = i_1, |z_{i_1}| = 1$, and for $i \neq i_1, |z_i| \leq 1$, therefore
$$d_{i_1} z_{i_1}^{2p} + \sum_{k=1}^{n} |b_{i_1,k}| z_{i_1}^{2p} + \sum_{k=1}^{n} b_{i_1,k} z_{i_1}^{2p-1} z_k \geq d_{i_1} \geq d_{i_1}^{(0)}.$$ 
(2.5.13)
Therefore
$$I \geq d_{i_1}^{(0)} + \sum_{i=1, i \neq i_1}^{n} \left( d_i z_i^{2p} + \sum_{k=1}^{n} |b_{ik}| z_i^{2p} - \sum_{k=1}^{n} |b_{ik}| z_i^{2p-1} \right),$$
(2.5.14)
$$= d_{i_1}^{(0)} + \sum_{i=1, i \neq i_1}^{n} \left( d_i z_i^{2p} - \sum_{k=1}^{n} |b_{ik}| (1 - |z_i|) z_i^{2p-1} \right).$$
We introduce
$$g : z \mapsto g(z) = -(1-z) z^{2p-1},$$
(2.5.15)
We know that $g$ is $C^1$ on $[0,1]$ and admits a minimum on $[0,1]$ at $z = 1 - \frac{1}{2p}$, as one can check that
$$g'(z) = (2pz - (2p - 1)) z^{2p-2}.$$ 
(2.5.16)
Therefore
$$I \geq d_{i_1}^{(0)} - \frac{1}{2p} \sum_{i=1, i \neq i_1}^{n} \sum_{k=1}^{n} |b_{ik}(x)|,$$
(2.5.17)
and this is true for all $x \in [0,L]$. Let us point out that there exists $p_1 > 0$ such that
$$\frac{1}{2p} \sum_{i=1, i \neq i_1}^{n} |b_{ik}| z_i < d_{i_1}^{(0)}, \forall p > p_1.$$ 
Here $p_1$ is a constant and does not depend on $x$. Hence we can conclude that $I > 0, \forall p > p_1, \forall x \in [0,L], \forall y \in \mathbb{R}^n$. Therefore (2.5.1) holds.

Now let us prove Theorem 2.4.1.

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Proof of Theorem 2.4.1. Let $T \in \mathbb{R}^*_+$. Let assume that $A$ and $B$ are of class $C^2$, and let $u$ be a $C^2$ solution of system (2.2.5), (2.2.10) such that $|u^p|_1 \leq \varepsilon$. Such solution exists for $\varepsilon$ small enough and $u_0 \in C^2([0, L], \mathbb{R}^n)$ which verifies the compatibility conditions (2.2.11) (see [13] in particular Theorem 4.21). We suppose here a $C^2$ regularity for technical reason but the final estimate will not depend on the $C^2$ norm and will be also true by density for $A$ and $B$ of class $C^1$ and for $u$ a $C^1$ solution. Recall that $\lambda_i$ are the eigenvalues of $A$ as defined in (2.2.7). We denote $s_i := \text{sgn}(\lambda_i(u, x))$ which only depends on $i$ from the hypothesis of non-vanishing eigenvalues and the continuity of $A$. We define:

$$W_{1, p} := \left( \int_0^L \sum_{i=1}^n f_i(x)^p u_i^{2p} e^{-2p\mu s, x} dx \right)^{1/2p},$$  

(2.5.19)

with $p \in \mathbb{N}^*$, and $f_i > 0$ on $[0, L]$ to be determined. Clearly $W_{1, p} > 0$ for $u \neq 0$, and $W_{1, p} = 0$ when $u \equiv 0$. If we differentiate $W_{1, p}$ with respect to time along the $C^2$ trajectories, we have

$$\frac{dW_{1, p}}{dt} = W_{1, p}^{-2p} \left[ \sum_{i=1}^n \lambda_i f_i(x)^p u_i^{2p} e^{-2p\mu s, x} \right]_0^L$$

$$- W_{1, p}^{-2p} \int_0^L \sum_{i=1}^n f_i(x)^p u_i^{2p-1} \left[ \sum_{k=1}^n M_{ik} u_k \right] e^{-2p\mu s, x} dx$$

$$- \frac{W_{1, p}^{-2p}}{2} \int_0^L \sum_{i=1}^n \left( \lambda_i f_i(x)^{p-1} f_i' u_i^{2p} + \frac{d}{dx} \left( \frac{\lambda_i}{p} f_i(x)^p u_i^{2p} \right) \right) e^{-2p\mu s, x} dx$$

(2.5.20)

$$- \mu W_{1, p}^{-2p} \int_0^L \sum_{i=1}^n \lambda_i |f_i'| u_i^{2p} e^{-2p\mu s, x} dx.$$  

We denote

$$I_2 := W_{1, p}^{-2p} \left[ \sum_{i=1}^n \lambda_i f_i(x)^p u_i^{2p} e^{-2p\mu s, x} \right]_0^L,$$

(2.5.23)

and

$$I_3 := W_{1, p}^{-2p} \int_0^L \sum_{i=1}^n f_i(x)^p u_i^{2p-1} \left( \sum_{k=1}^n M_{ik} u_k \right) e^{-2p\mu s, x} dx$$

$$- \frac{W_{1, p}^{-2p}}{2} \int_0^L \sum_{i=1}^n \lambda_i f_i(x)^{p-1} f_i' u_i^{2p} e^{-2p\mu s, x} dx.$$  

(2.5.24)

We supposed that $|u^p|_1 \leq \varepsilon$, where $\varepsilon > 0$ can be chosen arbitrarily small but, of course, independent of $p$. From (2.2.13) and denoting $\eta = C_1(T) \varepsilon$ we have: $|u|_0 \leq \eta$. Choosing $\varepsilon$ sufficiently small is thus equivalent to choosing $\eta$ sufficiently small, so we will rather choose $\eta$ in the following and this choice of $\eta$
will always be independent of \( p \). Besides, observe that there exists \( \eta_1 > 0 \) sufficiently small such that for all \( u_0 \in C^0([0, L], \mathbb{R}^n) \) such that \( |u_0| \leq \eta_1 \)

\[
\min_{x \in [0, L]} \left( \min_{i \in [1, n]} \left( (\lambda_i(u, x)) \right) \right) \geq \min_{x \in [0, L]} \left( \min_{i \in [1, n]} \left( \frac{|\Lambda_i(x)|}{2} \right) \right).
\]

(2.5.25)

Recall that \( \Lambda = \lambda(0, \cdot) \) and is defined in (2.2.4). As \([0, L]\) is a closed segment, and the \( |\Lambda_i| \) are strictly positive continuous functions we can define the positive constant \( \alpha_0 := \min_{x \in [0, L]} \left( \min_{i \in [1, n]} \left( \frac{|\Lambda_i(x)|}{2} \right) \right) > 0 \). We suppose from now on that \( \eta < \eta_1 \). Therefore from (2.5.22), (2.5.23), (2.5.24) and (2.5.25)

\[
\frac{dW_{1,p}}{dt} \leq -I_2 - \mu \alpha_0 W_{1,p} - I_3
\]

\[
-W_{1,p}^2 \int_0^L \sum_{i=1}^n f_i^p u_i^{2p-1} \left( \sum_{j=1}^n \langle V_{ik}(u, x), u \rangle u_{kx} \right) e^{-2p\mu s, x} dx
\]

\[
+ \frac{W_{1,p}}{2p} \int_0^L \sum_{i=1}^n \left( \frac{\partial \lambda_i}{\partial u} \partial_x \| u \|_1 \right) f_i(x)^p u_i^{2p} e^{-2p\mu s, x} dx.
\]

(2.5.26)

We now estimate the two last terms, starting by the last one. The \( \lambda_i \) are \( C^2 \) and in particular \( C^1 \) in \( u \) therefore

\[
\frac{W_{1,p}}{2p} \int_0^L \sum_{i=1}^n \left( \frac{\partial \lambda_i}{\partial u} \partial_x \| u \|_1 \right) f_i(x)^p u_i^{2p} e^{-2p\mu s, x} dx
\]

\[
\leq \frac{C_1}{2p} W_{1,p} + \frac{C_2}{2p} W_{1,p} |u|_1,
\]

where \( C_1 \) and \( C_2 \) are constants that depend on \( \eta \) and the system but are independent from \( p \) and \( u \) provided that \( |u|_1 < \eta \). Besides we have

\[
\frac{W_{1,p}^2}{2p} \int_0^L \sum_{i=1}^n f_i^p u_i^{2p-1} \left( \sum_{j=1}^n \langle V_{ik}(u, x), u \rangle u_{kx} \right) e^{-2p\mu s, x} dx \leq C_3 W_{1,p} |u|_1.
\]

(2.5.28)

where \( C_3 \) is a constant that does not depend on \( p \) and \( u \). Therefore (2.5.26) can be written as

\[
\frac{dW_{1,p}}{dt} \leq -I_2 - I_3 - \mu \alpha_0 \frac{C_1}{2p} W_{1,p} + (\frac{C_2}{2p} + C_3) W_{1,p} |u|_1.
\]

(2.5.29)

As \( \alpha_0 > 0 \), it is easy to see that there exists \( p_1 \in \mathbb{N}^* \) such that \( \forall p \geq p_1 \)

\[
\frac{dW_{1,p}}{dt} \leq -I_2 - I_3 - \frac{\mu \alpha_0}{2} W_{1,p} + C_4 W_{1,p} |u|_1.
\]

(2.5.30)

Here \( p_1 \) depends only on \( \alpha_0 \) and \( \eta \), while \( C_4 \) does not depend on \( p \) and \( u \). Before going any further, we see here that if we can manage to prove that \( I_2 > 0 \) and \( I_3 \geq 0 \) we may be able to conclude to the existence of a Lyapunov function that looks like a \( L^2 \) norm where \( p \) can be as large as we want and therefore we start to see the forthcoming basic \( C^1 \) Lyapunov function. We are now left with studying \( I_2 \) and \( I_3 \) which will correspond respectively to the boundary condition and the interior condition we mentioned in Section 2 and in Theorem 2.4.1.

Let us first deal with \( I_3 \):

\[
I_3 = \frac{W_{1,p}^2}{2p} \int_0^L \sum_{i=1}^n f_i^p u_i^{2p-1} \left( \sum_{k=1}^n M_{ik} u_k \right) - \frac{\lambda_i f_i}{2} f_i^{p-1} u_i^{2p} e^{-2p\mu s, x} dx.
\]

(2.5.31)
Let suppose that the system (2.4.1) admits a positive solution \((g_1, ... g_n)\) on \([0, L]\), which is the interior condition. Then we can write this as

\[
- \Lambda_i g'_i = 2\left( \sum_{k=1, k \neq i}^n |M_{ik}(0, x)|^{3/2} \sqrt{y_k} - M_{ii}(0, x)g_i \right) + h_i,
\]

(2.5.32)

where \(h_i\) are non-negative functions. By continuity (see for instance [105], in particular Theorem 2.1 in Chapter 5) there exists \(\sigma_1 > 0\) such that for all \(\sigma \in [0, \sigma_1]\) there exists a unique solution to

\[
- \Lambda_i f'_i = 2\left( \sum_{k=1, k \neq i}^n |M_{ik}(0, x)|^{3/2} \sqrt{f_{k,\sigma}} - M_{ii}(0, x)f_i \right) + h_i + \sigma,
\]

(2.5.33)

\(f_i(0) = g_i(0)\).

We denote \((f_1,\sigma, ..., f_n,\sigma)\) this solution, which is continuous with \(\sigma\). Therefore there exists \(\sigma_2 \in (0, \sigma_1]\) such that for all \(i \in [1, n]\), and all \(\sigma \in (0, \sigma_2]\), \(f_i,\sigma > 0\), on \([0, L]\) and

\[
- \Lambda_i f'_i,\sigma > 2\left( \sum_{k=1, k \neq i}^n |M_{ik}(0, x)|^{3/2} \sqrt{f_{k,\sigma}} - M_{ii}(0, x)f_i,\sigma \right).
\]

(2.5.34)

We choose now \(f_i := f_i,\sigma\) where \(\sigma \in (0, \sigma_2]\). As \(M\) and \(\lambda\) are continuous in \(u\), there exists \(\eta_2 > 0\) such that for \(|u|_0 < \eta_2\)

\[
- \lambda_i(u, x)f'_i > 2\sum_{k=1, k \neq i}^n |M_{ik}(u, x)|^{3/2} \sqrt{f_{k,\sigma}} - 2M_{ii}(u, x).
\]

(2.5.35)

Therefore from Lemma 2.5.1

\[
\sum_{i=1}^n \left( \frac{\lambda_i f'_i y_i^{2p} + \sum_{j=1}^n M_{jk} \sqrt{f_{k,\sigma}} y_k y_i^{2p-1}}{2f_i} \right) > 0, \quad \forall \, y = (y_i)_{i \in [1, n]} \in \mathbb{R}^n \setminus \{0\},
\]

(2.5.36)

applying this for \((y_i)_{i \in [1, n]} = (\sqrt{f_{ik}u_i})_{i \in [1, n]}\), it implies that

\[
W_{1,p}^{1-2p} \int_0^L \sum_{i=1}^n \left( -\lambda_i f'_i \frac{1}{2} u_i^{2p-1} u_i^{2p} + \sum_{k=1}^n M_{ik} u_k f_{i,\sigma} u_i^{2p-1} \right) dx \geq 0.
\]

(2.5.37)

Therefore by continuity, there exists a \(\mu_1 > 0\) such that \(\forall \mu \in [0, \mu_1]\)

\[
I_3 = W_{1,p}^{1-2p} \int_0^L \sum_{i=1}^n \left( -\lambda_i f'_i \frac{1}{2} u_i^{2p-1} u_i^{2p} + \sum_{k=1}^n M_{ik} u_k \right) e^{-2p\mu s, x} dx > 0.
\]

(2.5.38)

Now let us deal with \(I_2\), which will lead to the boundary condition. Recall that

\[
I_2 = \frac{W_{1,p}^{1-2p}}{2p} \left[ \sum_{i=1}^n \lambda_i(u(t, L), L)f_i(L) u_i^{2p}(t, L)e^{-2p\mu s, L} - \sum_{i=1}^n \lambda_i(u(t, 0), 0)f_i(0) u_i^{2p}(t, 0) \right].
\]

(2.5.39)
Recall that $m$ is the integer such that $\Lambda_i > 0$, for all $i \leq m$ and $\Lambda_i < 0$, for all $i > m$, we have

$$I_2 = \frac{W_{1,p}^{1-2p}}{2p} \left( \sum_{i=1}^{m} |\lambda_i(u(t, L), L)| f_i(L)^p u_i^2(t, L) e^{-2pL} \right.$$

$$- \sum_{i=1}^{m} |\lambda_i(u(t, 0), 0)| f_i(0)^p u_i^2(t, 0)$$

$$- \sum_{i=m+1}^{n} |\lambda_i(u(t, L), L)| f_i(L)^p u_i^2(t, L) e^{2pL}$$

$$+ \sum_{i=m+1}^{n} |\lambda_i(u(t, 0), 0)| f_i(0)^p u_i^2(t, 0) \left) \right.\right) \right). \tag{2.5.40}$$

We denote $K := G'(0)$ and we know that under assumption (2.4.2) there exists $\Delta = (\Delta_1, \ldots, \Delta_n)^T \in (\mathbb{R}^n_+)^n$ such that

$$\theta := \sup_{\|u\| \leq 1} \left( \sup_i \left( \sum_{j=1}^{n} \left( \Delta_i \lambda_j^{-1} \xi_j \right) \right) \right) < \inf_i \left( \frac{g_i(d_i)}{\Delta_i^2} \right) \sup_i \left( \frac{g_i(L-d_i)}{\Delta_i^2} \right). \tag{2.5.41}$$

where $(g_i)_{i \in [1,n]}$ denote the positive solution of (2.4.1) introduced previously in (2.5.32). Note that we have in fact $\theta = \sup_i (\sum_{j=0}^{n} |K_j|^{\frac{2}{\Delta_j}})$. Let:

$$\xi_i = \Delta_i u_i(t, L) \text{ for } i \in [1, m],$$

$$\xi_i = \Delta_i u_i(t, 0) \text{ for } i \in [m + 1, n]. \tag{2.5.42}$$

From (2.2.16) and using the fact that $G$ is $C^1$, we have

$$\left( \begin{array}{c} u_+ (t, 0) \\ u_-(t, L) \end{array} \right) = K \left( \begin{array}{c} u_+(t, L) \\ u_-(t, 0) \end{array} \right) + o \left( \left| \left( \begin{array}{c} u_+ (t, 0) \\ u_-(t, 0) \end{array} \right) \right| \right), \tag{2.5.43}$$

where $o(x)$ refers to a function such that $o(x)/|x|$ tends to 0 when $|u_0|$ tends to 0. Thus we get

$$I_2 = \frac{W_{1,p}^{1-2p}}{2p} \left( \sum_{i=1}^{m} \lambda_i(u(t, L), L) \frac{f_i(L)^p}{\Delta_i^{2p}} (u_i(t, L) \Delta_i)^{2p} e^{-2pL} \right.$$

$$+ \sum_{i=m+1}^{n} |\lambda_i(u(t, 0), 0)| f_i(0)^p (u_i(t, 0) \Delta_i)^{2p}$$

$$- \sum_{i=1}^{m} \lambda_i(u(t, 0), 0) \frac{f_i(0)^p}{\Delta_i^{2p}} \left( \sum_{k=1}^{n} K_{ik} \xi_k(t) \Delta_i \Delta_k + o(\xi) \right)^{2p}$$

$$- \sum_{i=m+1}^{n} \left( \sum_{k=1}^{n} K_{ik} \xi_k(t) \Delta_i \Delta_k + o(\xi) \right)^{2p} \frac{f_i(L)^p}{\Delta_i^{2p}} \right) \right). \right) \right). \tag{2.5.45}$$

As the $\lambda_i$ are $C^1$ in $u$ we have

$$I_2 = \frac{W_{1,p}^{1-2p}}{2p} \left( \sum_{i=1}^{m} \left( \lambda_i(L)+O(\xi) \right) \frac{f_i(L)^p}{\Delta_i^{2p}} (u_i(t, L) \Delta_i)^{2p} e^{-2pL} \right.$$

$$+ \sum_{i=m+1}^{n} \left( \lambda_i(0)+O(\xi) \right) \frac{f_i(0)^p}{\Delta_i^{2p}} (u_i(t, 0) \Delta_i)^{2p}$$

$$- \sum_{i=1}^{m} \left( \lambda_i(0)+O(\xi) \right) \frac{f_i(0)^p}{\Delta_i^{2p}} \left( \sum_{k=1}^{n} K_{ik} \xi_k(t) \Delta_i \Delta_k + o(\xi) \right)^{2p}$$

$$- \sum_{i=m+1}^{n} \left( \sum_{k=1}^{n} K_{ik} \xi_k(t) \Delta_i \Delta_k + o(\xi) \right)^{2p} \frac{f_i(L)^p}{\Delta_i^{2p}} \right) \right). \right) \right). \tag{2.5.46}$$
where $O(x)$ refers to a function such that $O(x)/|x|$ is bounded when $|u_0|$ tends to 0. Now let $t \in [0,T]$, there exists $\eta_3$ such that $\max_i(\xi^2(t)) = \xi_i^2$, to simplify the notations we introduce a $d_i$ such that $d_i = L$ for $i \leq m$ and $d_i = 0$ for $i \geq m + 1$. Then there exists a constant $C > 0$ independent of $u$ and $p$ such that

$$I_2 \geq \frac{W_1^{1-2p}}{2p} \left( (|A_{i_0}(d_{i_0})| - C|\xi_{i_0}|) \frac{f_{i_0}^p(d_{i_0})}{\Delta^2_{i_0}} \xi_{i_0}^{2p}(t) e^{-2p\mu d_{i_0}} ight)$$

$$- n \sup_{i \in [1,n]} \left( (|A_i(L - d_i)| + C|\xi_{i_0}|) \frac{f_{i_0}^p(L - d_i)}{\Delta^2_{i_0}} \xi_{i_0}^{2p} e^{2p\mu(L - d_i)} \right) (\theta + l(\xi_{i_0}))^{2p \xi_{i_0}^p}$$

where $l$ is a continuous and positive function which satisfies $l(0) = 0$. Thus

$$I_2 \geq \frac{W_1^{1-2p}}{2p} \left( (|A_{i_0}(d_{i_0})| - C|\xi_{i_0}|) \frac{f_{i_0}^p(d_{i_0})}{\Delta^2_{i_0}} \xi_{i_0}^{2p}(t) e^{-2p\mu d_{i_0}} ight)$$

$$- n \sup_{i \in [1,n]} \left( (|A_i(L - d_i)| + C|\xi_{i_0}|) \frac{f_{i_0}^p(L - d_i)}{\Delta^2_{i_0}} \xi_{i_0}^{2p} e^{2p\mu(L - d_i)} \right) (\theta + l(\xi_{i_0}))^{2p \xi_{i_0}^p}$$

Now, from (2.4.2) we have

$$\theta^2 < \frac{\inf_i \left( \frac{g_i(d_i)}{\Delta^2} \right)}{\sup_i \left( \frac{g_i(L - d_i)}{\Delta^2} \right)}$$

where $(g_i)_{i \in [1,n]}$ still denote the positive solution of (2.4.1). Remark that we set earlier $f_i := f_{i,\sigma}$ where $\sigma \in (0, \sigma_2]$ and can be chosen arbitrary small, and recall that the functions $f_{i,\sigma}$ are continuous in $\sigma$ on this neighbourhood of 0. Therefore there exists $\sigma \in (0, \sigma_2]$ such that

$$\theta^2 < \frac{\inf_i \left( \frac{f_i(d_i)}{\Delta^2} \right)}{\sup_i \left( \frac{f_i(L - d_i)}{\Delta^2} \right)}$$

But as the inequality is strict, there exist by continuity $\eta_3 \in (0, \eta_2)$, $p_3 > 0$ and $\mu_3$ such that for all $|u_0| < \eta_3$ and $p > p_3$

$$\left(\theta + l(\xi_{i_0})\right)^2 < \left( \frac{\inf_i |A_i(d_i)| - C|\xi_{i_0}|}{n \sup_i |A_i(L - d_i) + C|\xi_{i_0}|} \right)^{1/p} \frac{\inf_i \left( \frac{f_i(d_i)}{\Delta^2} \right)}{\sup_i \left( \frac{f_i(L - d_i)}{\Delta^2} \right)} e^{-4\mu L}, \quad \forall \sigma \in [0, \mu_3], \forall p \geq p_3. \quad (2.5.51)$$

Therefore from (2.5.51) and (2.5.48) $I_2 > 0$. We can conclude that there exist $p_4$ and $\mu > 0$

$$\frac{dW_1^p}{dt} \leq - \frac{\mu \alpha_0}{2} W_1^p + C_4 W_1^p |u_1|, \quad \forall p \geq p_4. \quad (2.5.52)$$

We now have our first estimate and we have seen appear both an interior condition and a boundary condition that explains the conditions that appear in Theorem (2.4.4). Yet there remains a potentially non-negative term in $|u_1|$ and the function we considered in (2.5.19) does not have the form of a basic $C^1$ Lyapunov function. The last step is now to convert $W_1^p$ in a basic $C^1$ Lyapunov function. Defining

$$W_{2, p} = \left( \int_0^L \sum_{i=1}^n f_i(x)^p (E u_i)^{2p} e^{-2p\mu s, x} dx \right)^{1/2p}, \quad (2.5.53)$$

where $E = E(u(t, x), x)$ is given by (2.3.8), and proceeding the same way and observing that, for $C^2$ solutions,

$$u_t + A(u, x)u_x + \left[ \frac{\partial A}{\partial u}(u, x)u_t \right] u_x + \frac{\partial B}{\partial u}(u, x)u_t = 0, \quad (2.5.54)$$
where \( \partial A/\partial u \) refers to the matrix with coefficients \( \sum_{k=1}^{n} \partial A_{ij}/\partial u_{k}(u, x_{j}) \partial t u_{k}(t, x) \), we can obtain similarly:

\[
\frac{dW_{p}}{dt} \leq -\frac{\mu_{0}}{2} W_{p} + C_{5} W_{2,p} |u|_{1}. 
\]  \hspace{1cm} (2.5.55)

In order to avoid overloading this article, the proof—which is very similar to the proof of (2.5.52)—is given in the Appendix (see 2.8.1).

Now let us define \( W_{p} := W_{1,p} + W_{2,p} \), there exists \( \eta_{4} > 0 \) (independent of \( p \)), \( \mu > 0 \), \( C \) (independent of \( p \) and \( u \)), and \( p_{5} \) such that, with \( |u|_{1} < \eta_{4} \),

\[
\frac{dW_{p}}{dt} \leq -\frac{\mu_{0}}{2} W_{p} + CW_{p} |u|_{1}, \hspace{0.5cm} \forall p \geq p_{5}. 
\]  \hspace{1cm} (2.5.56)

Here we see that this estimate does not depend on the \( C^{2} \) norm of the solution \( u \) and of the \( C^{2} \) norms of \( A \) and \( B \) and is therefore also true by density for solutions that are only of class \( C^{1} \) and for \( A \) and \( B \) also only \( C^{1} \). To be fully rigorous, this statement assumes the well-posedness of the system (2.2.5), (2.2.10), \( \eta, \mu, p_{5} \). To be fully rigorous, this statement assumes the well-posedness of the system (2.2.5), (2.2.10), \( \eta, \mu, p_{5} \). We choose such \( \eta, \mu, p_{5} \), and we define our basic \( C^{1} \) Lyapunov function candidate

\[
V := \left| \sqrt{\int_{1} u_{1} e^{-\mu_{x} x_{1}} x_{1}}, \ldots, \sqrt{\int_{n} u_{n} e^{-\mu_{x} x_{n}} x_{n}} \right|_{0} + \left| \sqrt{\int_{1} (E u_{1}) e^{-\mu_{x} x_{1}} x_{1}}, \ldots, \sqrt{\int_{n} (E u_{n}) e^{-\mu_{x} x_{n}} x_{n}} \right|_{0}. 
\]  \hspace{1cm} (2.5.57)

Similarly to the method used in (28) we can first choose \( \eta_{5} < \min(\eta_{1}, \eta_{2}, \eta_{3}, \eta_{4}) \) such that for all \( \eta < \eta_{5} \)

\[
|u|_{1} < \frac{\mu_{0}}{4C}. 
\]  \hspace{1cm} (2.5.58)

**Remark 2.5.1.** Recall that \( |u|_{1} \leq \eta \) and that for convenience we are choosing \( \eta \) the bound on \( |u|_{1} \) instead of choosing \( \varepsilon \), the bound on \( |u^{0}|_{1} \), but from (2.2.13) it is equivalent. Hence the previous only means choosing \( \varepsilon_{2} > 0 \) small enough, and such that for all \( \varepsilon < \varepsilon_{2} \)

\[
|u(0, \cdot)|_{1} < \frac{\mu_{0}}{4C_{1}(T)C}. 
\]  \hspace{1cm} (2.5.59)

where \( C_{1}(T) \) is the constant defined in (2.2.13).

Therefore from (2.5.56) and (2.5.58)

\[
\frac{dW_{p}}{dt} \leq -\frac{\mu_{0}}{4} W_{p}(t), \hspace{0.5cm} \forall p \geq p_{5}. 
\]  \hspace{1cm} (2.5.60)

Thus, using Gronwall Lemma, one has, for any \( p \geq p_{5} \) and any \( 0 \leq t' \leq t \leq T \),

\[
W_{p}(t) \leq W_{p}(t') e^{-\frac{\mu_{0}}{4}(t-t')}. 
\]  \hspace{1cm} (2.5.61)

Then, by definitions of \( W_{p} \) and \( V \)

\[
\lim_{p \to +\infty} W_{p}(t) = V^{2}(t), \hspace{0.5cm} \forall t \in [0, T], 
\]  \hspace{1cm} (2.5.62)

Therefore

\[
V(t) \leq V(t') e^{-\frac{\mu_{0}}{4}(t-t')}, \hspace{0.5cm} \forall 0 \leq t' \leq t \leq T. 
\]  \hspace{1cm} (2.5.63)

Therefore \( V \) is a basic \( C^{1} \) Lyapunov function with the associated constants \( \gamma = \frac{\mu_{0}}{4} \) and \( \eta = \eta_{5} \). \( \square \)
Proof of Theorem 2.4.2

Proof. The sufficient way is simply proven by using Theorem 2.4.1 with \( G \equiv 0 \) for instance. We are left with proving the necessary way. Let us suppose that there exists a basic \( C^1 \) Lyapunov function \( V \) induced by coefficients \( (f_1, \ldots, f_n) \) and \( \gamma \) and \( \eta_1 \) the constants associated such that \( V \) is a Lyapunov function for all \( u \) smooth solution that satisfies the compatibility conditions and such that \( |u_0| < \eta_1 \). Suppose now by contradiction that the system \([2.4.4]\) does not admit a solution \((g_1, \ldots, g_n)\) on \([0, L]\) such that for all \( i \in [1, n] \), \( g_i > 0 \). Then there exist \( x_0 \in [0, L] \) and \( i_0 \in [1, n] \) such that

\[
- \lambda_{i_0}(x_0)f'_{i_0}(x_0) < 2 \sum_{k=1, k \neq i_0}^n |M_{ik}(0, x_0)| \frac{f_{i_0}(x_0)}{\sqrt{f_k(x_0)}} - 2M_{i_0i_0}(0, x_0)f_{i_0}(x_0),
\]

(2.5.64)
as, if not, \((f_1, \ldots, f_n)\) would be a solution on \([0, L]\) to \([2.4.4]\) with \( f_i > 0 \), for all \( i \in [1, n] \). We can rewrite (2.5.64) simply as

\[
- \sum_{k=1, k \neq i_0}^n |M_{ik}(0, x_0)| \frac{f_{i_0}(x_0)}{\sqrt{f_k(x_0)}} - \frac{\lambda_{i_0}(x_0)f'_{i_0}(x_0)}{2f_{i_0}(x_0)} + M_{i_0i_0}(0, x_0) < 0.
\]

(2.5.65)

For simplicity we can assume without losing any generality that \( i_0 = 1 \). By continuity there exists \( \varepsilon > 0 \) such that \([2.5.65]\) is true on \([x_0 - \varepsilon, x_0 + \varepsilon] \cap [0, L] \). We actually can suppose without loss of generality that \( x_0 \in (0, L) \) and that \([x_0 - \varepsilon, x_0 + \varepsilon] \subseteq (0, L) \).

Then we take \( u_0^i = (-\eta_2, \eta_2) \) positive, where \( \eta_2 \) is a positive constant arbitrary so far, and define the vector \( u^0 \) by

\[
u_0^i := -u_0^i \left(1 - \frac{1}{k}\right) \text{sgn}(M_{i1}(0, x_0)), \quad \forall i \neq 1,
\]

(2.5.66)

where \( k \in \mathbb{N}^* \) is arbitrary and \( \text{sgn}(0) = 0 \). As the system is strictly hyperbolic, \( \min(|\lambda_i(x_0)|) \) is achieved at most for two \( i \in [1, n] \). If so, we denote \( i_0 \) and \( i_1 \) the corresponding index, and if \( i_0 \neq 1 \) and \( i_1 \neq 1 \) we can redefine \( u_0^i \) by

\[
u_0^i := -u_0^i \left(1 - \frac{1}{k_2}\right) \text{sgn}(M_{i_0}(0, x_0)),
\]

(2.5.67)

where \( k_2 \in \mathbb{N}^* \) with \( k_2 > k \). The goal of this redefinition is that in both cases we can choose \( k \) large enough so that

\[
(i \neq i_0) \Rightarrow \left| \frac{u_0^i}{\lambda_i(x_0)} \right| < \left| \frac{u_0^0}{\lambda_{i_0}(x_0)} \right|.
\]

(2.5.68)

We now define the initial condition by

\[
u_i(0, x) := \frac{u_0^i}{m} \chi(x) \frac{e^{-m(x-x_0)-c}}{\lambda_i(x)\sqrt{f_i(x)}},
\]

(2.5.69)

where \( \chi : [0, L] \to \mathbb{R} \) is a \( C^\infty \) function with compact support in \((0, L)\) to be determined, such that \( |\chi|\) is independent of \( m \in \mathbb{N}^* \) which will be set large enough and \( c \) is a constant independent from \( m \), also to be determined. In order to simplify the notations we will suppose here that \( \lambda_1 > 0 \), otherwise one only needs to replace \( e^{-m(x-x_0)-c} \) by \( e^{-\text{sgn}(\lambda_1)(m(x-x_0)+c)} \) to obtain the same result. Note here that the compatibility conditions are satisfied for this initial condition as the function and its derivatives vanish on the boundaries. From (2.5.66) and (2.5.69), we can choose \( \eta_2 \) small enough and independent of \( m \) such that \( |u(0, \cdot)| < \eta_1 \). Well-posedness of the system guarantees the existence and uniqueness of a solution \( y \) to the system \([2.2.5], [2.2.10]\) with such initial condition (see Theorem 2.2.1). For simplicity we will conduct the proof assuming that the system is linear, (i.e. \( \lambda_i u_i = \delta_i \), \( u_i(\cdot) = \delta_i \chi(\cdot) \), \( E(u, \cdot) = Id \), and \( M(u, \cdot) = M(0, \cdot) \)) although it is also not needed and is only to simplify the computations. A way to transform the proof for non-linear system is given in the Appendix (see 2.8.3).
Before going any further and selecting $\chi$, we shall first give the idea and explain our strategy. We want to select $\chi$ such that $|\sqrt{f_i}\partial_1u_1(0,\cdot),\ldots,\sqrt{f_n}\partial_1u_n(0,\cdot)|_0$ is achieved for $i = 1$ and $x = x_1$ close to $x_0$ and only for such $i$ and $x_1$. We also want $d/dt|\sqrt{f_i}\partial_1u_1(0,\cdot),\ldots,\sqrt{f_n}\partial_1u_n(0,\cdot)|_0(0)$ to exist and to be $O\left(d/dt\left(\sqrt{f_1}\partial_1u_1(0,x_0)/m\right)\right)$ such that $dV/dt(0)$ will exist and its sign will be given by the sign of $\sqrt{f_1}\partial_1^2u_1(0,x_0)$. Then we will show that this sign is positive.

Now let us select $\chi$ in order to achieve these goals. Rephrasing our first objective, we want that for all $i \neq 1$

$$\sqrt{f_i}(x_1) \left| \lambda_1(x_1)\partial_1u_1(0,x_1) + \sum_{j=1}^{n} M_{1j}u_j(0,x_1) \right| > \sup_{x \in [0,L]} \left( \sqrt{f_i}(x) \left| \lambda_1(x)\partial_1u_1(0,x) + \sum_{j=1}^{n} M_{1j}u_j(0,x) \right| \right),$$

(2.5.70)

while the maximum of $\sqrt{f_1}|\lambda_1\partial_1u_1(0,\cdot) + \sum_{j=1}^{n} M_{1j}u_j(0,\cdot)|$ is achieved only in $x_1$, close to $x_0$.

We search $\chi$ under the form

$$\chi = \phi(m(x-x_0)), \quad (2.5.71)$$

where $\phi$ is a positive $C^\infty$ function with compact support. And we search $\chi$ such that all the $|\sqrt{f_i}\partial_1u_i(0,\cdot)|$ admit their maximum at a single point in a small neighbourhood of $x_0$. In that case note that from (2.5.66) we would indeed get that for $m$ large enough $|\sqrt{f_i}\partial_1u_i(0,\cdot),\ldots,\sqrt{f_n}\partial_1u_n(0,\cdot)|_0$ is attained for $i = 1$ only and at a single point close to $x_0$. This will be shown rigorously later (see (2.5.70)). Now let us look at $\sqrt{f_i}\partial_1u_i(0,\cdot)$

$$\sqrt{f_i}\partial_1u_i(0,x) = -u_0^0 e^{-m(x-x_0)-c} \left[ -\chi(x) + \frac{\chi'(x)}{m} + \frac{\chi(x)\lambda_1 \sqrt{f_i}}{m} \left( \frac{1}{\lambda_1 \sqrt{f_i}} \right)' \right] + \frac{1}{m} \sum_{j=1}^{n} M_{ij} \left( \frac{u_0^j}{u_0^i} \right) \left( \sqrt{f_i} \frac{1}{\lambda_i} \chi(x) \right).$$

(2.5.72)

Using (2.5.71) and a change of variable $y = m(x-x_0)$, (2.5.72) becomes

$$\sqrt{f_i}\partial_1u_i(0,x) = -u_0^0 e^{-y-c} \left[ -\phi(y) + \phi'(y) \right] + \left( g_i \left( \frac{y}{m} + x_0 \right) + \sum_{j=0}^{n} f_{ij} \left( \frac{y}{m} + x_0 \right) \right) \phi(y),$$

(2.5.73)

where $g_i$ and $f_{ij}$ are $C^2$ bounded functions on $[0,L]$ independent of $m$. This comes from the fact that $A$ and $B$ are of class $C^3$. This hypothesis, that does not appear in Theorem 2.4.1, is used to apply the implicit function theorem later on (see (2.5.77) and (2.5.82)). Theorem 2.4.2 might also be proven with lower hypothesis on the regularity $A$ and $B$, however in most physical case $A$ and $B$ are $C^3$ even when the solutions of the system are much less regular. We can see that the coefficients of the equation (2.5.73) in $\phi$ and $\phi'$ depend on $m$ and are close to be constant for large $m$. One can show that there exists a function $\psi_0$ such that $\psi_0 \in C^3((-1,1))$, such that $|\psi_0(y) - \psi_0'(y)|e^{-y}$ has a unique maximum on $[-1,1]$ which is 1, and such that the second derivative of $|\psi_0(y) - \psi_0'(y)e^{-y}|$ does not vanish in this point, i.e. there exists a unique $y_1 \in (-1,1)$ such that

$$|\psi_0(y) - \psi_0'(y)|e^{-y} \leq 1 = |\psi_0(y_1) - \psi_0'(y_1)|e^{-y_1}, \quad \forall y \in [-1,1] \setminus \{y_1\},$$

(2.5.74)

$$\left( \psi_0 - \psi_0' \right)'(y_1) \neq 0. \quad (2.5.75)$$

The existence of this function $\psi_0$ is shown in the Appendix (see 2.8.2). We set $\phi: y \rightarrow \psi_0(y + y_1)$ and $c = y_1$. Therefore

$$e^{-y-c} \left[ -\phi(y) + \phi'(y) \right] = (-\psi_0(y + y_1) + \psi_0'(y + y_1))e^{-(y+y_1)},$$

(2.5.76)
which has a maximum absolute value for \( y = 0 \) with value equal to 1. Hence, there exists \( m_1 > 0 \) such that for all \( m > m_1 \) and all \( i \in [1, n] \)

\[
\exists ! x_i \in [x_0 - \varepsilon, x_0 + \varepsilon] : |\sqrt{f_i(x)} \partial_{t} u_i(0, x_i)| = \sup_{[0,L]} (|\sqrt{f_i} \partial_{t} u_i(0, \cdot)|),
\]

\[
- u_i^0 - \frac{C_i}{m} |u_i^0| \leq \sqrt{f_i(x)} \partial_{t} u_i(0, x_i) \leq - u_i^0 + \frac{C_i}{m} |u_i^0|,
\]

where \( C_i \) are constants that do not depend on \( m \). The unicity in (2.5.77) comes from the condition which ensures that the maximum stays unique when the function is slightly perturbed. We can actually replace \( C_i \) by \( C = \max_i(C_i) > 0 \). Therefore, there exists \( m_2 > m_1 \) such that for all \( m > m_2 \) and \( i \in [2, n] \)

\[
\sup_{[0,L]} (|\sqrt{f_i} \partial_{t} u_i(0, \cdot)|) \leq (1 - \frac{1}{k}) \left( 1 + \frac{C}{m} \right) u_i^0 < u_i^0 \left( 1 - \frac{C}{m} \right) \leq \sup_{[0,L]} (|\sqrt{f_i} \partial_{t} u_i(0, \cdot)|).
\]

Hence, as we announced earlier,

\[
|\sqrt{f_i} \partial_{t} u_i(0, \cdot), ..., \sqrt{f_n} \partial_{t} u_n(0, \cdot)|_0 = |\sqrt{f_i} \partial_{t} u_i(0, x)| \iff i = 1, x = x_1.
\]

Hence, as \( u_i^0 > 0 \) and from (2.5.77) and (2.5.78),

\[
|\sqrt{f_i} \partial_{t} u_i(0, \cdot), ..., \sqrt{f_n} \partial_{t} u_n(0, \cdot)|_0 = -\sqrt{f_1(x_1)} \partial_{t} u_1(0, x_1).
\]

Therefore, as the maximum is unique and the inequality of (2.5.79) is strict, and from (2.5.75) and the implicit function theorem, provided that \( m \) is large enough there exist \( t_1 > 0 \) and \( x_a \in C^1([0,t_1];[0,L]) \) such that

\[
|\sqrt{f_i} \partial_{t} u_1(t, \cdot), ..., \sqrt{f_n} \partial_{t} u_n(t, \cdot)|_0 = -\sqrt{f_1(x_a(t))} \partial_{t} u_1(t, x_a(t)), \forall t \in [0,t_1],
\]

\[
x_a(0) = x_1.
\]

We seek now to obtain a similar relation for \( |\sqrt{f_1} u_1(t, \cdot), ..., \sqrt{f_n} u_n(t, \cdot)|_0 \). One can show that it is possible to find \( \psi_0 \) that satisfies the previous hypothesis (2.5.74) and (2.5.75) and such that in addition, there exists \( y_2 \in [-1,1] \) such that

\[
|\psi_0(y)| e^{-y} < |\psi_0(y_2)| e^{-y_2}, \forall y \in [-1,1] \setminus \{y_2\},
\]

\[
|\psi_0(y_2) - \psi_0'(y_2)| > 0,
\]

and such that there exists \( m_3 > 0 \) such that for all \( m > m_3 \), if \( \sup_{y \in [-1,1]} (|\psi_0(y + y_1)| e^{-\frac{\psi_0(y + y_1)}{\lambda_c(x + y_2)}}) \) is achieved in \( y_m \in [-1,1] \), then

\[
|\psi_0(y_m + y_1) - \psi_0'(y_m + y_1)| > c_1,
\]

where \( c_1 \) is a positive constant that does not depend on \( m \). The example of \( \psi_0 \) provided in the Appendix is suitable. Thus with \( h_i(l, y) = \frac{u_i}{\lambda_c(y + x_2)} \phi(y) e^{-y - y_2} \) one has:

\[
\partial_{y} h_i(0, y_2 - y_1) = 0.
\]

Note that from (2.5.83), \( \psi_0(y_2) = \psi_0'(y_2) \), thus from (2.5.84)

\[
|\partial_{y} h_i(0, y_2 - y_1)| > 0.
\]

Therefore from the implicit function theorem, there exists \( m_4 > m_3 \) such that for all \( m > m_4 \) and each \( i \in [1, n] \) there exists a unique \( y_i \in [-1 - y_1, 1 - y_1] \) such that

\[
\partial_{y} h_i \left( \frac{1}{m}, y_i \right) = 0,
\]

\[
|y_i - (y_2 - y_1)| \leq \frac{C_a}{m},
\]
where $C_a$ is a constant independent of $m$. From (2.5.68) there exists $m_5 > m_4$ such that for all $m > m_5$,

$$| \frac{u^0_i}{\lambda_i \left( \frac{n}{m} + x_0 \right)} C_b < \frac{u^0_i}{\lambda_i \left( \frac{n}{m} + x_0 \right)} , \quad \forall i \neq i_0, \quad (2.5.90)$$

where $C_b > 1$ is a constant independent of $m$. From (2.5.89), we have for any $i \in [1, n]$

$$| \frac{\phi(y_i) e^{-y_i}}{\phi(y_i) e^{-y_i}} | \geq 1 - \frac{C_r}{m}, \quad (2.5.91)$$

where $C_r$ is a constant independent of $m$. Therefore there exists $m_6 > m_5$ such that for all $m > m_6$

$$\left| \frac{u^0_i}{\lambda_i \left( \frac{n}{m} + x_0 \right)} \phi(y_i) e^{-y_i} \right| < \left| \frac{u^0_i}{\lambda_i \left( \frac{n}{m} + x_0 \right)} \phi(y_i) e^{-y_i} \right| , \quad \forall i \neq i_0. \quad (2.5.92)$$

This means that for all $m > m_6$ there exists a unique $i_0 \in [1, n]$ and a unique $x_{a_0} \in [x_0 - \epsilon, x_0 + \epsilon]$ such that

$$| \sqrt{f_{i_0}(x_{a_0})} u_{i_0}(0, x_{a_0}) | = \sup_{v \in [1, n], x \in [0, L]} | \sqrt{f_{i_0}} u_{i_0}(0, \cdot) |, \quad (2.5.93)$$

Now if we denote $g(t, x) := \partial_x (\sqrt{f_{i_0}(x)} u_{i_0}(t, x) \text{sgn}(u_{i_0}(0, x_{a_0})))$, one has that

$$g(0, x_{a_0}) = 0, \quad (2.5.94)$$

hence

$$-\frac{\lambda_i'(x_{a_0})}{m \lambda_i(x_{a_0})} \chi(x_{a_0}) + \frac{\chi'(x_{a_0})}{m} = \chi(x_{a_0}). \quad (2.5.95)$$

Therefore

$$\partial_x g(0, x_{a_0}) = -\text{sgn}(\lambda_{i_0}) \frac{|u^0_{i_0}|}{m} e^{-m(x_{a_0} - x_0) - y_1} \left( \frac{1}{\lambda_{i_0}} \right)^\prime (x_{a_0}) \chi(x_{a_0}) + \frac{\lambda_{i_0}(x_{a_0})}{\lambda_i(x_{a_0})} \chi''(x_{a_0}) \frac{1}{\lambda_{i_0}(x_{a_0})}$$

$$+ 2 \chi'(x_{a_0}) \left( \frac{1}{\lambda_{i_0}} \right)' (x_{a_0}) + m \left( \frac{\chi'(x_{a_0})}{\lambda_{i_0}(x_{a_0})} \right)' (x_{a_0})$$

$$-m \left( \left( \frac{1}{\lambda_{i_0}} \right)'' (x_{a_0}) \chi(x_{a_0}) - m \chi'(x_{a_0}) \frac{1}{\lambda_{i_0}(x_{a_0})} + \chi'(x_{a_0}) \frac{1}{\lambda_{i_0}(x_{a_0})} \right). \quad (2.5.96)$$

Defining $c_0 := -\text{sgn}(\lambda_{i_0}) |u^0_{i_0}|$, which is a non-zero constant, we have from (2.5.71) and the definition of $\phi$

$$\partial_x g(0, x_{a_0}) = c_0 \left( \frac{1}{\lambda_{i_0}} \right)' (x_{a_0}) \left( \psi_0'(y_{i_0} + y_1) - 2 \psi_0'(y_{i_0} + y_1) + \psi_0(y_{i_0} + y_1) + O \left( \frac{1}{m^2} \right) + O \left( \frac{1}{m} \right) \right). \quad (2.5.97)$$

Observe that, by definition, $y_{i_0}$ maximises $| \psi_0(y + y_1) e^{-y_{i_0} - y_1} \chi_0(m + x_{a_0}) |$, therefore we have from (2.5.85) and (2.5.95)

$$| \partial_x g(0, x_{a_0}) | = |c_0 m | \left( \frac{e^{-y_{i_0} - y_1}}{\lambda_{i_0} \left( \frac{n}{m} + x_0 \right)} \left( \psi_0'(y_{i_0} + y_1) - \psi_0(y_{i_0} + y_1) + O \left( \frac{1}{m} \right) \right) \right), \quad (2.5.98)$$

Hence, as the inequality (2.5.92) is strict and from the implicit function theorem, there exists $m_7 > m_6$ such that for all $m > m_7$, $x_0 \in C \left( \{0, t_2\}; \{0, L\} \right)$ and $i_0 \in [1, n]$ such that

$$| \sqrt{f_{i_0}} u_{i_0}(t, \cdot) | = \sqrt{f_{i_0}(x_0(t))} u_{i_0}(t, x_0(t)) \text{sgn}(u_{i_0}(0, x_{a_0})), \quad \forall t \in [0, t_2], \quad x_3(0) = x_{a_0}. \quad (2.5.99)$$
Hence $V$ is $C^1$ on $[0, t_3)$ where $t_3 = \min(t_1, t_2) > 0$ and, denoting $s_{a_0} := \text{sgn}(u_{io}(0, x_{a_0}))$, we have from the definition of $V$, (2.5.82) and (2.5.99)
\[
\frac{dV}{dt}(0) = -\sqrt{f_1(x_1)}\partial_t u_1(0, x_1) - \frac{\partial}{\partial x} (\sqrt{f_1}\partial_t u_1(0, \cdot))(x_1) \frac{dx_{a_0}}{dt}(0)
+ s_{a_0} \left( \sqrt{f_1(x_{a_0})}\partial_t u_{ia}(t, x_{a_0}) + \frac{\partial}{\partial x} \left( \sqrt{f_1} u_{io}(0, \cdot) \right) (x_{a_0}) \frac{dx_{b}}{dt}(0) \right).
\] (2.5.100)

But now observe that for a fixed $m$, $x_{a_0}$ is an interior maximum thus
\[
\frac{d}{dx} (\sqrt{f_1} u_{io}(0, \cdot))(x_{a_0}) = 0.
\] (2.5.101)

Also as \( \frac{d}{dx} (\sqrt{f_1} \partial_t u_1(0, \cdot))(x_1) = 0 \), we have
\[
\frac{dV}{dt}(0) = -\sqrt{f_1(x_1)}\partial^2_t u_1(0, x_1) + s_{a_0} \sqrt{f_1(x_{a_0})}\partial_t u_{io}(t, x_{a_0}).
\] (2.5.102)

Besides as $\phi$ has compact support in $[-1 - y_1, 1 - y_1]$, we have
\[
\left| e^{m(x-x_0)+y_1} \chi(x) \right| \leq e^1 \| \chi \|_\infty,
\] (2.5.103)

and the right-hand side does not depend on $m$, thus
\[
\lim_{m \to +\infty} \left| \frac{e^{m(x-x_0)+y_1}}{m} \chi(x) \right| = 0,
\] (2.5.104)

uniformly on $[0, L]$ and therefore in particular for $x_{a_0}$ (even though $x_{a_0}$ might depend on $m$). We denote
\[
V_2 := -\sqrt{f_1(x_a(t))}\partial_t u_1(t, x_a(t)).
\] (2.5.105)

Using (2.5.69) and \( \frac{d}{dx} (\sqrt{f_1} \partial_t u_1(0, \cdot))(x_1) = 0 \), we have
\[
\frac{dV_2}{dt}(0) = -\sqrt{f_1(x_1)}\partial_t^2 u_1(0, x_1)
- \sqrt{f_1(x_1)}\partial_t (-\lambda_1 \partial_x u_1(\cdot, x_1) - \sum_{j=1}^{n} M_{ij} u_j(\cdot, x_1))(0)
- \sqrt{f_1(x_1)}(-\lambda_1 \partial_x u_1(0, x_1)) - \sum_{j=1}^{n} M_{ij} \partial_t u_j(0, x_1)
- \sqrt{f_1(x_1)}(\lambda_1 \sqrt{f_1})' \partial_t u_1(0, x_1) - \sum_{j=1}^{n} M_{ij} \partial_t u_j(0, x_1)
- \sqrt{f_1(x_1)}(\lambda_1 \frac{f_1'}{2f_1} \partial_t u_1(0, x_1) - \sum_{j=1}^{n} M_{ij} \partial_t u_j(0, x_1)).
\] (2.5.106)

And from (2.5.72) and (2.5.78)
\[
\frac{dV_2}{dt}(0) = u_{i0}^0 \left( \lambda_1 \frac{f_1'}{2f_1} \left( 1 + O \left( \frac{1}{m} \right) \right) \right)
- \sum_{j=1}^{n} M_{ij}(0, x_1) \frac{u_{ji}^0}{\sqrt{f_j(x_1)}} \left( 1 + O \left( \frac{1}{m} \right) + \sqrt{f_j(x_j)} \partial_t u_j(x_j) - \sqrt{f_j(x_1)} \partial_t u_j(x_1) \right).
\] (2.5.107)
We know that if $M_{ij}(0, x_0) \neq 0$, then there exists $m_8 \in \mathbb{N}^*$ such that for all $m > m_8$, $\text{sgn}(M_{ij}(0, x_0)) = \text{sgn}(M_{ij}(0, x_1))$. We denote by $\mathcal{N}$ the subset of $j \in \{1, ..., n\}$ such that $M_{ij}(0, x_0) = 0$. Therefore from (2.5.107) and (2.5.66)

$$\frac{dV_2}{dt}(0) = u_1^0 \left( \frac{\lambda_1 f_1^1}{2f_1^1} - M_{11}(0, x_1) + \sum_{j=2, j \in \mathcal{N}^c}^n |M_{1j}(0, x_1)| \left( 1 - \frac{1}{k} \right) \frac{\sqrt{f_1}}{\sqrt{f_j}} \right)$$

$$+ O \left( \frac{1}{m} \right) + \sum_{j=0}^n C_j \left( \sqrt{T_j(x_j)} \partial_t u_j(x_j) - \frac{\sqrt{T_j(x_1)}}{u_j} \partial_t u_j(x_1) \right)$$

(2.5.108)

where $C_j$ are constants that do not depend on $m$. Now, keeping in mind (2.5.102), we are going to add $s_{aa} \sqrt{f_{10}(x_{a0})} \partial_t u_{a0}(0, x_{a0})$ to obtain $dV/dt$ at $t = 0$. But first observe that using (2.5.101) and (2.5.103)

$$\sqrt{f_{10}(x_{a0})} \partial_t u_{a0}(0, x_{a0}) = \sqrt{f_{10}(x_{a0})} (-\lambda_1 \partial_x u_{a0}(0, x_{a0}) - \sum_{j=1}^n M_{a0j} u_j(0, x_{a0}))$$

$$= \sqrt{f_{10}(x_{a0})} \left( \frac{\lambda_1}{\sqrt{f_{10}(x_{a0})}} u_{a0}^0 \chi(x_{a0}) e^{-m(x_{a0} - x_0) - y_1} \sum_{j=1}^n M_{a0j} u_j^0 \chi(x_{a0}) e^{-m(x_{a0} - x_0) - y_1} \lambda_1 \sqrt{f_{10}(x_{a0})} \right)$$

$$= O \left( \frac{1}{m} \right).$$

(2.5.109)

Therefore

$$\frac{dV}{dt}(0) = \frac{dV_2}{dt}(0) + O \left( \frac{1}{m} \right)$$

$$= u_1^0 \left( \frac{\lambda_1 f_1^1}{2f_1^1} - M_{11}(0, x_1) + \sum_{j=2, j \in \mathcal{N}^c}^n |M_{1j}(0, x_1)| \left( 1 - \frac{1}{k} \right) \frac{\sqrt{f_1}}{\sqrt{f_j}} \right)$$

$$+ O \left( \frac{1}{m} \right) + \sum_{j=0}^n C_j \left( \sqrt{T_j(x_j)} \partial_t u_j(x_j) - \frac{\sqrt{T_j(x_1)}}{u_j} \partial_t u_j(x_1) \right) + O \left( \frac{1}{m} \right).$$

(2.5.110)

And from (2.5.72) and the definition of $x_j$

$$\lim_{m \to +\infty} \left( \frac{\sqrt{T_j(x_j)} \partial_t u_j(x_j) - \sqrt{T_j(x_1)} \partial_t u_j(x_1)}{u_j} \right) = 0.$$  

(2.5.111)

Note that $x_1$ and $x_j$ both depend on $m$ and tend to $x_0$ when $m$ goes to infinity. Also we know that for all $m > m_2$, we have $x_1 \in [x_0 - \varepsilon, x_0 + \varepsilon]$. Thus from (2.5.65),

$$\lim_{m \to +\infty} \left[ \frac{\lambda_1(x_1) f_1^j(x_1)}{2f_1(x_1)} - M_{11}(0, x_1) + \sum_{j=2, j \in \mathcal{N}^c}^n |M_{1j}(0, x_1)| \left( 1 - \frac{1}{k} \right) \frac{\sqrt{f_1(x_1)}}{\sqrt{f_j(x_1)}} \right] > 0.$$  

(2.5.112)

Therefore there exists $m_9 > 0$ such that for all $m > m_9$

$$\frac{dV}{dt}(0) > 0.$$  

(2.5.113)

But we know from (2.3.14) that

$$\frac{dV}{dt}(0) \leq -\gamma V(0) < 0.$$  

(2.5.114)
Note that (2.5.114) is true as $V$ is $C^1$ in $[0,t_1)$ and from (2.3.14), for any $t \in [0,t_1)$,
\[
\frac{V(t) - V(0)}{t} \leq V(0)e^{-\gamma t} - 1
\]
which, letting $t \to 0$, gives (2.5.114) and a contradiction. This ends the proof of Theorem 2.4.2.

\section{Further details}

The previous results were derived for the $C^1$ norm but actually they can be extended to the $C^p$ norm, for $p \in \mathbb{N}^*$, with the same conditions. Namely we can extend the definition of \textit{basic $C^p$ Lyapunov function} for $p \in \mathbb{N}^*$ by replacing $V$ in Definition 2.3.3 by
\[
V(u(t,\cdot)) = \sum_{k=0}^{p} \left| \sqrt{f_k(E\partial_t^k u(t,\cdot))]_1, ..., \sqrt{f_n(E\partial_t^k u(t,\cdot))]_n} \right|_0.
\]

Defining the $p-1$ compatibility conditions as in \cite{13} at (4.136) (see also (4.137)-(4.142)), the well-posedness still holds \cite{13} and we can state:

\textbf{Theorem 2.6.1.} \textit{Let a quasilinear hyperbolic system be of the form (2.2.5), (2.2.10), with $A$ and $B$ of class $C^p$, $\Lambda$ defined as in (2.2.4) and $M$ as in (2.3.7), if}

1. \textit{(Interior condition) the system}
\[
\Lambda_i f_i' \leq -2 \left( -M_{ii}(0,x)f_i + \sum_{k=1,k\neq i}^{n} |M_{ik}(0,x)| \frac{f_i^{3/2}}{\sqrt{f_k}} \right),
\]
\textit{admits a solution $(f_1, ..., f_n)$ on $[0,L]$ such that for all $i \in [1,n]$, $f_i > 0$.}

2. \textit{(Boundary condition) there exists a diagonal matrix $\Delta$ with positive coefficients such that}
\[
\|\Delta G^2(0)\Delta^{-1}\|_\infty \leq \frac{\inf_i \left( \frac{f_i(d_i)}{\Delta_i} \right)}{\sup_i \left( \frac{L(L-d_i)}{\Delta_i} \right)},
\]
\textit{where $d_i = L$ if $\Lambda_i > 0$, and $d_i = 0$ otherwise.}

\textit{Then there exists a basic $C^p$ Lyapunov function for the system (2.2.5), (2.2.10).}

\textbf{Theorem 2.6.2.} \textit{Let a quasilinear hyperbolic system be of the form (2.2.5) with $A$ and $B$ of class $C^{p+2}$, there exists a control of the form (2.2.10) such that there exists a basic $C^p$ Lyapunov function if and only if}
\[
\Lambda_i f_i' \leq -2 \left( -M_{ii}(0,x)f_i + \sum_{k=1,k\neq i}^{n} |M_{ik}(0,x)| \frac{f_i^{3/2}}{\sqrt{f_k}} \right),
\]
\textit{admits a solution $(f_1, ..., f_n)$ on $[0,L]$ such that for all $i \in [1,n]$, $f_i > 0$.}

A proof of this is included in the Appendix (see 2.8.4).

This article therefore fills the blank about the exponential stability for the $C^p$ norm for quasilinear hyperbolic systems with non-zero source term using a Lyapunov approach, for any $p \in \mathbb{N}^*$.

We introduced the notion of \textit{basic $C^1$ Lyapunov function} that can be seen as natural Lyapunov function for the $C^1$ norm. For general quasilinear hyperbolic systems we gave a sufficient interior condition on the system and a sufficient boundary condition such that there exists a basic $C^1$ Lyapunov function that ensure exponential stability of the system for the $C^1$ norm. We also showed that the interior condition is necessary for the existence of such basic $C^1$ Lyapunov function. Therefore in some cases, there cannot exist such basic $C^1$ Lyapunov function whatever the boundary conditions are.
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2.8 Appendix

2.8.1 Bound on the derivative of \( W_{2,p} \)

2.8.1.1 Derivative of \( W_{2,p} \)

Recall that we have from (2.5.53)

\[
W_{2,p} = \left( \int_0^L \sum_{i=1}^n f_i(x) \partial_i (E u_i)_{2^p} e^{-2p\mu s, x} dx \right)^{1/2p},
\]

where \( E = E(u(t, x), x) \) given by (2.3.8)–(2.3.9) and that \( u \) satisfies the following equation

\[
u_{tt} + A(u, x)u_{tx} + \left[ \frac{\partial A}{\partial u}(u, x) \right] u_x + \frac{\partial B}{\partial u}(u, x) \mu_s = 0, \tag{2.8.1}\]

where \( \partial A/\partial u. u_t \) is the matrix with coefficients \( \sum_{k=1}^n \partial A_{ij}/\partial u_k(u, x). \partial_t u_k(t, x) \). We can again differentiate \( W_{2,p} \) with respect to time along the trajectories which are of class \( C^2 \) (recall that we are proving the estimate (2.5.56) for \( C^2 \) solutions first). Using integration by parts as previously:

\[
\frac{dW_{2,p}}{dt} = - \frac{W_{2,p}^{1-2p}}{2p} \left[ \sum_{i=1}^n \lambda_i f_i(x) \partial_i (E u_i)_{2^p} e^{-2p\mu s, x} \right]_0^L
\]

\[
- \frac{W_{2,p}^{1-2p}}{2p} \int_0^L \sum_{i=1}^n f_i(x) \partial_i (E u_i)_{2^p-1} \left( \left( E \left( D_a + \frac{\partial B}{\partial u}(u, x) \right) . u_t \right)_i \right) \]  

\[
- \left( \frac{\partial E}{\partial u}. u_t \right) + \lambda \left( \frac{\partial E}{\partial u} . u_x \right) + \lambda (\partial_x E) u_t \right)_i \right] e^{-2p\mu s, x} dx
\]

\[
+ \frac{W_{2,p}^{1-2p}}{2} \int_0^L \sum_{i=1}^n \left( \lambda_i (u, x) f_i(x) \partial_i (E u_i)_{2^p} \right) \]  

\[
e^{2p\mu s, x} \int_0^L \sum_{i=1}^n \left( \partial_x (u, x) f_i(x) \partial_i (E u_i)_{2^p} \right) e^{-2p\mu s, x} dx
\]

\[
- \mu W_{2,p}^{1-2p} \int_0^L \sum_{i=1}^n |\partial f_i(x)| \partial_i (E u_i)_{2^p} e^{-2p\mu s, x} dx,
\]

where \( D_a \) is the matrix with coefficient \( \sum_{k=1}^n (\partial A_{ik}/\partial u_k)(u_x) \), so that \( D_a . u_t = \left[ \frac{\partial A}{\partial u}(u, x) . u_x \right] u_x \). Observe that \( E \) is \( C^2 \) and invertible by definition (given by (2.3.8)–(2.3.9)), thus \( u_t = E^{-1}(E u_t) \). We can therefore denote, similarly as previously

\[
I_{21} := \frac{W_{2,p}^{1-2p} L}{2p} \left[ \sum_{i=1}^n \lambda_i f_i(x) \partial_i (E u_i)_{2^p} e^{-2p\mu s, x} \right]_0^L,
\]

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where $R = (R_{ij})_{(i,j) \in [1,n]}^2$ is defined as $R := E(D_{u} + \frac{\partial B}{\partial t}) E^{-1}$. As $E$ is $C^1$ and its inverse is continuous, and from (2.3.9), there exists a constant $C_0$ independent of $u$ (and $p$) such that

$$\max_{(i,j) \in [1,n]^2} \left| \left( \frac{\partial E}{\partial u} u_i \right) E^{-1} + \left( \frac{\partial E}{\partial u} u_x \right) E^{-1} + (\partial_x E) E^{-1} \right|_{ij} \leq C_0 |u|_1. \tag{2.8.5}$$

Note that we used (2.3.9) and the fact that $\partial_x (E(0, x)) = 0$. Thus, similarly as for (2.5.30), we have

$$\frac{dW_{2,p}}{dt} \leq -I_{31} - (\mu \alpha_0 - \frac{C_0}{2p}) W_{2,p} + C_5 W_{2,p} |u|_1, \tag{2.8.6}$$

where $C_5$ and $C_6$ are constants that do not depend on $p$ or $u$ provided that $|u|_1 < \eta$ for $\eta$ small enough but independent of $p$. Recall that $\alpha_0$ is defined in Section 2.5 right before (2.5.26). Just as previously, a sufficient condition such that there exist $p_1 \in \mathbb{N}^*$, $\eta_1 > 0$ and $\mu_1$ such that $I_{31} > 0$ for $\mu < \mu_1$, $p > p_1$ and $|u|_1 < \eta_1$ is

$$-\lambda \frac{f'_i}{f_i} > 2 \sum_{k=1, k \neq i}^n |R_{ik}(u, x)| \sqrt{\frac{f_i}{f_k}} - 2R_{ii} \tag{2.8.7}$$

But we have from the definition of $D_{u}$, (2.3.9) and (2.3.7):

$$E \left( D_{u} + \frac{\partial B}{\partial u} \right) E^{-1} = \frac{\partial B}{\partial u}(0, x) + O(|u|_1) = M(0, x) + O(|u|_1), \tag{2.8.8}$$

and recall that in the proof $(f_1, ..., f_n)$ have been selected such that

$$-\Lambda \frac{f'_i}{f_i} > 2 \sum_{k=1, k \neq i}^n |M_{ik}(0, x)| \sqrt{\frac{f_i}{f_k}} - 2M_{ii}(0, x). \tag{2.8.9}$$

Thus from (2.8.8) and (2.8.9) there exist $\eta_2 > 0$, $p_1 \in \mathbb{N}^*$ and $\mu_1$ such that if $\mu < \mu_1$, $p > p_1$ and $|u|_1 < \eta_2$, then $I_{31} > 0$. It remains to deal with $I_{21}$. As $E$ is $C^1$, and from (2.3.9),

$$(E u_i) = u_i + (u, \mathcal{V}) u_i, \tag{2.8.10}$$

where $\mathcal{V} = \mathcal{V}(u(t, x), x)$ is continuous on $B_{q_0} \times [0, L]$. Using (2.8.10) together with (2.8.3) and proceeding exactly as previously for $I_2$, we get

$$I_{21} = \frac{W_{2,p}^{1-2p}}{2p} \left( \sum_{i=1}^m \lambda_i u_i(t, L) f_i(L)^p ((u_i)_i(t, L) + ((u(t, L), \mathcal{V}) u_i(t, L)))^{2p} e^{-2\mu L} \right.\left. - \sum_{i=1}^m \lambda_i u_i(t, 0) f_i(0)^p ((u_i)_i(t, 0) + ((u(t, 0), \mathcal{V}) u_i(t, 0)))^{2p} \right.\left. - \sum_{i=m+1}^n |\lambda_i u_i(t, L)| f_i(L)^p ((u_i)_i(t, L) + ((u(t, L), \mathcal{V}) u_i(t, L)))^{2p} e^{2\mu L} \right.\left. + \sum_{i=m+1}^n |\lambda_i u_i(t, 0)| f_i(0)^p ((u_i)_i(t, 0) + ((u(t, 0), \mathcal{V}) u_i(t, 0)))^{2p} \right). \tag{2.8.11}$$

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Recall that $K = G'(0)$ and $\Delta = (\Delta_1, \ldots, \Delta_n)^T \in (\mathbb{R}^*_+)^n$ are chosen such that

$$\theta := \sup_{\|\xi\| \leq 1} \left( \sup_i \left( \sum_{j=1}^m (\Delta_i K_{ij} \Delta_j^{-1}) |\xi_j| \right) \right) < \frac{\inf_i \left( \frac{f_i(d_i)}{\Delta_i} \right)}{\sup_i \left( \frac{f_i(l_i-d_i)}{\Delta_i} \right)}.$$  

(2.8.12)

We denote again

$$\xi_i := \Delta_i(u_i)_i(t, L) \text{ for } i \in [1, m],$$  

(2.8.13)

$$\xi_i := \Delta_i(u_i)_i(t, 0) \text{ for } i \in [m+1, n].$$  

(2.8.14)

From the fact that $G$ and $u$ are $C^1$, we can differentiate (2.2.10) with respect to time, and we have

$$\left( \begin{array}{c} (u_i)_+(t, 0) \\ (u_i)_{-}(t, L) \end{array} \right) = K \left( \begin{array}{c} (u_i)_+(t, L) \\ (u_i)_{-}(t, 0) \end{array} \right) + o \left( \left( \begin{array}{c} (u_i)_+(t, L) \\ (u_i)_{-}(t, 0) \end{array} \right) \right),$$  

(2.8.15)

where $o(x)$ refers to a function such that $o(x)/|x|$ tends to 0 when $|u|_1$ tends to 0. Thus

$$I_{21} = \frac{W_{2, p}^{1-2p}}{2p} \left( \sum_{i=1}^m \lambda_i(u(t, L), L) \frac{f_i(L)^p}{\Delta_i^{2p}} (\Delta_i + o(|\xi|) e^{-2p\mu L} \right.$$  

$$+ \sum_{i=m+1}^n |\lambda_i(u(t, 0), 0)| \frac{f_i(0)^p}{\Delta_i^{2p}} (\Delta_i + o(|\xi|) e^{-2p\mu L} \right.$$  

$$- \sum_{i=1}^m \lambda_i(u(t, 0), 0) \frac{f_i(0)^p}{\Delta_i^{2p}} \sum_{k=1}^n K_{ik} \xi_k(t) \frac{\Delta_i}{\Delta_k} + o(|\xi|) e^{-2p\mu L} \right.$$  

$$- \sum_{i=m+1}^n |\lambda_i(u(t, L), L)| \frac{f_i(L)^p}{\Delta_i^{2p}} \sum_{k=1}^n K_{ik} \xi_k(t) \frac{\Delta_i}{\Delta_k} + o(|\xi|) e^{-2p\mu L} \right).$$  

(2.8.16)

We end by proceeding exactly as for $I_2$. Therefore under assumption (2.4.3), there exist $p_3$, $\mu_3$, and $\eta_3 > 0$ such that for $\mu < \mu_3$ and $|u|_1 < \eta_3$, $I_{21} < 0$. Therefore, as stated in the main text, there exist $\eta_4$, $p_5$ and $\mu$ such that for all $p > p_5$ and $|u|_1 < \eta_4$

$$\frac{dW_{2, p}}{dt} \leq -\frac{\mu\alpha_0}{2} W_{2, p} + C_7 W_{2, p} |u|_1.$$  

(2.8.17)

### 2.8.2 Existence of $\psi_0$

We want to find a function $\psi_0$ that is $C^1$ with compact support in $[-1, 1]$ such that there exists a unique $y_1 \in (-1, 1)$ such that

$$|\psi_1(y) - \psi_1(y)| e^{-y} < |\psi_1(y_1) - \psi_1(y_1)| e^{-y_1}, \quad \forall y \in [-1, 1] \setminus \{y_1\}.$$  

(2.8.18)

Let $\chi$ be a positive $C^1$ with compact support in $[-1, 1]$ such that

$$\chi \equiv 1 \text{ on } \left[ -\frac{1}{2}, \frac{1}{2} \right],$$  

$$|\chi| \leq 1 \text{ on } [-1, 1],$$  

(2.8.19)

$$|\chi'| \leq 3 \text{ on } \left[ -1, -\frac{1}{2} \right] \cup \left[ \frac{1}{2}, 1 \right],$$

and let us define $f : y \to e^{-n_1 y^2}$ where $n_1 \in \mathbb{N}^*$ will be chosen later on. We have

$$(f(y) - f'(y)) e^{-y} = e^{-n_1 y^2 - y(1 + 2n_1 y)}.$$  

(2.8.20)
Therefore
\[ |f(y) - f'(y)|e^{-y} \leq e^{-\frac{n}{2} + 1}(1 + 2n_1) \text{ on } [-1, \frac{1}{2}] \cup \left(\frac{1}{2}, 1\right]. \tag{2.8.21} \]

As \(\lim_{n \to \infty} e^{-\frac{n}{2} + 1}(1 + 2n) = 0\) we can choose \(n_1 \geq 1\) large enough such that
\[ e^{-\frac{n}{2} + 1}(1 + 2n_1) \leq \frac{1}{3}. \tag{2.8.22} \]

Now let us consider \(\psi_1 = \chi f\), one has
\[ |\psi_1(y) - \psi_1'(y)|e^{-y} = |\chi(y)f(y) - f'(y)e^{-y} - \chi'(y)f(y)e^{-y}|. \tag{2.8.23} \]

Therefore from (2.8.19), (2.8.21), and (2.8.22), we have on \([-1, \frac{1}{2}] \cup \left(\frac{1}{2}, 1\right]
\[ |\psi_1(y) - \psi_1'(y)|e^{-y} \leq \frac{1}{3} + \frac{3}{9} < 1. \tag{2.8.24} \]

As \(g : y \to |\psi_1(y) - \psi_1'(y)|e^{-y}\) has compact support on \([-1, 1]\) we can define \(d\) as
\[ d := \sup_{y \in [-1, 1]} (|\psi_1(y) - \psi_1'(y)|e^{-y}), \tag{2.8.25} \]

and \(d\) is attained in at least one point. But as \(g(0) = 1\) and as from (2.8.24) \(|g| < 1\) on \([-1, \frac{1}{2}] \cup \left(\frac{1}{2}, 1\right], \(d\) is attained only on \((-\frac{1}{2}, \frac{1}{2})\), and on \((-\frac{1}{2}, \frac{1}{2})\) we have
\[ |\psi_1(y) - \psi_1'(y)|e^{-y} = |f(y) - f'(y)e^{-y}|. \tag{2.8.26} \]

Let us show now that \(|f(y) - f'(y)e^{-y}|\) admits a unique maximum on \((-\frac{1}{2}, \frac{1}{2})\). We know that \(|f(y) - f'(y)e^{-y}|\) attains a maximum \(d \geq 1\) on \((-\frac{1}{2}, \frac{1}{2})\) and when it attains this maximum \((|f(y) - f'(y)e^{-y}|)\) vanishes, therefore
\[ e^{-n_1y^2 - y}(2n_1 - 4n_1^2y^2 - 1 - 4n_1y) = 0, \tag{2.8.27} \]

hence
\[ 4n_1^2y^2 + 4n_1y + (1 - 2n_1) = 0. \tag{2.8.28} \]

This equation has only two solutions: \(y_\pm = \frac{-1 \pm \sqrt{2n_1}}{2n_1}\) but
\[ |f(y_-) - f'(y_-)|e^{-y_-} = \sqrt{2n_1}e^{\frac{1 - 2n_1 + 1 + \sqrt{2n_1}}{4n_1}} > \sqrt{2n_1}e^{\frac{1 - 2n_1 - 1 - \sqrt{2n_1}}{4n_1}} = |f(y_+) - f'(y_+)|e^{-y_+}. \tag{2.8.29} \]

Therefore \(|f(y) - f'(y)e^{-y}|\) admits its maximum on \((-\frac{1}{2}, \frac{1}{2})\) at most one time. But we also know that it does admit a maximum on \((-\frac{1}{2}, \frac{1}{2})\), hence and from (2.8.24)
\[ \exists y_1 \in (-1, 1) : |\psi_1(y) - \psi_1'(y)|e^{-y} < |\psi_1(y_1) - \psi_1'(y_1)|e^{-y_1}, \forall y \in [-1, 1] \setminus \{y_1\}. \tag{2.8.30} \]

Now we just need to normalize the function and define \(\psi_0 := \frac{1}{y}\psi_1\) where \(d\) is given in (2.8.25) to obtain the desired function \(\psi_0\).

Observe that this function also satisfies (2.5.83), (2.5.84) and (2.5.85): Let \(y \in (-1/2, 1/2)\), then \(\psi_0(y)e^{-y}\) is positive and one has
\[ (\psi_0(y)e^{-y + 2d})' = \frac{1}{d}(-1 - 2n_1 y)e^{-y - n_1 y^2}, \tag{2.8.31} \]

thus on \((-1/2, 1/2)\), \(|\psi_0|e^{-y + 2d}\) has a unique maximum achieved in \(y_2 = -1/2n_1\). Now let \(y \in [-1, 1] \setminus (-1/2, 1/2)\), we have from (2.8.22)
\[ |\psi_0(y)|e^{-y} \leq e^{\frac{1 - 2n_1}{d}} \leq e^{\frac{-2n_1}{d}} = \psi_0(y_2)e^{-y_2}. \tag{2.8.32} \]
Hence the function admit a unique maximum on $[-1,1]$ and (2.5.83) is verified. And from (2.8.22) we have
\[
\psi_0(y_2) - \psi_0''(y_2) = (-2n_1 + 2) > 0. 
\] (2.8.33)

This implies (2.5.84) and we are left with proving (2.5.85). Let again $y + y_1 \in [-1,1] \setminus (-1/2,1/2)$, for $i \in [1,n]$ and $m$ large enough, from (2.8.22)
\[
\left| \psi_0(y + y_1) \frac{e^{-y_1 y_1}}{\lambda_i(\frac{m}{m} + x_0)} \right| \leq e^{-\frac{n_i y_1 y_1}{\lambda_i(\frac{m}{m} + x_0)}} \left( \frac{e^{-y_1 y_1}}{\lambda_i(\frac{m}{m} + x_0)} \right) < \left| \psi_0(0) \frac{e^{-y_1 y_1}}{\lambda_i(\frac{m}{m} + x_0)} \right| ,
\] (2.8.34)

which means that
\[
\sup_{[-1,1]} \left| \psi_0(y + y_1) \frac{e^{-y_1 y_1}}{\lambda_i(\frac{m}{m} + x_0)} \right| 
\] can only be achieved on $(-1/2 - y_1,1/2 - y_1)$. But we also know that on $[-1/2,1/2]$, \( \psi_0 = d^{-1}f \). Therefore let be a \( y_m \) maximizing sup_{[-1,1]} \left| \psi_0(y + y_1) \frac{e^{-y_1 y_1}}{\lambda_i(\frac{m}{m} + x_0)} \right|, \) we know that \( y_m \) exists as $[-1,1]$ is a compact, that \( y_m \) is an interior maximum and we have
\[
\partial_y(e^{-n_i(y + y_1)^2} \frac{e^{-y_1 y_1}}{d\lambda_i(\frac{m}{m} + x_0)})(y_m) = 0.
\] (2.8.35)

Hence
\[
2n_1 \left( (y_m + y_1) + \frac{\lambda_i(\frac{m}{m} + x_0)}{m\lambda_i(\frac{m}{m} + x_0)} \right) \frac{e^{-n_i(y_m + y_1)} - e^{-n_i y_m - y_1}}{d\lambda_i(\frac{m}{m} + x_0)} = 0,
\] (2.8.36)

thus
\[
(y_m + y_1) = -\frac{1}{2n_1} - \frac{\lambda_i(\frac{m}{m} + x_0)}{(2n_1)m\lambda_i(\frac{m}{m} + x_0)}.
\] (2.8.37)

All it remains to show is that for $m$ large enough we have (2.5.85). Let us compute \( \psi_0''(y_m + y_1) \)
\[
\psi_0''(y_m + y_1) = d^{-1}f''(y_m + y_1) = f(y_m + y_1)(-2n_1 + 4n_1^2(y_m + y_1)^2)
\]
\[
= f(y_m + y_1)(-2n_1 + 1 + \left( \frac{\lambda_i(\frac{m}{m} + x_0)}{m\lambda_i(\frac{m}{m} + x_0)} \right)^2 + \left( \frac{\lambda_i(\frac{m}{m} + x_0)}{m\lambda_i(\frac{m}{m} + x_0)} \right)).
\] (2.8.38)

Therefore there exists \( m_3 > 0 \) such that for all \( m > m_3, \)
\[
|\psi_0''(y_m + y_1) - \psi_0(y_m + y_1)| > e^{-\frac{m}{2}}(2n_1 - 3),
\] (2.8.39)

and as we chose \( n_1 \) large enough, \( C := e^{-\frac{m}{2}}(2n_1 - 3) > 0. \) This ends the proof of the existence of \( \psi_0. \)

2.8.3 Adapting proof of Theorem 2.4.2 in the nonlinear case

For all \( u(0, \cdot) \in \mathcal{B}_n, \) we can still define
\[
u_i(0, x) = \frac{\nu_0}{m} \chi(x) e^{-m(x-x_0)-y_1}. \] (2.8.40)

which is the analogous of (2.5.69) in the proof of Theorem 2.4.2. If there are two index \( i_0 \) and \( i_1 \) such that \( \min(|\Lambda_i(x_0)|) \) is achieved we can still redefine \( u_i^0 \) as in (2.5.67). Observe then that if (2.5.64) is satisfied, then there exists \( \eta_3 > 0 \) such that if \( |u| < \eta_3 \) then
\[
- \lambda_i(u, x_0)f''(x_0) < 2 \sum_{k=1,k \neq i_0}^n |M_{i_0k}(u, x_0)| \frac{\eta_3^2}{f''(x_0)} - 2M_{i_0i_0}(u, x_0)f''(x_0). \] (2.8.41)
From (2.8.40), (2.5.72) becomes

\[ \sqrt{f_i} \partial_i u_i(0, x) = - u_i^0 e^{-m(x-x_0) - y_1} \left[ - \chi(x) + \frac{\chi'(x)}{m} + \frac{\chi(x) \lambda_t}{m} \left( \frac{1}{\lambda_t \sqrt{f_i}} \right)' \right] \]

\[ + \frac{1}{m} \sum_{j=1}^{n} M_{ij}(u, x) \left( \frac{u_j^0}{u_i^0} \right) \left( \frac{1}{\sqrt{f_j}} \right) \frac{1}{\lambda_j} \chi(x) \]

\[ + \frac{n}{m} \sum_{j=1}^{n} (V_{ij}(u(x), u(0, x))) \left( \frac{u_j^0}{u_i^0} \right) \left( \frac{1}{\sqrt{f_j}} \right) \frac{1}{\lambda_j} \chi(x) \left[ - \chi(x) + \frac{\chi'(x)}{m} + \frac{\chi(x) \lambda_j}{m} \left( \frac{1}{\lambda_j \sqrt{f_j}} \right)' \right]. \]

(2.8.42)

where \( V_{ij} \) are \( C^2 \) functions as we assume that \( A \) is of class \( C^3 \). Therefore

\[ \sqrt{f_i} \partial_i u_i(0, x) = - u_i^0 e^{-m(x-x_0) - y_1} \left[ - \phi(y) \left( \frac{1}{m} g_i(u(0, y/m + x_0), y/m + x_0) \right) \right. \]

\[ \left. + \phi'(y) \left( \frac{1}{m} h_i(u(0, y/m + x_0), y/m + x_0) \right) \right]. \]

(2.8.44)

Now, after the change of variable (2.5.71), one has

\[ \sqrt{f_i} \partial_i u_i(0, x) = - u_i^0 e^{-y-y_1} \left[ - \phi(y) \left( \frac{1}{m} g_i(0, y/m + x_0), y/m + x_0) \right) \right. \]

\[ \left. + \phi'(y) \left( \frac{1}{m} h_i(0, y/m + x_0), y/m + x_0) \right) \right]. \]

(2.8.45)

where \( Z(u, x) = (Z_{ij})_{(i,j) \in [1,n]^2} := u \cdot V(u, x), \) with \( V \) given by (2.8.10). In addition one also has

\[ u_i(0, \frac{y}{m} + x_0) = \frac{\phi(y)}{\Lambda_i(\frac{y}{m} + x_0) \sqrt{f_i(\frac{y}{m} + x_0)}}, \quad \forall i \in [1,n]. \]

(2.8.46)

Thus, the function \( y \to u(0, y/m + x_0) \) is \( O(1/m) \) in the \( C^2 \) norm, which means that \( g_i(u(0, y/m + x_0), y/m + x_0) \) and \( g_i(u(0, y/m + x_0), y/m + x_0) \) are \( O(1) \) in the \( C^2 \) norm when \( m \) tends to \( +\infty \). Similarly, \( Z \) is a \( C^2 \)
function as $E$ is a $C^3$ function (recall that $A$ is $C^3$ for Theorem 2.4.2, and there exists a constant $C$ independent of $u$ and $m$ such that $\max_{(x,y)\in [1,n]^2} |Z_j(x,u(0,y/m+x_0),0,y/m+x_0)| \leq C|u(0,y/m+x_0)|$. This, with (2.8.46), implies that the terms which involves $Z$ in (2.8.43) are all $O(1/m)$ in the $C^2$ norm. Therefore we can process similarly as previously for the existence of $(x_i)_{i=1,n}$, $t_1$ and $x_a \in C^4([0,t_1])$ such that

$$V_2(t) = |\sqrt{f_1}E\partial_t u_1(t,\cdot), \sqrt{f_2}E\partial_t u_2(t,\cdot)|_0 = -\sqrt{f_1}(x_a(t))(E\partial_t u_1(t,x_a(t)), \forall t \in [0,t_1], \tag{2.8.47}$$

The only thing that remains to be checked is whether we still have the existence of $x_b \in C^4([0,t_2])$ for some $t_2$ positive and independent of $m$. Existence of a unique $i_0$ and $x_{a_0} \in [x_0 - \varepsilon, x_0 + \varepsilon]$ such that

$$|\sqrt{f_1}(x_{a_0})u_{a_0}(0,x_{a_0})| = \sup_{i \in [1,n], \varepsilon \in [0,L]} |\sqrt{f_1} u_i(0,\cdot)| \tag{2.8.48}$$

is granted by the same argument as previously. As $u(0,x)$ is defined exactly as in the linear case, we still have for our choice of $\chi$

$$\partial_x g(0,x_{a_0}) \neq 0. \tag{2.8.49}$$

This implies the existence of $x_b \in C^4([0,t_2])$ for some $t_2$ positive and independent of $m$.

If we look now at the computation of $dV_2/dt(0)$ and $dV_1/dt(0)$, one has, proceeding as in Section 2.5 and using (2.3.8)

$$\frac{dV_2}{dt}(0) = -\sqrt{f_1}(x_1) \left( E\partial_t^2 u + \left( \frac{\partial E}{\partial u} \right) \partial_t u \right)_1(0, x_1)
\begin{align*}
= +\sqrt{f_1}(x_1) \left( \lambda_1(E\partial_t^2 u_1)(0, x_1) + (E \left( D_a + \frac{\partial B}{\partial u} \right) E^{-1} E\partial_t u_1)(0, x_1) \right) \\
- \sqrt{f_1}(x_1) \left( \left( \frac{\partial E}{\partial u} \right) \partial_t u \right) E^{-1} E\partial_t u_1 \right)_1(0, x_1)
\end{align*} \tag{2.8.50}$$

where $R = E(D_a + \frac{\partial B}{\partial u} E^{-1}$. Therefore from the definition of $u(0,x)$ given by (2.8.40), one has

$$\frac{dV_2}{dt}(0) = -\sqrt{f_1}(x_1) \left( -\left( \lambda_1 + l(0,0,x_1, x_1) \right) \left( \partial_x (E\partial_t u_1)(0, x_1) + \left( \frac{R(0,0, x_1, x_1)}{m} \right) E\partial_t u_1 \right)_1(0, x_1) \right)
\begin{align*}
- \sqrt{f_1}(x_1) \left( \left( \frac{\partial E}{\partial u} \right) \partial_x u + \partial_x E \right) \partial_t u_1 \right)_1(0, x_1) + \left( \left( \frac{\partial E}{\partial u} \right) \partial_t u \right) E^{-1} E\partial_t u_1 \right)_1(0, x_1)
\end{align*} \tag{2.8.51}$$

where $l$ and $v$ are bounded functions on $B_{q_i} \times [0,L]$ with a bound independent of $m$ from (2.5.103). Hence, using this together with (2.3.9) and noting that $R(0,0) = M(0, x)$,

$$\frac{dV_2}{dt}(0) = -\sqrt{f_1}(x_1) \left( \left( \frac{\partial E}{\partial u} \right) \partial_x u + \partial_x E \right) \partial_t u_1 \right)_1(0, x_1) + O(|u|^2_1)
\begin{align*}
- \sum_{j=1}^n (M_{ij}(0, x_1) + \frac{v_{ij}(0, x_1)}{m})(E\partial_t u_1)(0, x_1) + O(|u|^2_1)
\end{align*} \tag{2.8.52}$$

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where the $O$ does not depend on $u_0^0$ but only on an upper bound of $u_0^0$ (we can choose $\eta_0$ for instance). Observe that, from (2.8.40), (2.8.45), and the fact that $Z = O(1/m)$ for the $C^2$ norm, we can proceed as previously and we obtain (2.5.108) with an additional $O(|u|^2)$. Similarly as previously we can obtain

$$\sqrt{f_n(x_0)\partial_t u_n(0, x_0)} = O \left( \frac{1}{m} \right). \tag{2.8.53}$$

The rest of the proof to get (2.5.110), (2.5.112) can then be done identically as all the relations used in the proof still hold in the nonlinear case. But actually looking at (2.5.110), (2.5.112), together with (2.8.43), (2.8.46), (2.8.52), (2.8.53), there exists $a > 0$ independent of $m$ and $C$ independent of $m$ and $u_0^0$ such that for any $m > m_9$,

$$\frac{dV_1}{dt} + \frac{dV_2}{dt} \geq au_1^0 - C(|u_1^0|^2). \tag{2.8.54}$$

Thus, there exists $\eta_2 > 0$ independent of $m$ such that, for any $m > m_9$,

$$\frac{dV}{dt} > 0 \tag{2.8.55}$$

which ends the proof in the nonlinear case.

### 2.8.4 Extension of the proof to the $C^q$ norm

To be able to extend the proof for the $C^q$ norm one should first define the corresponding compatibility conditions of order $q - 1$ that are given for instance in [13] at (4.136) and see also (4.137)-(4.142). Then one only needs to realize that if we now consider the state $y = (u, \partial_t u, ..., \partial_t^{q-1} u)$, $y$ is still the solution of a quasilinear hyperbolic system of the form

$$y_t + A_1(y, x)y_x + M_1(y, x)y + C = 0 \tag{2.8.56}$$

where $|C_1|_0 = O (|u, ..., \partial_t^{q-1} u|^0)$, and where the principal matrix $A_1$ verifies

$$A_1 = \begin{pmatrix} A(u, x) & (0) & \cdots & (0) \\ (0) & A(u, x) & \cdots & (0) \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{2.8.57}$$

and is therefore block diagonal with blocks that are all $A$ as previously. Similarly $M_1(0, x)$ is also block diagonal with blocks that are all $M(0, x)$. Therefore if we consider the following functions

$$W_{k+1, p} = \left( \int_0^L \sum_{i=1}^n f_i(x)^p (E \partial_t^k u_j)^{2p} e^{-2ps_x \cdot dx} \right)^{1/2p} \tag{2.8.58}$$

for all $k \in [0, q]$, where $f_i$ are chosen as previously, and if we perform as previously (Section 2.5 and Appendix 2.8.1), we have existence of $C_k > 0$ constants, $\eta_k > 0$ and $p_k \in \mathbb{N}^*$ such that for all $p > p_k$ and $|y|_1 < \eta_k$ we have relations of the type

$$\frac{dW_{k+1, p}}{dt} \leq -\frac{\mu a_0}{2} W_{k+1, p} + C_{k+1} \sum_{i=1}^k W_{r+1, p}|y|_1. \tag{2.8.59}$$

for all $k \in [0, q]$. Thus denoting $W_p = \sum_{k=0}^q W_{k+1, p}$, there exists $C > 0$ constant, $p_l \in \mathbb{N}^*$ and $\eta_l > 0$ such that for all $p > p_l$ and $|y|_1 < \eta_l$,

$$\frac{dW_p}{dt} \leq -\frac{\mu a_0}{2} W_p + CW_p|y|_1. \tag{2.8.60}$$

and we could perform as previously to obtain the exponential decay of $V = \sum_{k=0}^q V_{k+1}$ where $V_{k+1} = |\sqrt{f_1}(\partial_t^k u)_1, ..., \sqrt{f_n}(\partial_t^k u)_n|_0$ and therefore stability for the $C^q$ norm.
Chapter 3

On boundary stability of inhomogeneous $2 \times 2$ 1-D hyperbolic systems for the $C^1$ norm.

This chapter is taken from the following article (also referred to as [107]):


Abstract. We study the exponential stability for the $C^1$ norm of general $2 \times 2$ 1-D quasilinear hyperbolic systems with source terms and boundary controls. When the propagation speeds of the system have the same sign, any nonuniform steady-state can be stabilized using boundary feedbacks that only depend on measurements at the boundaries and we give explicit conditions on the gain of the feedback. In other cases, we exhibit a simple numerical criterion for the existence of basic $C^1$ Lyapunov function, a natural candidate for a Lyapunov function to ensure exponential stability for the $C^1$ norm. We show that, under a simple condition on the source term, the existence of a basic $C^1$ (or $C^p$, for any $p \geq 1$) Lyapunov function is equivalent to the existence of a basic $H^2$ (or $H^q$, for any $q \geq 2$) Lyapunov function, its analogue for the $H^2$ norm. Finally, we apply these results to the nonlinear Saint-Venant equations. We show in particular that in the subcritical regime, when the slope is larger than the friction, the system can always be stabilized in the $C^1$ norm using static boundary feedbacks depending only on measurements at the boundaries, which has a large practical interest in hydraulic and engineering applications.

3.1 Introduction

Hyperbolic systems are widely studied, as their ability to model physical phenomena gives rise to numerous applications. The $2 \times 2$ hyperbolic systems, in particular, are very interesting at two extends: on the one hand they are the simplest systems that present a coupling, and on the other hand, by modeling the systems of two balance laws, they represent a huge number of physical systems from fluid dynamics in rivers and shallow waters [42], to road traffic [8], signal transmission, laser amplification [86], etc. In order to use these models in industrial or practical applications, the question of their stability or their possible stabilization is fundamental. While for linear 1-D systems, or nonlinear 1-D systems without source term, many results exist (see in particular [14] Section 4.5 [48] [136]) the question of the stabilization in general for $2 \times 2$ 1-D nonlinear systems has often been treated for the $H^p$ norm and only few results exist for the more natural $C^1$ (or $C^p$) norm when a source term occurs. In [106], however, were presented some results for the $C^p$ stability ($p \geq 1$) of general $n \times n$ quasilinear hyperbolic system using basic Lyapunov functions for the $C^p$ norm. In this article we consider the stability for the $C^1$ norm of $2 \times 2$ general quasilinear 1-d hyperbolic systems. We show several results and we use them to study the exponential stability of the general nonlinear Saint-
Venant equations for the $C^1$ norm. Firstly introduced in 1871 by Barré de Saint-Venant and used to model flows under shallow water approximation, the Saint-Venant equations can be derived from the Navier-Stokes equations and have been widely used in the last centuries in many areas such as agriculture, river regulation, and hydraulic electricity production. For instance they are used in Belgium for the control of the Meuse and Sambre river (see [54], [71]). Their indisputable usefulness in the field of fluid mechanics or in engineering applications makes them a well-studied example in stability theory ([14], [110], [71]) although their stability for the $C^1$ norm by means of boundary controls seems to be only known so far in the particular case when when both the slope and the friction are sufficiently small (or equivalently the size of the river is sufficiently small) [169].

We first show that the results presented in [106] can be simplified for $2 \times 2$ systems in conditions that are easier to check in practice. In particular, any $2 \times 2$ quasilinear hyperbolic system with propagation speeds of the same sign can be stabilized by means of static boundary feedback and we give here explicit conditions on the gain of the feedbacks to achieve such result. In the general case we also give a simple linear numerical criterion to design good boundary controls and estimate the limit length above which stability is not guaranteed anymore.

Then we deduce a link between the $H^p$ stability and $C^q$ stability under appropriate boundary control for any $p \geq 2$ and $q \geq 1$. In particular we give a practical way to construct a basic Lyapunov function for the $C^1$ norm from a basic quadratic Lyapunov function for the $H^2$ norm and reciprocally.

Finally, we use these results to study the $C^1$ stability of the general nonlinear Saint-Venant equations taking into account the slope and the friction. We show that when the friction is stronger than the slope the system can always be made stable for the $C^1$ norm by applying appropriate boundary controls that are given explicitly. When the slope is higher than the friction, however, there always exists a length above which the system do not admit a basic $C^1$ Lyapunov function that would ensure the stability, whatever the boundary controls are. This results is all the more interesting that it has been shown that there always exists a basic quadratic $H^2$ Lyapunov function ensuring the stability for the $H^2$ norm under suitable boundary controls (see [110]).

Nevertheless in that last case the results given in this article allow to find good Lyapunov function numerically and estimate the limit length under which the stability can be guaranteed. We provide at the end of this chapter numerical computations of this limit for the Saint-Venant equations that illustrate that for most applications the stability can be guaranteed by means of explicit static boundary feedback. This article is organised as follows: in Section 3.2 we present several properties of $2 \times 2$ quasilinear hyperbolic system, as well as some useful definitions and we review some existing results. Section 3.3 present the main results for the general case and for the particular case of the Saint-Venant equations. Section 3.4 is devoted to the proof of the results in the general case and to the link between the $H^p$ and $C^q$ stability, while the proofs of the results about the Saint-Venant equations are given in Section 3.5. Finally, we provide some numerical computations in Section 3.6 and some comments in Section 3.7.

3.2 General considerations and previous results

3.2.1 General considerations

A $2 \times 2$ quasilinear hyperbolic system can be written in the form:

$$Y_t + F(Y)Y_x + D(Y) = 0,$$

$$B(Y(t, 0), Y(t, L)) = 0.$$

As the goal of this study is to deal with the exponential stability of the system around a steady-state we assume that there exists $Y^*$ a steady-state that we aim at stabilizing. Note that this steady-state is not necessarily uniform and can potentially have large variations of amplitude. As we are looking at the local stability around this steady-state, we study $F$ and $D$ on $U = B_{Y^*, \eta_0}$, the ball of radius $\eta_0$ centered in $Y^*$ in the space of the continuous functions endowed with the $L^\infty$ norm, for some $\eta_0$ small enough to be precised. We assume that the system is strictly hyperbolic around $Y^*$ with non vanishing propagation speeds, i.e. non
vanishing eigenvalues of $F(Y)$, then $F(Y^*)$ is diagonalisable and denoting by $N$ a matrix of eigenvector we introduce the following change of variables:

$$u = N(x)(Y - Y^*)$$

(3.2.3)

and the system (3.2.1)-(3.2.2) is equivalent to

$$u_t + A(u,x)u_x + B(u,x) = 0,$$

(3.2.4)

$$B(N^{-1}(0)u(t,0) + Y^*(0),N^{-1}(L)u(t,L) + Y^*(L)) = 0,$$

where

$$A(u,x) = N(x)F(Y^* + N^{-1}u)N^{-1}(x),$$

(3.2.5)

$$A(0,x) = \begin{pmatrix} A_1(x) & 0 \\ 0 & A_2(x) \end{pmatrix},$$

(3.2.6)

and $B$ is given in Appendix 3.8.1. Let us assume that $F$ and $D$ are $C^1$ on $B_{\mathbb{R}^n} \times [0,L]$ (see Appendix 3.8.1). As $Y^*$ is a stationary state, one has $B(0,\cdot) \equiv 0$ and $B$ can be written:

$$B(u,x) = M(u,x).u.$$  

(3.2.7)

Therefore the system (3.2.1)-(3.2.2) is now equivalent to

$$u_t + A(u,x)u_x + M(u,x)u = 0,$$

(3.2.8)

$$B(N^{-1}(0)u(t,0) + Y^*(0),N^{-1}(L)u(t,L) + Y^*(L)) = 0.$$  

We can suppose without loss of generality that $A_1 \geq A_2$. As the system is strictly hyperbolic with non-vanishing eigenvalues we can denote by $u_+$ the components associated with positive eigenvalues, i.e. $A_1 > 0$, and $u_-$ the component associated with negative eigenvalues. We focus now on boundary conditions of the form:

$$\begin{pmatrix} u_+(0) \\ u_-(0) \end{pmatrix} = G \begin{pmatrix} u_+(L) \\ u_-(L) \end{pmatrix},$$

(3.2.9)

For the rest of the article, unless otherwise stated, we will assume that $F$, $D$ and $G$ are $C^1$ when dealing with the $C^1$ norm and that $F,D$ and $G$ are $C^2$ when dealing with the $H^2$ norm. We also introduce the associated first order compatibility condition on an initial condition $u^0$:

$$\begin{pmatrix} u_+(0) \\ u_-(0) \end{pmatrix} = G \begin{pmatrix} u_+(L) \\ u_-(L) \end{pmatrix},$$

(3.2.10)

$$\begin{pmatrix} (A(u^0(0),0)\partial_x u^0(0) + B(u^0(0),0))_+ \\ (A(u^0(L),L)\partial_x u^0(L) + B(u^0(L),L))_- \end{pmatrix} = G' \begin{pmatrix} u_+(L) \\ u_-(0) \end{pmatrix}$$

$$\times \begin{pmatrix} (A(u^0(L),L)\partial_x u^0(L) + B(u^0(L),L))_+ \\ (A(u^0(0),0)\partial_x u^0(0) + B(u^0(0),0))_- \end{pmatrix}.$$  

With these boundary conditions the incoming information is a function of the outgoing information which enables the system to be well-posed (see [140, 170] or [14] in particular Theorem 6.4).

**Theorem 3.2.1.** Let $T > 0$, there exists $\delta(T) > 0$ and $C(T) > 0$ such that for any $u_0 \in C^1([0,L])$ satisfying the compatibility conditions (3.2.10) and

$$|u_0|_1 \leq \delta,$$

(3.2.11)

the system (3.2.8)-(3.2.9) with initial condition $u_0$ has a unique maximal solution $u \in C^1([0,T] \times [0,L])$ and we have the estimate:

$$|u(t,\cdot)|_1 \leq C_1(T)|u(0,\cdot)|_1, \forall t \in [0,T],$$

(3.2.12)

where $|\cdot|_1$ is the $C^1$ norm that is recalled later on in Definition 3.2.1. Moreover if $u_0 \in H^2([0,L])$ and

$$\|u_0\|_{H^2([0,L])} \leq \delta,$$

(3.2.13)

then the solution $u$ belongs to $C^0([0,T], H^2(0,L))$.  

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3.2.2 Context and previous results

Exponential stability of $2 \times 2$ hyperbolic systems.

- In [13] (see Theorem 4.3) and [48] respectively it has been shown that when there is no source term, i.e. $M \equiv 0$, it is always possible to guarantee the exponential stability of the system (3.2.8) with boundary controls of the form (3.2.9), both for the $H^p$ and the $C^q$ norm (with $p \geq 2$ and $q \geq 1$). Moreover, when the system is linear, this is also true for the $L^2$ and $C^0$ norm.

- In [11] the authors study a linear $2 \times 2$ system and found a necessary and sufficient interior condition to have existence of quadratic Lyapunov function for the $L^2$ norm with a boundary control of the form (3.2.9) when the system (3.2.8) is linear (Theorem 3.4.1). However it is straightforward to extend this results to the existence of a basic quadratic Lyapunov function for the $H^p$ norm with $p \geq 2$ when the system (3.2.8) is nonlinear (see Theorem 3.4.2). At it is mentioned in [11] the existence of a basic quadratic Lyapunov function for the $H^p$ norm implies the exponential stability of the system in the $H^p$ norm.

- In [106] the author gives a necessary and sufficient condition on (3.2.8) such that there exists a basic $C^1$ Lyapunov function with a boundary control of the form (3.2.9), guaranteeing therefore the stability for the $C^1$ norm of the system (3.2.8)-(3.2.9). The results can be extended with the same condition to the $C^p$ norm, with $p \geq 1$.

Exponential stability of the Saint-Venant Equations. The Saint Venant equations correspond to a system of the form (3.2.8) where the eigenvalues of $A$ satisfy $\lambda_1 \lambda_2 < 0$ when the flow is in the fluvial regime and $\lambda_1 \lambda_2 > 0$ when in the torrential regime. The stability of the Saint-Venant equations has been well-studied in the past twenty years and, to our knowledge, the most advanced contribution in the area would refer, but not exclusively, to the following:

- In [16] the authors show that when there is no slope, i.e. $C \equiv 0$, there always exists a Lyapunov function in the fluvial regime (i.e. the eigenvalues satisfy $\lambda_1 \lambda_2 < 0$) for the $H^p$ norm for the nonlinear system under boundary controls of the form (3.2.9) and they give an explicit example. In [110] the authors show that this is true even when the slope is arbitrary.

- In [21] it is found, through a time delay approach, a necessary and sufficient condition for the stability of the linearized system under proportional integral control.

- In [195] and [65, 194] the authors use a backstepping method to stabilize respectively a linear $2 \times 2$ 1-d hyperbolic systems and a nonlinear $2 \times 2$ 1-d hyperbolic systems. These results cover in particular the linearized Saint-Venant equations and the nonlinear Saint-Venant equations. However, in both cases this method gives rise to full-state feedback laws that are harder to implement in practice than static feedback laws depending only on the measurements at the boundaries.

In this article we intend to show that there always exists a Lyapunov function ensuring exponential stability in the $C^1$ (and actually $C^p$) norm under boundary controls of the form (3.2.9) when the system is in the fluvial regime and the slope is smaller than the friction. However, in the fluvial regime when the slope is larger than the friction, there exists a maximal length $L_{\text{max}}$ beyond which there never exists a basic $C^1$ Lyapunov function whatever the boundary controls are. Nevertheless, this maximal length $L_{\text{max}}$ can be estimated numerically and can be shown to be large enough to ensure the feasibility of nearly all hydraulic applications.

Notations and definitions. We recall the definition of the $C^1$ norm:
Definition 3.2.1. Let \( U \in C^0([0, L], \mathbb{R}^2) \), its \( C^0 \) norm \( |U|_0 \) is defined by:
\[
|U|_0 = \max(\|U_1\|_\infty, \|U_2\|_\infty),
\]
and if \( U \in C^1([0, L], \mathbb{R}^2) \), its \( C^1 \) norm \( |U|_1 \) is defined by
\[
|U|_1 = |U|_0 + |\partial_x U|_0.
\]

We recall the definition of exponential stability for the \( C^1 \) (resp. \( H^2 \)) norm:

**Definition 3.2.2.** The null steady-state \( u^* \equiv 0 \) of the system (3.2.8)–(3.2.9) is said exponentially stable for the \( C^1 \) (resp. \( H^2 \)) norm if there exists \( \gamma > 0, \delta > 0 \) and \( C > 0 \) such that for any \( u_0 \in C^1([0, L]) \) (resp. \( H^2([0, L]) \)) satisfying the compatibility conditions (3.2.10) and such that \( \|u_0\|_{C^1([0, L])} \leq \delta \) (resp. \( \|u_0\|_{H^2([0, L])} \leq \delta \)), the system (3.2.8)–(3.2.9) has a unique solution \( u \in C^1([0, +\infty) \times [0, L]) \) (resp. \( u \in C^2([0, +\infty) \times [0, L]) \cap C^0((0, +\infty), H^2((0, L))) \)) and
\[
\|u(t, \cdot)\|_{C^1([0, L])} \leq Ce^{-\gamma t}\|u_0\|_{C^1([0, L])}, \quad \forall t \in [0, +\infty)
\]
and
\[
\|u(t, \cdot)\|_{H^2([0, L])} \leq Ce^{-\gamma t}\|u_0\|_{H^2([0, L])}, \quad \forall t \in [0, +\infty)
\]

**Remark 3.2.1.** The exponential stability of the steady state \( u^* \equiv 0 \) of the system (3.2.8)–(3.2.9) is equivalent to the exponential stability of the steady-states \( Y^* \) (3.2.16) could in fact even be seen as a definition of the exponential stability of \( Y^* \). We see here one of the interests of the change of variables given by (3.2.3); from the stabilization of a potentially nonuniform steady-state the problem is reduced to the stabilization of a null steady-state.

We now recall the definition of two useful tools. The first one deals with the basic \( C^1 \) Lyapunov functions described in [106]:

**Definition 3.2.3.** We call basic \( C^1 \) Lyapunov function for the system (3.2.8), (3.2.9) the function \( V : C^1([0, L]) \to \mathbb{R}_+ \) defined by:
\[
V(U) = \sqrt{f_1}U_1, \sqrt{f_2}U_2|_0 + |(A(U, \cdot)U_x + B(U, \cdot))_1\sqrt{f_1}, (A(U, \cdot)U_x + B(U, \cdot))_2\sqrt{f_2}|_0,
\]
where \( f_1 \) and \( f_2 \) belong to \( C^1([0, L], \mathbb{R}_+^*) \), and such that there exists \( \gamma > 0 \) and \( \eta > 0 \) such that for any \( u \in C^1([0, L]) \) solution of the system (3.2.8), (3.2.9) with \( |u^0|_1 \leq \eta \) and for any \( T > 0 \):
\[
\frac{dV(u)}{dt} \leq -\gamma V(u),
\]
in a distributional sense on (0, T). In that case \( f_1 \) and \( f_2 \) are called coefficients of the basic \( C^1 \) Lyapunov function.

**Remark 3.2.2.** Note that for any \( u \in C^1([0, L] \times [0, T]) \) solution of (3.2.8), one has
\[
V(u(t, \cdot)) = \sqrt{f_1}u_1(t, \cdot), \sqrt{f_2}u_2(t, \cdot)|_0 + |(u_1(t, \cdot))_1\sqrt{f_1}, (u_2(t, \cdot))_2\sqrt{f_2}|_0.
\]
The previous definition (3.2.17) of \( V \) is only stated to show that \( V \) is in fact a function on \( C^1([0, L]) \) and therefore only depends on \( t \) through \( u \). Besides, one could wonder why using a weight \( \sqrt{f_i} \) instead of \( f_i \) in the definition. The goal is to facilitate the comparison with the existing definition of basic quadratic Lyapunov functions for the \( L^2 \) (resp. \( H^2 \)) norm introduced by Jean-Michel Coron and Georges Bastin in [111] and recalled below.
**Definition 3.2.4.** We call basic quadratic Lyapunov function for the $L^2$ norm (resp. for the $H^2$ norm) and for the system (3.2.8), (3.2.9) the function $V$ defined on $L^2(0,L)$ (resp. $H^2(0,L)$) by:

\[
V(U) = \int_0^L q_1 U_1^2 + q_2 U_2^2 dx
\]

(resp. $V(U) = \int_0^L q_1 U_1^2 + q_2 U_2^2 dx$)

\[
+ \int_0^L \left( (A(U,x) U_x + B(U,x)) U_1^2 q_1 + (A(U,x) U_x + B(U,x)) U_2^2 q_2 \right) dx
\]

\[
+ \int_0^L q_1 (\partial_U A.[A.U_x + B].U_x + A \frac{d}{dx} (A(U,x) U_x + B(U,x))) + \partial_U B.[A.U_x + B]) U_1^2
\]

\[
+ q_2 (\partial_U A.[A.U_x + B].U_x + A \frac{d}{dx} (A(U,x) U_x + B(U,x)) + \partial_U B.[A.U_x + B]) U_2^2 dx
\]

where $q_1$ and $q_2$ belong to $C^1([0,L], \mathbb{R}^*_+)$ and such that there exists $\gamma > 0$ and $\eta > 0$ such that for any $u \in L^2(0,L)$ (resp. $H^2(0,L)$) solution of the system (3.2.8), (3.2.9) with $|u(t)|_{L^2(0,L)} \leq \eta$ (resp. $|u(t)|_{H^2(0,L)} \leq \eta$) and any $T > 0$

\[
\frac{dV(u(t))}{dt} \leq -\gamma V(u(t)),
\]

in a distributional sense on $(0,T)$. The function $q_1$ and $q_2$ are called coefficients of the basic quadratic Lyapunov function.

**Remark 3.2.3.** As for the basic $C^1$ Lyapunov functions, note that for any $u \in C^0([0,T], H^2(0,L))$ solution to (3.2.8) the expression (3.2.20) of a basic quadratic Lyapunov function for the $H^2$ norm becomes

\[
V(U) = \int_0^L q_1 U_1^2 + q_2 U_2^2 dx
\]

\[
+ \int_0^L (u_1)^2 q_1 + (u_2)^2 q_2 dx
\]

\[
+ \int_0^L (u_1)^2 q_1 + (u_2)^2 q_2 dx
\]

which justifies the expression (3.2.20).

**Remark 3.2.4.** (Lyapunov functions and stability)

- The existence of a basic $C^1$ Lyapunov function for a quasilinear hyperbolic system implies the exponential stability for the $C^1$ norm of this system. A proof for the general case is given in [106].

- Similarly the existence a basic quadratic Lyapunov function for the $L^2$ (resp. $H^2$) norm implies the exponential stability of the system for the $L^2$ (resp. $H^2$) norm (see for instance the proof in [114] and in particular (4.50)).

Finally we introduce the following notations, useful for the rest of the article,

\[
\varphi_1 = \exp \left( \int_0^x \frac{M_{11}(0,s)}{\Lambda_1} ds \right),
\]

\[
\varphi_2 = \exp \left( \int_0^x \frac{M_{22}(0,s)}{\Lambda_2} ds \right),
\]

\[
\varphi = \frac{\varphi_1}{\varphi_2},
\]

(3.2.23)
\[ a = \varphi M_{12}(0, \cdot), \]
\[ b = (M_21(0, \cdot))/\varphi. \]  
(3.2.24)

While the function \( \varphi_1 \) and \( \varphi_2 \) represent the influence of the diagonal terms of \( M(0, \cdot) \) that would lead to an exponential variation of the amplitude on \([0, L] \) in the absence of coupling between \( u_1 \) and \( u_2 \), the function \( a \) and \( b \) represent the coupling term of \( M(0, \cdot) \) after a change of variables on the system to remove the diagonal coefficients of \( M \) (see (3.4.1) and (3.4.4)).

We can now state the main results.

### 3.3 Main results

#### 3.3.1 Stability of a general \( 2 \times 2 \) hyperbolic system for the \( C^1 \) norm

**Theorem 3.3.1.** Let a \( 2 \times 2 \) quasilinear hyperbolic system of the form (3.2.8) be such that \( \Lambda_1 \Lambda_2 > 0 \). Assume that

\[
G'(0) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \quad \text{where}
\]

\[
k_1^2 < \exp \left( \int_0^L 2 M_{11}(0,s)/|\Lambda_1| - 2 \max \left( \frac{a(s)}{\Lambda_1} \bigg| \frac{b(s)}{\Lambda_2} \right) ds \right),
\]

\[
k_2^2 < \exp \left( \int_0^L 2 M_{22}(0,s)/|\Lambda_2| - 2 \max \left( \frac{a(s)}{\Lambda_1} \bigg| \frac{b(s)}{\Lambda_2} \right) ds \right).
\]

Then there exists a basic \( C^1 \) Lyapunov function and a basic quadratic \( H^2 \) Lyapunov function. In particular, the null steady-state \( u^* \equiv 0 \) of the system (3.2.8)–(3.2.9) is exponentially stable for the \( C^1 \) and the \( H^2 \) norms.

This theorem is a direct consequence of Theorem 3.1 in [106] and will be proven in Appendix 3.8.3. From this theorem, when the eigenvalues of the hyperbolic system have the same sign, the coupling between the two equations does not raise any obstruction to the stability in the \( H^2 \) and in \( C^1 \) norm, so this case poses no challenge. We will therefore focus on the case where the eigenvalues have opposite signs, and without loss of generality we can assume that \( \Lambda_1 > 0 \) and \( \Lambda_2 < 0 \).

**Theorem 3.3.2.** Let a \( 2 \times 2 \) quasilinear hyperbolic system be of the form (3.2.8), where \( A \) and \( B \) are \( C^3 \) functions with \( \Lambda_1 > 0 \) and \( \Lambda_2 < 0 \). There exists a control of the form (3.2.9) such that there exists a basic \( C^1 \) Lyapunov function, if and only if

\[
d'_1 = \frac{|a(x)|}{\Lambda_1} d_2,
\]

\[
d'_2 = -\frac{|b(x)|}{\Lambda_2} d_1,
\]

admit a positive solution \( d_1, d_2 \) on \([0, L] \) or equivalently

\[
\eta' = \left| \frac{a}{\Lambda_1} \right| + \left| \frac{b}{\Lambda_2} \right| \eta^2,
\]

\[
\eta(0) = 0,
\]

admits a solution on \([0, L] \), where \( a \) and \( b \) are defined in (3.2.24). Moreover if one of the previous condition is verified and

\[
G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix}
\]

with \( k_2^2 < \varphi(L)^2 \left( \frac{d_2(L)}{d_1(L)} \right)^2 \) and \( k_1^2 < \left( \frac{d_1(0)}{d_2(0)} \right)^2 \),

(3.3.5)

where \( d_1 \) and \( d_2 \) are any positive solution of (3.3.2)–(3.3.3), then the system (3.2.8)–(3.2.9) is exponentially stable for the \( C^1 \) norm.
Remark 3.3.1. This result can be used in general to find good Lyapunov functions numerically and to estimate the limit length under which the stability is guaranteed by solving linear ODEs which are quite simple to handle.

The third equivalence together with the criterion given in [11] (recalled in Section 3.4) can be used to show a link between the $H^2$ and $C^1$ stability. This link is given in the following corollary.

Corollary 1. Let a $2 \times 2$ quasilinear hyperbolic system be of the form (3.2.8), (3.2.9), where $A$ and $B$ are $C^3$ functions and such that $\Lambda_1 > 0$ and $\Lambda_2 < 0$.

1. If there exists a basic $C^1$ Lyapunov function then there exists a boundary control of the form (3.2.9) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm. Moreover, if $M_{12}(0, \cdot)M_{21}(0, \cdot) \geq 0$, then the converse is true.

2. In particular if the system (3.2.8), (3.2.9) admits a basic $C^1$ Lyapunov function and

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \varphi(L)^2 \left( \frac{d_2(L)}{d_1(L)} \right)^2 \text{ and } k_1^2 < \left( \frac{d_1(0)}{d_2(0)} \right)^2,$$

(3.3.6)

where $d_1$ and $d_2$ are positive solutions of (3.3.2)–(3.3.3), then under the same boundary control there exists a basic quadratic Lyapunov function for the $H^2$ norm. Conversely if the system admits a basic quadratic $H^2$ Lyapunov function and $M_{12}(0, \cdot)M_{21}(0, \cdot) \geq 0$ and

$$G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \text{ with } k_2^2 < \left( \frac{\varphi(L)}{\eta(L)} \right)^2 \text{ and } k_1^2 < \eta(0)^2,$$

(3.3.7)

where $\eta$ is a positive solution of

$$\eta' = \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|} \eta^2,$$

(3.3.8)

then there exists a basic $C^1$ Lyapunov function.

Remark 3.3.2. • The existence of a positive solution to (3.3.8) is guaranteed by [11] when there exists a basic quadratic Lyapunov function for the $H^2$ norm. This result is recalled in Theorem 3.4.3.

• The converse of 1. is wrong in general. An example where the system admits a basic quadratic $H^2$ Lyapunov function but no basic $C^1$ Lyapunov function, whatever are the boundary controls, is provided in Appendix 3.8.2.

• To our knowledge the only such link that existed so far consists in the trivial case where $B \equiv 0$ and where there consequently always exists both a basic quadratic $H^2$ Lyapunov function and a basic $C^1$ Lyapunov function. This link can be in fact extended to the $H^p$ and $C^q$ stability with $p \geq 2$ and $q \geq 1$ with the same condition (see Section 6).

This theoretical link can be complemented by the following practical theorem that enables to construct basic quadratic $H^2$ Lyapunov functions from basic $C^1$ Lyapunov functions and conversely when possible.

Theorem 3.3.3. If there exists a boundary control of the form (3.2.8) such that there exists a basic $C^1$ Lyapunov function with coefficients $g_1$ and $g_2$, then for any $0 < \epsilon < \min_{[0,L]}((\varphi_2/\varphi_1)\sqrt{g_1/g_2})/L$ there exists a boundary control of the form (3.2.9) such that

$$\frac{1}{\Lambda_1} \left( \sqrt{\frac{g_2}{g_1}} \varphi_1 \varphi_2 - \varphi_1^2 \varphi_2 \epsilon Ld \right) \text{ and } \frac{1}{|\Lambda_2|} \sqrt{\frac{g_2}{g_1}} \varphi_1 \varphi_2$$

(3.3.9)
are coefficients of a basic quadratic Lyapunov functions for the $H^2$ norm, where $I_d$ refers to the identity function.

If there exists a basic quadratic $H^2$ Lyapunov function with coefficients $(q_1, q_2)$ and if $M_{12}(0, \cdot)M_{21}(0, \cdot) \geq 0$, then for all $A \geq 0$ and $\varepsilon > 0$ there exists a boundary control of the form (3.2.9) such that $g_1$ and $g_2$ defined by:

$$g_1(x) = A \exp \left( 2 \int_0^x \frac{M_{11}(0, \cdot)}{\Lambda_1} - \left| \frac{M_{12}(0, \cdot)}{\Lambda_1} \right| \sqrt{\frac{|\Lambda_1 q_1|}{|\Lambda_2 q_2|}} ds - \varepsilon x \right),$$

$$g_2 = \frac{|\Lambda_2| q_2}{\Lambda_1 q_1} g_1,$$

induce a basic $C^1$ Lyapunov function.

### 3.3.2 Stability of the general Saint-Venant equations for the $C^1$ norm

We introduce the nonlinear Saint-Venant equations with a slope and a dissipative source term resulting from the friction:

$$\partial_t H + \partial_x (HV) = 0,$$

$$\partial_t V + \partial_x \left( \frac{V^2}{2} + gH \right) + \left( \frac{kV^2}{H} - C \right) = 0,$$

where $k > 0$ is the constant friction coefficient, $g$ is the acceleration of gravity, and $C$ is the constant slope coefficient. We denote by $(H^*, V^*)$ the steady-state around which we want to stabilize the system, and we assume $gH^* - V^{*2} > 0$ such that the propagation speeds have opposite signs, i.e. the system is in fluvial regime (see [2] in particular (63)). The case where the propagation speeds have same sign raises no difficulty and is treated by Theorem 3.1. We show two results depending on whether the slope or the friction is the most influent.

**Theorem 3.3.4.** Consider the nonlinear Saint-Venant equations (3.3.12) with the boundary control:

$$h(t, 0) = b_1 v(t, 0),$$

$$h(t, L) = b_2 v(t, L),$$

such that

$$b_1 \in \left( -\frac{H^*(0)}{V^*(0)} - \frac{V^*(0)}{g} \right) \quad \text{and} \quad b_2 \in \mathbb{R} \setminus \left[ -\frac{H^*(L)}{V^*(L)}, -\frac{V^*(0)}{g} \right].$$

If $kV^{*2}(0)/H^*(0) > C$, this system admits a basic $C^1$ Lyapunov function and the steady-state $(H^*, V^*)$ is exponentially stable for the $C^1$ norm.

**Remark 3.3.3.** It could seem surprising at first that the condition (3.3.14) that appears is the same as the condition that appears for the existence of a basic quadratic $H^2$ Lyapunov function (see [110]). This is an illustration of the second part of Corollary 1.

**Theorem 3.3.5.** Consider the nonlinear Saint-Venant equations (3.3.12) on a domain $[0, L]$. If $kV^{*2}(0)/H^*(0) < C$ then:

1. There exists $L_1 > 0$ such that if $L < L_1$, there exists boundary controls of the form (3.2.9) such that the system admits a basic $C^1$ Lyapunov function and $(H^*, V^*)$ is exponentially stable for the $C^1$ norm.

2. There exists $L_2 > 0$ independent from the boundary control such that, if $L > L_2$, the system does not admit a basic $C^1$ Lyapunov function.

**Remark 3.3.4.** This last result is all the more interesting since it has been shown that for any $L > 0$ the system always admits a basic quadratic $H^2$ Lyapunov function (see [110]).
3.4 \( C^1 \) stability of a \( 2 \times 2 \) quasilinear hyperbolic system and link with basic quadratic \( H^2 \) Lyapunov functions

In this section we prove Theorem 3.3.2, Corollary 1 and Theorem 3.3.3. For convenience in the computations, let us first introduce the following change of variables to remove the diagonal coefficients of the source term:

\[
\begin{align*}
z_1(t,x) &= \varphi_1(x)u_1(t,x), \\
z_2(t,x) &= \varphi_2(x)u_2(t,x),
\end{align*}
\]

(3.4.1)

where \( \varphi_1 \) and \( \varphi_2 \) are given by (3.2.23). This change of variables can be found in [11] and is inspired from [127, Chapter 9]. Then the system (3.2.8) becomes

\[
z_t + A_2(z,x)z_x + M_2(z,x)z = 0,
\]

(3.4.2)

where

\[
A_2(0,x) = A(0,x),
\]

(3.4.3)

\[
M_2(0,x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix},
\]

(3.4.4)

with \( a \) and \( b \) given by (3.2.24), and (3.2.9) becomes:

\[
(z_+\!(0)) = G_1(z_+(L) - z_-(0)).
\]

(3.4.5)

where \( G_1 \) as the same regularity than \( G \). Showing the existence of a basic \( C^1 \) Lyapunov function (resp. a basic quadratic \( H^2 \) Lyapunov function) for the system (3.2.8)–(3.2.9) is obviously equivalent to showing the existence of a basic \( C^1 \) Lyapunov function (resp. a basic quadratic \( H^2 \) Lyapunov function) for the system (3.4.2), (3.4.5), and the stability of the steady-state \( u^* \equiv 0 \) in (3.2.8)–(3.2.9) is equivalent to the stability of the steady-state \( z^* \equiv 0 \) in (3.4.2), (3.4.5). We now state two useful Lemma that can be found for instance in [105]:

Lemme 3.4.1. Let \( n \in \mathbb{N}^+ \). Consider the ODE problem

\[
\begin{align*}
y' &= f(x,y,s), \\
y(0) &= y_0,
\end{align*}
\]

(3.4.6)

where \( y_0 \in \mathbb{R}^n \). If \( f \in C^0([0,L] \times \mathbb{R}^n \times \mathbb{R}^n) \) and is locally Lipschitz in \( y \) for any \( s \in \mathbb{R} \), then for all \( s \in \mathbb{R} \), \( (x,s) \to y_s(x) \) is continuous on \( \{(x,s) \in \mathbb{R}^2 : s \in \mathbb{R}, x \in I_s\} \).

Lemme 3.4.2. Let \( L > 0 \) and let \( g \) and \( f \) be continuous functions on \([0,L] \times \mathbb{R}^n \) and locally Lipschitz with respect to their second variable such that

\[
g(x,y) \geq f(x,y) \geq 0, \quad \forall (x,y) \in [0,L] \times \mathbb{R}^n.
\]

(3.4.7)

If there exists a solution \( y_1 \) on \([0,L] \) to

\[
\begin{align*}
y_t' &= g(x,y_1), \\
y_1(0) &= y_0,
\end{align*}
\]

(3.4.8)

with \( y_0 \in \mathbb{R}^n \), then there exists a solution \( y \) on \([0,L] \) to

\[
\begin{align*}
y_t' &= f(x,y), \\
y(0) &= y_0,
\end{align*}
\]

(3.4.9)

and in addition \( 0 \leq y \leq y_1 \) on \([0,L] \).
Let us now prove Theorem 3.3.2, which is mainly based on the results in [106].

**Proof of Theorem 3.3.2.** Let a \(2 \times 2\) quasilinear hyperbolic system be of the form (3.4.2). Using Theorem 3.2 in [106] on (3.4.2) we know that there exists a boundary control of the form of (3.2.9) such that there exists a basic \(C^1\) Lyapunov function if and only if:

\[
\begin{align*}
    f_1' &\leq -2|a(x)| \frac{f_1^{3/2}}{\Lambda_1 \sqrt{f_2}}, \\
    f_2' &\geq 2|b(x)| \frac{f_2^{3/2}}{|\Lambda_2| \sqrt{f_1}}, \\
\end{align*}
\]  

(3.4.10)

admit a solution on \([0,L]\) with \(f_1 > 0\) and \(f_2 > 0\) on \([0,L]\). But as \(f_1\) and \(f_2\) are positive this is equivalent to say that:

\[
\begin{align*}
    \left( \frac{1}{\sqrt{f_1}} \right)' &\geq \frac{|a(x)|}{\Lambda_1} \frac{1}{\sqrt{f_2}}, \\
    \left( \frac{1}{\sqrt{f_2}} \right)' &\leq -\frac{|b(x)|}{|\Lambda_2|} \frac{1}{\sqrt{f_1}}, \\
\end{align*}
\]  

(3.4.11) (3.4.12)

Denoting \(d_1 = 1/\sqrt{f_1}\) and \(d_2 = 1/\sqrt{f_2}\) and checking that \((f_1, f_2) \in \mathbb{R}^*_+\) is equivalent to \((d_1, d_2) \in \mathbb{R}^*_+\), the existence of a solution with positive components to (3.4.10) is equivalent to having a solution with positive components on \([0,L]\) to the system:

\[
\begin{align*}
    d_1' &\geq \frac{|a(x)|}{\Lambda_1} d_2, \\
    d_2' &\leq -\frac{|b(x)|}{|\Lambda_2|} d_1, \\
\end{align*}
\]  

(3.4.13)

Let us show that this is equivalent to the existence of a solution with positive components on \([0,L]\) to the system (3.3.2)-(3.3.3). One way is obvious: if there exists a solution with positive components to (3.3.2)-(3.3.3) then it is also a solution with positive components to (3.4.13). Let us show the other way: suppose that there exists a solution \((d_1, d_2)\) to (3.4.13) with positive components on \([0,L]\). Then:

\[
\left( \frac{d_1}{d_2} \right)' \geq \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|} \left( \frac{d_1}{d_2} \right)^2, \\
\]  

(3.4.14)

Hence from Lemma 3.4.2 the system:

\[
\begin{align*}
    \eta' &= \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|} \eta^2, \\
    \eta(0) &= \frac{d_1(0)}{d_2(0)}, \\
\end{align*}
\]  

(3.4.15) (3.4.16)

admits a solution on \([0,L]\). We can now define \(g_2\) as the unique solution of:

\[
\begin{align*}
    g_2' &= -\eta \frac{|b(x)|}{\Lambda_2} g_2, \\
    g_2(0) &= d_2(0) > 0, \\
\end{align*}
\]  

(3.4.17)

and \(g_1 = \eta g_2\). Thus \(g_1\) and \(g_2\) exist on \([0,L]\), and take only positive values and

\[
\begin{align*}
    g_1' &= \frac{|a(x)|}{\Lambda_1} g_2, \\
    g_2' &= -\frac{|b(x)|}{\Lambda_2} g_1. \\
\end{align*}
\]  

(3.4.18) (3.4.19)
Therefore this system admits a solution \((g_1, g_2)\) with positive components on \([0, L]\). This ends the proof of the first equivalence.

To prove the second equivalence, note from the previous that if there exists a solution to (3.4.13) with positive components on \([0, L]\) then there exists a function \(\eta\) on \([0, L]\) such that:

\[
\eta' = \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|} \eta^2,
\]

(3.4.20)

\(\eta(0) > 0\).

Therefore by comparison the system :

\[
\eta' = \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|} \eta^2,
\]

(3.4.21)

\(\eta(0) = 0\), admits a solution on \([0, L]\).

Conversely, if (3.4.21) admits a solution on \([0, L]\) then there exists \(\varepsilon > 0\) such that:

\[
\eta' = \frac{|a(x)|}{\Lambda_1} + \frac{|b(x)|}{|\Lambda_2|} \eta^2,
\]

(3.4.22)

\(\eta(0) = \varepsilon\), admits a solution \(\eta_\varepsilon\) on \([0, L]\). Defining as previously \(g_2\) the unique solution of:

\[
g_2' = -\eta_\varepsilon \frac{b(x)}{|\Lambda_2|} g_2, \\
g_2(0) = \eta_\varepsilon(0) > 0, \\
\]

(3.4.23)

and \(g_1 = \eta_\varepsilon g_2\), then \(g_1\) and \(g_2\) and \((g_1, g_2)\) is solution on \([0, L]\) of the system (3.3.2)-(3.3.3). This ends the proof of the second equivalence.

It remains now only to prove that if one of the previous conditions is verified, and if the boundary conditions \((3.2.9)\) satisfy (3.3.5), then the system (3.2.8)-(3.2.9) is exponentially stable for the \(C^1\) norm. Suppose that the system (3.3.2)-(3.3.3) admits a solution \((d_1, d_2)\) on \([0, L]\) where \(d_1\) and \(d_2\) are positive, then from the previous, (3.4.10) admits a solution \((f_1, f_2)\) on \([0, L]\) where \(f_1 = d_1^{-1/2}\) and \(f_2 = d_2^{-1/2}\) are positive. Therefore, as \(y \to y_0^{1/2}/\sqrt{g_2}\) is \(C^1\) and hence locally Lipshitz on \(\mathbb{R}_+\), and from Lemma 3.4.1 there exists \(\sigma_1 > 0\) such that for all \(0 \leq \sigma < \sigma_1\) there exists on \([0, L]\) a solution \((f_{1,\sigma}, f_{2,\sigma})\) of the system

\[
f'_{1,\sigma} = -2\frac{a(x)}{\Lambda_1} \frac{f_{1,\sigma}^{3/2}}{\sqrt{f_{2,\sigma}}} - \sigma, \\
f'_{2,\sigma} = 2\frac{b(x)}{|\Lambda_2|} \frac{f_{2,\sigma}^{3/2}}{\sqrt{f_{1,\sigma}}} + \sigma,
\]

(3.4.24)

with \(f_{1,\sigma} > 0\) and \(f_{2,\sigma} > 0\). Note that from the proof of Theorem 3.1 in [106] (see in particular (4.29),(4.37),(4.45) and note that \(K := G'(0)\), when these \(f_{1,\sigma}\) and \(f_{2,\sigma}\) exist, one only needs to show the following condition to have a basic \(C^1\) Lyapunov function:

\[
\exists \alpha > 0, \exists \mu > 0, \exists p_1 \geq 0 : \forall p \geq p_1, \\
\Lambda_1(L)f_{1,\sigma}(L)^p(z_1(t, L))^{2p}e^{-2\mu p L} - |\Lambda_2(L)|f_{2,\sigma}(L)^p \left(G'_1(0), \left(\frac{z_1(t, L)}{z_2(t, 0)}\right)^{2p} + \alpha(z_1^{2p} + z_2^{2p}) \right) \\
+ |\Lambda_2(0)|f_{2,\sigma}(0)^p z_2^{2p}(t, 0) - \Lambda_1(0)f_{1,\sigma}(0)^p \left(G'_1(0), \left(\frac{z_1(t, L)}{z_2(t, 0)}\right)^{2p} + \alpha(z_1^{2p} + z_2^{2p}) \right) > 0
\]

(3.4.25)
where $G_1$ is given by (3.4.5). As (3.4.25) only needs to be true for one particular $\sigma > 0$ and using that $f_1, \sigma$ and $f_2, \sigma$ are continuous with $\sigma$ and $f_i, \sigma = f_i$ for $i \in \{1, 2\}$, by continuity one only needs to show that:

$$\exists \alpha > 0, \exists \mu > 0, \exists p_1 \geq 0 : \forall p \geq p_1, \quad \Lambda_1(L) f_1(L) p^2 e^{-2p\mu L} - |\Lambda_2(L)| f_2(L) p (z_1(t, L))^2 e^{2p\mu L}$$

$$+ |\Lambda_2(0)| f_2(0)p^2 z_2^2(t, 0) - \Lambda_1(0)f_1(0)p^2 (z_1(t, L))^2 e^{2p\mu L} > \alpha(z_1^2 + z_2^2).$$

(3.4.26)

Now under hypothesis (3.3.5) and with the change of variables (3.4.1) we have

$$G_1'(0) = \begin{pmatrix} 0 & k_1 \\ \omega(L) & 0 \end{pmatrix}.$$

(3.4.27)

Therefore the condition (3.4.20) becomes:

$$\exists \alpha > 0, \exists \mu > 0, \exists p_1 \geq 0 : \forall p \geq p_1, \quad \Lambda_1(L)f_1(L)e^{-2p\mu L} - k_1^2 \varphi^{-2}(L)f_2(L)k_2^2 < f_1(0),$$

$$f_1(0)k_1^2 < f_2(0).$$

(3.4.29)

Therefore by continuity there exists $\alpha > 0, \mu > 0$ and $p_1 \geq 0$ such that

$$\forall p \geq p_1, \quad e^{2p\mu L}|\Lambda_2(L)|^{1/p} \varphi^{-2}(L)f_2(L)k_2^2 < f_1(0) |\Lambda_1(L)|^{1/p} e^{-2p\mu L},$$

and

$$(\Lambda_1(0))^{1/p} f_1(0)k_1^2 < f_2(0) |\Lambda_2(0)|^{1/p},$$

(3.4.30)

and therefore (3.4.28) is verified, hence the system admits a basic $C^1$ Lyapunov function and is exponentially stable for the $C^1$ norm. This ends the proof of Theorem 3.3.2.

**Remark 3.4.1.** This theorem has a theoretical interest as it gives a simple criterion to ensure the stability of the system, but it has also a numerical interest. By computing numerically $d_1$ and seeking the first point where it vanishes, one can find the limit length $L_{\max}$ above which there cannot exist a basic $C^1$ Lyapunov function and under which the stability is guaranteed. Then the coefficients of the boundary feedback control can also be designed numerically using $d_1$ and $d_2$ thus computed. Moreover, finding $d_1$ and $d_2$ only consists in solving two linear ODEs and is therefore computationally very easy to achieve. An example is given with the Saint-Venant equations in Section 3.5 to illustrate this statement. Finally the second equivalence is useful to show Corollary 4.

Before proving Corollary 4 let us first state the following theorem dealing with the stability in the $L^2$ norm of linear hyperbolic systems:

**Theorem 3.4.1 (III).** Let a linear hyperbolic system be of the form:

$$z_t + \begin{pmatrix} \Lambda_1(z) & 0 \\ 0 & \Lambda_2(z) \end{pmatrix} z_x + \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} z = 0,$$

(3.4.31)

with $\Lambda_1 > 0$ and $\Lambda_2 < 0$. There exists a boundary control of the form (3.2.9) such that there exists a basic quadratic Lyapunov function for the $L^2$ norm and for this system if and only if there exists a function $\eta$ on $[0, L]$ solution of:

$$\eta' = \frac{b}{|\Lambda_2|} \eta^2 + \frac{a}{\Lambda_1},$$

$$\eta(0) = 0.$$

(3.4.32)
Besides for any $\sigma > 0$ such that
\[
\eta'_{\sigma} = \left| \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|} \eta^2_{\sigma} \right|, \\
\eta_{\sigma}(0) = \sigma,
\] (3.4.33)
has a solution $\eta_{\sigma}$ on $[0, L]$ then
\[
G'_1(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix}
\] with $l_1 < \eta^2_{\sigma}(0)$ and $l_2 < \frac{1}{\eta_{\sigma}(L)}$, (3.4.34)
are suitable boundary conditions such that there exists a quadratic $L^2$ Lyapunov function for the system
(3.4.31), (3.4.34).

Such Lyapunov function guarantees the global exponential stability in the $L^2$ norm for a linear system under suitable boundary controls of the form (3.2.9). This result can be extended to the stability in the $H^2$ norm when the system is nonlinear, namely we have:

**Theorem 3.4.2.** Let a quasilinear hyperbolic system be of the form (3.4.2) with $\Lambda_1 > 0$ and $\Lambda_2 < 0$, where the $\Lambda_i$ are defined in (3.2.6). There exists a boundary control of the form (3.4.3) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm for this system if and only if there exists a function $\eta$ on $[0, L]$ solution of:
\[
\eta' = \left| \frac{b}{|\Lambda_2|} \eta^2 + \frac{a}{|\Lambda_1|} \right|, \\
\eta(0) = 0.
\] (3.4.35)

Besides for any $\sigma > 0$ such that
\[
\eta'_{\sigma} = \left| \frac{a}{\Lambda_1} + \frac{b}{|\Lambda_2|} \eta^2_{\sigma} \right|, \\
\eta_{\sigma}(0) = \sigma,
\] (3.4.36)
has a solution $\eta_{\sigma}$ on $[0, L]$ then
\[
G'_1(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix}
\] with $l_1 < \eta^2_{\sigma}(0)$ and $l_2 < \frac{1}{\eta_{\sigma}(L)}$, (3.4.37)
are suitable boundary conditions such that there exists a quadratic $H^2$ Lyapunov function for the system
(3.4.2), (3.4.37).

The proof of this theorem is straightforward and is given in Appendix 3.8.3. Knowing Theorem 3.4.2 we can prove Corollary 1.

**Proof of Corollary 1.** Let a $2 \times 2$ quasilinear hyperbolic system be of the form (3.4.2). Let us suppose that there exists a boundary control such that there exists a basic $C^1$ Lyapunov function. Then from Theorem 3.3.2 there exists $\eta_1$ solution on $[0, L]$ of:
\[
\eta'_1 = \left| \frac{a}{|\Lambda_1|} + \frac{b}{|\Lambda_2|} \eta^2_1 \right|, \\
\eta_1(0) = 0.
\] (3.4.38)
and from Lemma 3.4.2 there also exists $\eta$ solution on $[0, L]$ of
\[
\eta' = \left| \frac{a}{|\Lambda_1|} + \frac{b}{|\Lambda_2|} \eta^2 \right|, \\
\eta(0) = 0.
\] (3.4.39)

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Therefore, from Theorem 3.4.2 there exists a boundary control of the form (3.2.9) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm for this system.

Let us suppose now that $M_{12}(0,\cdot)M_{22}(0,\cdot) \geq 0$, then from (3.2.2) $ab \geq 0$. Thus (3.4.39) and (3.4.38) are the same equations and therefore, from Theorems 3.3.2 and Theorem 3.4.2 if there exists a boundary control of the form (3.4.5) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm, then there also exists a boundary control of the form (3.4.5) such that the system admits a basic $C^1$ Lyapunov function. This ends the proof of the first part of Corollary 1.

Let us now show the second part of Corollary 1. As previously, from (3.4.1) we only need to show the result for the equivalent system (3.4.2), (3.4.5). Observe first that from Theorem 3.4.2, for any $\sigma > 0$ such that

$$\eta_2'(0) = \left| \frac{a}{|A_1|} + \frac{b}{|A_2|} \eta_2^2 \right|,$$

(3.4.40)

has a solution $\eta_2$ on $[0, L]$, then

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix}$$

with $l_1^2 < \eta_2^2(0)$ and $l_2^2 < \frac{1}{\eta_2^2(L)}$,

(3.4.41)

are suitable boundary conditions such that there exists a basic quadratic $H^2$ Lyapunov function for the system (3.4.2), (3.4.5), where $G_1$ is given by (3.4.5). Now, let us suppose that there exists a basic $C^1$ Lyapunov function for the system (3.4.2), (3.4.5). From the first part of Corollary 1 there exists a boundary control of the form (3.4.5) such that there exists a basic quadratic Lyapunov function for the $H^2$ norm. Let us suppose that

$$G'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix}$$

with $k_2^2 < \varphi(L)^2 \left( \frac{d_2(L)}{d_1(L)} \right)^2$ and $k_1^2 < \left( \frac{d_1(0)}{d_2(0)} \right)^2$,

(3.4.42)

where $d_1$ and $d_2$ are positive solutions of (3.3.2)-(3.3.3). Note that defining $\eta_3 = d_1/d_2$ and $\sigma = \eta_3(0) = d_1(0)/d_2(0) > 0$ the condition (3.4.42) is equivalent to

$$G_1'(0) = \begin{pmatrix} 0 & l_1 \\ l_2 & 0 \end{pmatrix}$$

with $l_1^2 < \eta_3^2(0)$ and $l_2^2 < \frac{1}{\eta_3^2(L)}$.

(3.4.43)

As from (3.3.2)-(3.3.3),

$$\eta_3 = \frac{d_1}{d_2} - \frac{d_1 d_2}{d_2} = \left| \frac{a}{|A_1|} + \frac{b}{|A_2|} \eta_3^2 \right|,$$

(3.4.44)

and $\eta_3(0) = \sigma$.

then from Lemma 3.4.2 the problem (3.4.40) has a solution $\eta_2$ on $[0, L]$ and $\eta_2(L) \geq \eta_3(L)$. Therefore $G'(0)$ also satisfies (3.4.41). Hence there exists a basic quadratic Lyapunov function for the $H^2$ norm and the system is exponentially stable for the $H^2$ norm.

Let us now show the other way. Suppose that $M_{12}(0,\cdot)M_{22}(0,\cdot) \geq 0$ and that the system admits a basic quadratic Lyapunov function for the $H^2$ norm. Then from Theorem 3.4.2 and by continuity of the solutions with respect to the initial conditions there exists $\sigma > 0$ such that:

$$\eta_2'(0) = \left| \frac{a}{|A_1|} + \frac{b}{|A_2|} \eta_2^2 \right|,$$

(3.4.45)

has a solution on $[0, L]$, that we denote $\eta_2$. From hypothesis (3.3.7) there exists such $\sigma > 0$ such that the condition (3.3.7) is still satisfied with $\eta_2$. We can define

$$d_2 = \exp \left( - \int_0^x \eta_2(s) \left| \frac{b(s)}{\lambda_2(s)} \right| ds \right),$$

(3.4.46)
As \(|ab| = ab\), then \((d_1, d_2)\) is a solution of \((3.3.2)\)–\((3.3.3)\) and \(d_1 > 0\) and \(d_2 > 0\), and from \((3.3.7)\) and \((3.4.46)\)–\((3.4.47)\).

\[
d_1 = \eta_2 d_2. \tag{3.4.47}
\]

Hence from Theorem 3.3.2 there exists a basic \(C^1\) Lyapunov function. This ends the proof of Corollary 3.3.3. \(\square\)

**Proof of Theorem 3.3.3.** Let us first note from \((3.4.1)\) that \((g_1, g_2)\) are the coefficients of a basic \(C^1\) Lyapunov function for the system \((3.2.8)\). If and only if \((f_1, f_2)\) are the coefficients of a basic \(C^1\) Lyapunov for the system \((3.4.2)\) with

\[
f_i = \frac{g_i}{\varphi_i}, \quad i \in \{1, 2\}, \tag{3.4.49}
\]

Therefore, we will first prove the result for \((3.4.2)\) and then use the change of coordinates \((3.4.1)\) and the transformation \((3.4.49)\) to come back to the system \((3.2.8)\). Let a \(2 \times 2\) quasilinear hyperbolic system be of the form \((3.2.9)\) such that there exists a basic \(C^1\) Lyapunov function with coefficients \(f_1\) and \(f_2\). Then from \([108]\) (see in particular Theorem 3.2), one has:

\[
f'_1 \leq \frac{2|a(x)| f_{1}^{3/2}}{\Lambda_1} \sqrt{f_2}, \tag{3.4.50}
\]

\[
f'_2 \geq \frac{2|b(x)| f_{2}^{3/2}}{|\Lambda_2|} \sqrt{f_1}.
\]

Now let us denote \(d_1 = f_1^{-1/2}\) and \(d_2 = f_2^{-1/2}\), then

\[
d'_1 \geq \frac{|a(x)|}{\Lambda_1} d_2, \tag{3.4.51}
\]

\[
d'_2 \leq \frac{|b(x)|}{|\Lambda_2|} d_1. \tag{3.4.52}
\]

Therefore:

\[
- \left( \frac{d_1}{d_2} \right)' \left( \frac{d_2}{d_1} \right)' \geq \left( \frac{b}{\Lambda_2} \left( \frac{d_1}{d_2} \right)^2 + \frac{a}{\Lambda_1} \right) \left( \frac{b}{\Lambda_2} + \frac{a}{\Lambda_1} \right) \left( \frac{d_2}{d_1} \right)^2. \tag{3.4.53}
\]

Hence:

\[
- \left( \frac{d_1}{d_2} \right)' \left( \frac{d_2}{d_1} \right)' \geq \left( \frac{b}{\Lambda_2} \left( \frac{d_1}{d_2} \right)^2 + \frac{a}{\Lambda_1} \right) \left( \frac{d_2}{d_1} \right)^2. \tag{3.4.54}
\]

Now let us take \(\varepsilon > 0\) such that \(\varepsilon L < \min_{[0, L]} (d_2/d_1)\). Note that from \((3.4.49)\) this is equivalent to \(\varepsilon L < \min_{[0, L]} (\varphi_2/\varphi_1 \sqrt{g_1/g_2})\), where \(g_1\) and \(g_2\) are the coefficients of the basic \(C^1\) Lyapunov function for the original system \((3.2.8)\). We have:

\[
\left( \frac{d_1(x)}{d_2(x)} \right)' \geq 0 \tag{3.4.55}
\]

and

\[
- \left( \frac{d_1(x)}{d_2(x)} \right)' \left( \frac{d_2(x)}{d_1(x)} - \varepsilon x \right)' \geq \left( \frac{b(x)}{\Lambda_2(x)} \left( \frac{d_1(x)}{d_2(x)} \right)^2 + \frac{a(x)}{\Lambda_1(x)} \left( \frac{d_2(x)}{d_1(x)} - \varepsilon x \right) \right)^2. \tag{3.4.56}
\]

It can be shown (see \([11]\) or \([14]\) in particular Theorem 6.10 for more details) that this condition implies that

\[
\frac{1}{\Lambda_1} \left( \frac{d_2}{d_1} - \varepsilon Id \right) \quad \text{and} \quad \frac{d_1}{|\Lambda_2| d_2}.
\]

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are the coefficients of a basic quadratic Lyapunov function for the $H^2$ norm for the system \([3.4.2]\) for some boundary controls of the form \([3.4.5]\). Equivalently this means that, for some boundary controls of the form \([3.4.6]\),
\[
V(t) = \int_0^L \frac{1}{A_1} \left( \sqrt{\frac{f_1(x)}{f_2(x)}} - \varepsilon x \right) z_1^2(t,x) + \frac{1}{|A_2|} \sqrt{\int f_2 f_1 z_2^2(t,x)dx} \\
+ \int_0^L \frac{1}{A_1} \left( \frac{f_1(x)}{f_2(x)} \right) (\partial_t z_1)^2(t,x) + \frac{1}{|A_2|} \sqrt{\int f_2 f_1 (\partial_{tt} z_2)^2(t,x)dx} \\
+ \int_0^L \frac{1}{A_1} \left( \frac{f_1(x)}{f_2(x)} \right) (\partial_{tt} z_1)^2(t,x) + \frac{1}{|A_2|} \sqrt{\int f_2 f_1 (\partial_{tt}^2 z_2)^2(t,x)dx}
\] (3.4.57)
is a Lyapunov function for the $H^2$ norm. Therefore using \((3.4.49)\) and performing the inverse change of coordinates to go from \([3.4.2]\) to \([3.2.8]\), $V$ can also be written as
\[
V(t) = \int_0^L \frac{1}{A_1} \left( \sqrt{\frac{g_1}{g_2}} - \frac{\varphi_1 \varphi_2 \varepsilon x}{\varphi_2} \right) \varphi_1 \varphi_2 u_1^2(t,x) + \frac{1}{|A_2|} \sqrt{\int g_2 g_1 \varphi_1 \varphi_2 u_2^2(t,x)dx} \\
+ \int_0^L \frac{1}{A_1} \left( \frac{g_1}{g_2} \right) \varphi_1 \varphi_2 (\partial_t u_1(t,x))^2 + \frac{1}{|A_2|} \sqrt{\int g_2 g_1 \varphi_1 \varphi_2 (\partial_{tt} u_2(t,x))^2dx} \\
+ \int_0^L \frac{1}{A_1} \left( \frac{g_1}{g_2} \right) \varphi_1 \varphi_2 (\partial_{tt}^2 u_1(t,x))^2 + \frac{1}{|A_2|} \sqrt{\int g_2 g_1 \varphi_1 \varphi_2 (\partial_{tt}^2 u_2(t,x))^2dx},
\] (3.4.58)
where $g_1$ and $g_2$ are the coefficients of the basic $C^1$ Lyapunov function of the system \([3.2.8]\), and this concludes the proof of the first part of Theorem \([3.3.3]\).

To show the second part of Theorem \([3.3.3]\) suppose that $M_{12} M_{21} \geq 0$. Therefore from \([3.2.24]\), $ab \geq 0$. Suppose also that \((l_1,l_2)\) are the coefficients of a basic quadratic Lyapunov functions for the $H^2$ norm for the system \([3.4.2]\), \([3.4.5]\). Define $h_i = |A_i| l_i$ and
\[
f_1(x) = A \exp \left( - \int_0^x 2 \frac{|a|}{A_1} \sqrt{\frac{h_1}{h_2}} ds - \varepsilon x \right),
\] (3.4.59)
\[
f_2 = \frac{h_2}{h_1} f_1,
\] (3.4.60)
where $A > 0$ and $\varepsilon > 0$ are taken arbitrarily. We have:
\[
f_1' = -2 \frac{|a|}{A_1} \sqrt{\frac{h_1}{h_2}} f_1 - \varepsilon f_1 < -2 \frac{|a|}{A_1} \sqrt{\frac{h_1}{h_2}} f_1 = -2 \frac{|a|}{A_1} \sqrt{f_1^{3/2}},
\] (3.4.61)
and
\[
\left( \frac{h_2}{h_1} \right)'^2 = \left( \frac{h_2}{h_1} \right)' \left( \frac{h_2}{h_1} \right)' \geq 4h_2' \left( \frac{a}{A_1} + b \frac{h_2}{h_1} \right)^2.
\] (3.4.62)
Besides $l_i$ are the coefficients of a basic quadratic Lyapunov function for the $H^2$ norm for \([3.4.2]\) therefore (see [11], in particular (41)-(43)) $h_1' < 0$, $h_2' > 0$ and
\[
h_2' \left( \frac{a}{A_1} + b \frac{h_2}{h_1} \right)^2.
\] (3.4.63)
Let us denote
\[
I_1 := \left( h_2' \left( \frac{a}{A_1} + b \frac{h_2}{h_1} \right)^2 \right)^{1/2} - \left| \frac{a}{A_1} + b \frac{h_2}{h_1} \right| > 0.
\] (3.4.64)
Thus from (3.4.61), (3.4.62) and (3.4.63):

\[ f'_2 \geq 2 \frac{h_2}{h_1} \left| \frac{a}{\Lambda_1} + \frac{b}{\Lambda_2} \right| f_1 + 2 \frac{h_2}{h_1} f_1 f_1 f'_1 \frac{h_2}{h_1} \]

(3.4.65)

Assuming now that \( \varepsilon < 2 \min_{[0,L]} (f_1 \sqrt{h_1/h_2}) \), we have, as \( |ab| = ab \) and \( h_1 \) and \( h_2 \) are positive,

\[ f'_2 > 2 \frac{h_2}{h_1} \frac{|b| h_2}{|\Lambda_2| h_1} f_1 \]

(3.4.66)

Therefore from (3.4.65) and (3.4.66) and Theorem 3.1 in [105], there exists a boundary control of the form (3.4.5) such that \((f_1, f_2)\) induce a basic \( C^1 \) Lyapunov function for the system (3.4.2). Performing the inverse change of coordinates and using (3.4.1) there exists a boundary control of the form (3.2.9) such that \((f_1, f_2)\) induce a \( C^1 \) basic Lyapunov function, where

\[ g_1(x) = A \exp \left( 2 \int_0^x \frac{\langle M_{11}(0, \cdot), \cdot \rangle - |M_{12}(0, \cdot)|}{\Lambda_1} \sqrt{\frac{\langle A \rangle q_1}{|\Lambda_2| q_2}} ds - \varepsilon x \right), \]

(3.4.67)

\[ g_2 = \frac{|\Lambda_2| q_2}{|\Lambda_1| q_1} g_1, \]

(3.4.68)

and \((q_1, q_2)\) are the coefficients inducing a basic quadratic Lyapunov function for the \( H^2 \) norm for the system (3.2.8)–(3.2.9). This ends the proof of Theorem 3.3.3.

\[ \square \]

### 3.5 An application to the Saint-Venant equations

In this section we will show Theorem 3.3.4 and Theorem 3.3.5. Before proving these results, we recall some properties of the Saint-Venant equations. The steady-states \((H^*, V^*)\) of (3.3.12) are the solutions of:

\[ \partial_x (H^* V^*) = 0, \]

\[ \partial_x \left( \frac{V^{*2}}{2} + g H^* \right) = \left( C - \frac{k V^{*2}}{H^*} \right). \]

(3.5.1)

Under the assumption of physical fluvial (also called subcritical) regime, i.e. \( 0 < V^* < \sqrt{g H^*} \), these equations reduce to:

\[ H_x^* = -H^* \frac{V_x^*}{V^*}, \]

\[ V_x^* = V^* \frac{k V^{*2}}{g H^* - V^{*2}} - C. \]

(3.5.2)

and have a unique maximal solution for a given \( H^*(0) \) and \( V^*(0) \) verifying \( V^*(0) < \sqrt{g H^*(0)} \). Observe now that there are three different cases:

- \( \frac{k V^{*2}}{H^*(0)} > C \), i.e. the friction is larger than the slope. In this case \( V^* \) is an increasing function, \( H^* \) is a decreasing function and therefore the friction stays larger than the slope on the whole domain, i.e. \( k V^{*2}/H^* > C \).
Therefore, it is enough to look at the difference between the friction and the slope at the initial point:

$$h = H - H^* \text{ and } v = V - V^*.$$  \hspace{1cm} (3.5.3)

Assuming subcritical regime, i.e. $V^* < \sqrt{gH^*}$, the Saint-Venant equations (3.3.12) can be transformed using the transformation described by (3.2.1)–(3.2.2) in

$$\partial_t u + A(u, x)\partial_x u + M(u, x)u = 0,$$  \hspace{1cm} (3.5.4)

where

$$A_1 = V^* + \sqrt{gH^*},$$  \hspace{1cm} (3.5.5)

$$A_2 = V^* - \sqrt{gH^*},$$  \hspace{1cm} (3.5.6)

$$M(0, \cdot) = \frac{kV^{*2}(0)}{H(0)} \left( \frac{3}{4(\sqrt{gH^*+V^*})+\frac{1}{V^*}} + \frac{1}{V^*} - \frac{1}{2\sqrt{gH^*}} - \frac{3}{4(\sqrt{gH^*+V^*})+\frac{1}{V^*}} + \frac{1}{V^*} + \frac{1}{2\sqrt{gH^*}} \right)$$

$$- C \left( \frac{4(\sqrt{gH^*+V^*})}{3} - \frac{4(\sqrt{gH^*+V^*})}{4(\sqrt{gH^*+V^*})} \right).$$  \hspace{1cm} (3.5.7)

Observe that the system is indeed strictly hyperbolic under small perturbations as $A_1 > 0$ and $A_2 < 0$. The derivation of $A_1, A_2$ and $M(0, \cdot)$ will not be detailed here but is quite straightforward and the expression (3.5.6)–(3.5.8) can be found for instance in [11] Section 1.4.2.

Proof of Theorem 3.3.4 Let us suppose that the flow is in the fluvial regime on $[0, L]$, therefore

$$V^*(x) < \sqrt{gH^*(x)}, \forall x \in [0, L],$$  \hspace{1cm} (3.5.9)

and suppose that $kV_0^{*2}/H_0^* > C$. From the previous we have $kV^{*2}/H^* > C$ on $[0, L]$. Thus from (3.5.8) $M_{12}(0, \cdot) \geq 0$ and $M_{21}(0, \cdot) \geq 0$. Before going any further let us note that we know from [110] that there exists a basic quadratic $H^2$ function for the system (3.5.4) for some boundary controls of the form (3.4.5), therefore Corollary [1] applies and there exists a boundary control of the form (3.4.5) such that there exists a basic $C^1$ Lyapunov function for this system.

Moreover from [110] (see Lemma 3.1), we know that

$$\eta = \frac{\varphi_1|A_2|}{\varphi_2A_1},$$  \hspace{1cm} (3.5.10)

is a positive solution of:

$$\eta' = \frac{b}{|A_2|}\eta^2 + \frac{a}{A_1}.$$  \hspace{1cm} (3.5.11)
Therefore from the second part of Corollary \[4\] one has that if

\[ G'(0) = \begin{pmatrix} 0 & k_1 \\ k_2 & 0 \end{pmatrix} \]  

with \( k_1^2 < \varphi^2(L) \eta^2(L) \) and \( k_2^2 < \eta^2(0) \),

where \( G \) is given by \[3.2.9\] and \( \eta \) by \[3.5.10\], then the system \[3.5.4\], \[3.2.9\] admits a basic \( C^1 \) Lyapunov function and is exponentially stable for the \( C^1 \) norm.

It is therefore enough to show that the boundary conditions \[3.3.13\] under hypothesis \[3.3.14\] are equivalent to boundary conditions of the form \[3.2.9\] satisfying the previous condition \[3.5.12\]–\[3.5.13\] in the new system \[3.5.4\], obtained by the change of variables \[3.5.5\]. Observe that from \[3.3.13\] we have

\[ v(t,0) + h(t,0) \sqrt{g H^*(0)} = k_1 \left( v(t,0) - h(t,0) \sqrt{g H^*(0)} \right), \]

\[ v(t,L) - h(t,L) \sqrt{g H^*(L)} = k_2 \left( v(t,L) + h(t,L) \sqrt{g H^*(0)} \right), \]

where

\[ k_1 := \left( \frac{1 + \sqrt{g H^*(0)b_1}}{1 - \sqrt{g H^*(0)b_1}} \right), \]  

\[ \frac{1}{k_2} := \left( \frac{1 + \sqrt{g H^*(L)b_2}}{1 - \sqrt{g H^*(L)b_2}} \right), \]

and therefore:

\[ u_1(t,0) = k_1 u_2(t,0), \]

\[ u_2(t,L) = k_2 u_1(t,L). \]  

Therefore after the change of variables given by \[3.5.5\], the boundary conditions \[3.3.13\] are equivalent to boundary conditions of the form \[3.2.9\] satisfying \[3.5.12\]. All it remains to do is to prove that under the hypothesis \[3.3.14\], the boundary conditions \[3.5.17\] also satisfy the condition \[3.5.13\]. Now observe that from \[3.5.10\], \[3.5.15\] and \[3.5.16\], the condition \[3.5.13\] becomes:

\[
\left( \frac{1 + \sqrt{g H^*(0)b_1}}{1 - \sqrt{g H^*(0)b_1}} \right)^2 < \left( \frac{\Lambda_2(0)}{\Lambda_1(0)} \right)^2, \]

\[
\left( \frac{1 + \sqrt{g H^*(L)b_2}}{1 - \sqrt{g H^*(L)b_2}} \right)^2 > \left( \frac{\Lambda_2(L)}{\Lambda_1(L)} \right)^2, \]

which is equivalent to

\[
\left( 1 - \left( \frac{\Lambda_2(0)}{\Lambda_1(0)} \right)^2 \right) + \left( 1 - \left( \frac{\Lambda_2(0)}{\Lambda_1(0)} \right)^2 \right) \left( \sqrt{g H^*(0)b_1} \right)^2 + 2 \left( 1 + \left( \frac{\Lambda_2(0)}{\Lambda_1(0)} \right)^2 \right) \sqrt{g H^*(0)b_1} < 0, \]

\[
\left( 1 - \left( \frac{\Lambda_2(L)}{\Lambda_1(L)} \right)^2 \right) + \left( 1 - \left( \frac{\Lambda_2(L)}{\Lambda_1(L)} \right)^2 \right) \left( \sqrt{g H^*(L)b_2} \right)^2 + 2 \left( 1 + \left( \frac{\Lambda_2(L)}{\Lambda_1(L)} \right)^2 \right) \sqrt{g H^*(L)b_2} > 0, \]
and from (3.5.6) and (3.5.7) this is equivalent to having
\[ b_1 \in \left( -\frac{H^*(0)}{V^*(0)}, -\frac{V^*(0)}{g} \right) \quad \text{and} \quad b_2 \in \mathbb{R} \setminus \left[ -\frac{H^*(L)}{V^*(L)}, -\frac{V^*(0)}{g} \right]. \] (3.5.22)

which is exactly (3.3.14). Therefore under boundary conditions (3.3.13) and hypothesis (3.3.14) the system admits a basic \( C^1 \) Lyapunov function and is therefore stable for the \( C^1 \) norm, this ends the proof.

**Proof of Theorem 3.3.5.** Let us suppose that \( C - kV_0^2/H_0^* > 0 \) and \( gH_0^* > V_0^* \). Then there exists a unique maximal solution \((H^*, V^*)\) to the equations (3.5.2). Let us prove that this solution is defined on \([0, L_0)\). Denoting \( L_0 \in (0, \infty] \) the limit such that the maximal solution is defined on \([0, L_0)\), we have from the beginning of this section, in particular (3.5.9), that for all \( 0 \leq x < L_0 \), \( H^* \) and \( V^* \) are continuous, positive, and:
\[ \frac{kV^*}{H^*} > 0, \] (3.5.23)
\[ gH^* > V^*. \] (3.5.24)

Therefore from (3.5.2), \( H^* \) is an increasing function and \( V^* \) is a decreasing function. Besides, as \( H^*V^* \) remains constant, both \( H^* \) and \( V^* \) remain positive. From (3.5.2) we can get an estimate on the growth of \( H^* \):
\[ H^*_x(1 - \frac{V^*_2}{gH^*}) = \frac{C}{g} - \frac{kV^*_2}{gH^*}, \] (3.5.25)
therefore
\[ C - \frac{kV^*_2}{gH_0^*} \leq H^*_x(1 - \frac{V^*_2}{gH^*}) \leq \frac{C}{g}, \] (3.5.26)

hence
\[ 0 < \frac{C}{g} - \frac{kV^*_2}{gH_0^*} \leq H^*_x \leq \frac{C}{g(1 - \frac{V^*_2}{gH^*_x})}. \] (3.5.27)

Thus \( H^*_x \) is bounded. Hence \( L_0 = \infty \), as \( H^* \) is an increasing function and cannot explode in finite length and as \( V^* = H_0^*V_0^*/H^* \) from (3.5.2).

Consider now the Saint-Venant equations transformed into the system (3.5.4) with (3.5.5)–(3.5.8). The first part of the theorem is straightforward from Theorem 3.3.2 as there exists \( L_1 > 0 \) such that (3.3.4) admits a solution on \([0, L_1)\). Let us now suppose by contradiction that for any \( L > 0 \) there exists a basic \( C^1 \) Lyapunov function for the system. Then from Theorem 3.3.2 for any \( L > 0 \) there exists a solution \( \eta \) on \([0, L] \) of
\[ \eta' = \begin{cases} \frac{a}{|A_1|} + \frac{b}{|A_2|} \eta^2, \\ \eta(0) = 0, \end{cases} \] (3.5.28)
where \( a, b, A_1 \) and \( A_2 \) are given by (3.2.24), (3.5.6) and (3.5.7). From (3.5.27), \( H \) goes to \( +\infty \) when \( x \) goes
Therefore there exists $x_1 \in (0, \infty)$ such that for all $x > x_1$

$$\left| \frac{a}{\Lambda_1} \right| \geq \frac{C_2 C}{5gH^*}, \quad \left| \frac{b}{\Lambda_2} \right| \geq \frac{C}{5C_2 gH^*}.$$
Let $L > x_1$, by assumption equation (3.5.28) has a solution $\eta$ defined on $[0, L]$ and from (3.5.35) and (3.5.36), for all $x > x_1$

$$\eta' \geq \frac{C}{5gH^*} (C_2 + \frac{\eta^2}{C_2}) \geq \frac{C_3C}{5gH^*} (1 + \eta^2),$$

where $C_3 = \min \left( C_2, \frac{1}{C_2} \right)$. From (3.5.27) right-hand side and (3.5.37) we have

$$\frac{\eta'}{(1 + \eta^2)} \geq \frac{C_3 \left( 1 - \frac{V_0^2}{gH_0^*} \right)}{5 \left( x + \frac{1 - \frac{V_0^2}{gH_0^*}}{c^2} gH_0^* \right)},$$

hence

$$\int_{x_1}^{x} \frac{\eta'}{(1 + \eta^2)} dx \geq \int_{x_1}^{x} \frac{C_3 \left( 1 - \frac{V_0^2}{gH_0^*} \right)}{5 \left( x + \frac{1 - \frac{V_0^2}{gH_0^*}}{c^2} gH_0^* \right)} dx.$$

Thus

$$\arctan(\eta(x)) - \arctan(\eta(x_1)) \geq \frac{C_3}{5} \left( 1 - \frac{V_0^2}{gH_0^*} \right) \ln \left( \frac{x + \frac{1 - \frac{V_0^2}{gH_0^*}}{c^2} gH_0^*}{x_1 + \frac{1 - \frac{V_0^2}{gH_0^*}}{c^2} gH_0^*} \right).$$

Note that the right-hand side does not depend on $\eta$ and $L$ and that

$$\lim_{x \to +\infty} \frac{C_3}{5} \left( 1 - \frac{V_0^2}{gH_0^*} \right) \ln \left( \frac{x + \frac{1 - \frac{V_0^2}{gH_0^*}}{c^2} gH_0^*}{x_1 + \frac{1 - \frac{V_0^2}{gH_0^*}}{c^2} gH_0^*} \right) = +\infty.$$

Therefore, as this is true for any $L > 0$ we can choose $L$ such that

$$\frac{C_3}{5} \left( 1 - \frac{V_0^2}{gH_0^*} \right) \ln \left( \frac{L + \frac{1 - \frac{V_0^2}{gH_0^*}}{c^2} gH_0^*}{x_1 + \frac{1 - \frac{V_0^2}{gH_0^*}}{c^2} gH_0^*} \right) \geq \frac{\pi}{2}.$$

By hypothesis, there still exist a function $\eta$ that verifies (3.5.40), is positive and defined on $[0, L]$ with this choice of $L$. Hence, $\arctan(\eta(L)) < \pi/2$. But, as $\eta$ is positive, $\arctan(\eta(x_1)) > 0$ so we have from (3.5.40) and (3.5.42)

$$\arctan(\eta(L)) > \frac{\pi}{2}.$$

Hence we have a contradiction. Therefore there exists $L_2 > 0$ such that for any $L > L_2$ there do not exist a basic $C^1$ Lyapunov function whatever the boundary conditions are. This ends the proof.

### 3.6 Numerical estimation

From Theorem 3.3.4 when $kV_0^2/H_0^* \geq C$ there exists explicit static boundary controls under which the general Saint-Venant equations are exponentially stable for the $C^1$ norm, whatever the length of the channel. When $kV_0^2/H_0^* < C$ no such explicit result exists but in practice we can however use Theorem 3.3.2 to find the limit length under which stability can be guaranteed. We provide here some numerical estimations under reasonable conditions ($Q^* = 1 \text{ m}^2\cdot\text{s}^{-1}$, $V_0^* = 0.5 \text{ m} \cdot \text{s}^{-1}$, $k = 0.002$). On Figure 3.1 one can see that for a 50km channel with a constant slope such that $C = 2kV_0^2/H_0^*$, $\eta$ exists and there is no problem. In Figure 3.2 we extended the channel until the limit length $L_{\text{max}}$ for this system and it appears that $L_{\text{max}} > 10^4 \text{km}$.
and that $H^*(L_{\text{max}}) > 100m$ which is quite unrealistic in current hydraulic applications. This suggest that for nearly all hydraulic applications it will be possible to design boundary conditions such that there exists a basic $C^1$ Lyapunov function that ensures the stability of the system for the $C^1$ norm.

![Figure 3.1](image1.png)

*Figure 3.1*: In the $x$-axis is represented the length of the water channel, in blue the height of the water for the stationary state and in red the value of $\eta$.

![Figure 3.2](image2.png)

*Figure 3.2*: In the $x$-axis is represented the length of the water channel, in blue the height of the water in stationary state and in red the value of $\eta$, one can see the limit length $L_{\text{max}}$ at which $\eta$ explodes.

### 3.7 Further details

The previous results were derived for the $C^1$ and the $H^2$ norm but they can actually be extended to the $C^q$ and the $H^p$ norm with the same conditions, for any $p \in \mathbb{N}^* \setminus \{1\}$ and $q \in \mathbb{N}^*$. To show that, one only needs
to realize that Theorem 3.1 and 3.2 in [106] and Theorem 3.4.1 (and therefore Theorem 3.4.2) are true for the $C^q$ and the $H^p$ norm with the same conditions, for any $p \in N^* \setminus \{1\}$ and $q \in N^*$.

In conclusion we gave explicit conditions on the gain of the feedbacks to get exponential stability for the $C^1$ norm for $2 \times 2$ quasilinear hyperbolic systems with propagation speeds of the same sign. In the general case we derived a simple criterion for the existence of basic $C^1$ Lyapunov functions and a practical way to derive admissible static feedback gains when this criterion is satisfied, simply by solving an ODE. We showed that under some conditions on the coefficients of the source term the existence of a $H^p$ and $C^q$ basic Lyapunov function for any $p \in N^* \setminus \{1\}$ and $q \in N^*$ are equivalent and that, in the general case, the existence of a $C^q$ Lyapunov function for any $q \in N^*$ for some appropriate boundary controls implies the existence of a basic quadratic $H^p$ Lyapunov function for any $p \in N^* \setminus \{1\}$ for some appropriate boundary controls. Finally we showed that when the friction is larger than the slope the general nonlinear Saint-Venant equations can be stabilized for the $C^1$ norm by means of simple pointwise feedback, and we gave explicit conditions on the feedbacks. When the slope is larger than the friction no such general result can be shown. However, we showed that for nearly all applications the Saint-Venant equations can be stabilized by means of such simple feedbacks.

3.8 Appendix

3.8.1 Explicit form of $B$ and regularity of $A$ and $B$

Applying the transformation (3.2.3) on the system (3.2.8), $B$ is given by:

$$B(u, x) = N(x)(F(Y)(Y_\nu + (N^{-1}(x))'u) + D(Y)).$$  \hspace{1cm} (3.8.1)

As $Y^*$ is a steady-state, it verifies the equation:

$$F(Y^*) \partial Y^* = -G(Y^*).$$  \hspace{1cm} (3.8.2)

Thus if we suppose that $F$ and $G$ are $C^p$ on $B_{Y^*, s}$, where $p \in N^*$, as $F$ is strictly hyperbolic with non-vanishing eigenvalues, $Y^*$ is $C^{p+1}$ on $[0, L]$. Therefore, using (3.2.5) and (3.8.1), $A$ and $B$ are also $C^p$ on $B_{0, s} \times [0, L]$.

3.8.2 Counter example of the converse of Corollary 1 in general

As mentioned earlier, from [110] we know that for any $L > 0$ the system (3.5.4) corresponding to the Saint-Venant equations with boundary conditions (3.4.5) admits a basic quadratic $H^2$ Lyapunov function. However from Theorem 3.3.3 we know that there exists $L_{max}$ such that for $L > L_{max}$ the system does not admit a $C^1$ Lyapunov function whatever the boundary control is. This is a counter example of the Corollary 1 when one cannot ensure that $M_{12}$ and $M_{21}$ have the same sign.

3.8.3 Proof of Theorem 3.3.1

In this Theorem we rely mainly on Theorem 3.1 of [106]. Let a quasilinear $2 \times 2$ hyperbolic system of the form (3.2.8)–(3.2.9) be with $\Lambda_1, \Lambda_2 > 0$. Without loss of generality we can assume that $\Lambda_1 > 0$ and $\Lambda_2 > 0$. As previously this system is equivalent to the system (3.4.2), (3.4.5) (see Section 3.5) and the existence of a basic $C^1$ (resp. basic quadratic $H^2$) Lyapunov function for this system is equivalent to the existence of a basic $C^1$ (resp. basic quadratic $H^2$) Lyapunov function for the system (3.2.8)–(3.2.9). Let us suppose that

$$G'(0) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

such that

$$k_1^2 < \exp \left( \int_0^L \frac{2M_{11}(0, s)}{\Lambda_1} - 2 \max \left( \frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2} \right) ds \right),$$  \hspace{1cm} (3.8.3.1)

$$k_2^2 < \exp \left( \int_0^L \frac{2M_{22}(0, s)}{\Lambda_2} - 2 \max \left( \frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2} \right) ds \right),$$
then from \([3.4.1]\) we have

\[
G_1'(0) = \begin{pmatrix} l_1 & 0 \\ 0 & l_2 \end{pmatrix}
\]

such that

\[
l_1^2 < \exp \left( -2 \int_0^L \max \left( \frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2} \right) ds \right),
\]

\[
l_2^2 < \exp \left( -2 \int_0^L \max \left( \frac{|a(s)|}{\Lambda_1}, \frac{|b(s)|}{\Lambda_2} \right) ds \right),
\]

where \(G_1\) is defined in \([3.4.5]\). Let us define \(f_1 = f_2 = f\) by

\[
f(x) = \exp \left( -2 \int_0^x \max \left( \frac{|a(s)|}{\Lambda_1(s)}, \frac{|b(s)|}{\Lambda_2(s)} \right) ds \right), \forall x \in [0, L].
\]

Then \(f\) is positive and \(C^1\) on \([0, L]\) and

\[
f' \leq -2 \frac{|a(x)|}{\Lambda_1(x)} f^{3/2},
\]

and \(f' \leq -2 \frac{|b(x)|}{\Lambda_2(x)} f^{3/2}.
\]

Besides, as \(G_1'(0)\) is diagonal, if we define \(\rho_\infty : M \to \min(\|DM\Delta^{-1}\|_\infty : \Delta \in D^+\) where \(D^+\) is the space of diagonal \(2 \times 2\) matrix with positive coefficients we have from \([3.8.4]\):

\[
\rho_\infty(G_1'(0)) = \max(l_1, l_2) < \sqrt{\frac{f(L)}{f(0)}}.
\]

Therefore, from \([3.8.6]\) and \([3.8.7]\) and Theorem 3.1 in \([106]\), there exists a basic \(C^1\) Lyapunov function and therefore the system \([3.1.1]-[3.1.2]\) is stable for the \(C^1\) norm.

Let us now show the stability for the \(H^2\) norm by showing that there exists a basic quadratic \(H^2\) Lyapunov function for the system \([3.4.4], [3.4.5]\). From \([3.8.7]\) and by continuity we know that there exists \(\sigma > 0\) such that

\[
\max(l_1, l_2) < \sqrt{\frac{g(L)}{g(0)}}.
\]

where \(g\) is defined by

\[
g(x) = \exp \left( -2 \int_0^x \max \left( \frac{|a(s)|}{\Lambda_1(s)}, \frac{|b(s)|}{\Lambda_2(s)} \right) ds \right) - \sigma, \forall x \in [0, L].
\]

We want now to be able to apply Theorem 6.6 in \([14]\) which would give the result. Note that if we now define \(q_1 = g/\Lambda_1\) and \(q_2 = g/\Lambda_2\) we have

\[
- \begin{pmatrix} (\Lambda_1 q_1)' & 0 \\ 0 & (\Lambda_2 q_2)' \end{pmatrix} + \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}^T \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}
\]

\[
= \begin{pmatrix} g' \left( \frac{b}{\Lambda_2} + \frac{b}{\Lambda_1} \right) \\ g' \left( \frac{a}{\Lambda_2} + \frac{a}{\Lambda_1} \right) \end{pmatrix},
\]

and this matrix is positive definite as \(g' < 0\) and:

\[
g'^2 - g^2 \left( \frac{b}{\Lambda_2} + \frac{a}{\Lambda_1} \right)^2 > \left( 4 \max \left( \frac{|a|}{\Lambda_1}, \frac{|b|}{\Lambda_2} \right) - \left( \frac{b}{\Lambda_2} + \frac{a}{\Lambda_1} \right)^2 \right) g'^2 \geq 0.
\]
Besides
\[
\begin{pmatrix}
q_1(L)A_1(L) & 0 \\
0 & q_2(L)A_2(L)
\end{pmatrix}
- G_1'(0)^T
\begin{pmatrix}
q_1(0)A_1(0) & 0 \\
0 & q_2(0)A_2(0)
\end{pmatrix}
G_1'(0)
= \begin{pmatrix}
g(L) - (1 - \sigma)l_2^2 & 0 \\
0 & g(L) - (1 - \sigma)l_2^2
\end{pmatrix}
\] (3.8.12)
is positive semi-definite from (3.8.8). Therefore from Theorem 6.6 in [14] we know that the system admits a basic quadratic $H^2$ Lyapunov function and is stable for the $H^2$ norm. This ends the proof of Theorem 3.3.1.

**Remark 3.8.1.** Although the existence of a basic quadratic $H^2$ Lyapunov function is not stated directly in Theorem 6.6 in [14] one can easily check that the theorem actually proves the $H^2$ stability by showing that there exists a basic quadratic $H^2$ Lyapunov function as defined in Definition 3.2.4 (see in particular Lemma 6.8 in [14]).

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Part II

Physical equations and density-velocity systems
Chapter 4

A quadratic Lyapunov function for Saint-Venant equations with arbitrary friction and space-varying slope

This chapter is taken from the following article (also referred to as [110]):

Abstract. The exponential stability problem of the nonlinear Saint-Venant equations is addressed in this chapter. We consider the general case where an arbitrary friction and space-varying slope are both included in the system, which lead to non-uniform steady-states. An explicit quadratic Lyapunov function as a weighted function of a small perturbation of the steady-states is constructed. Then we show that by a suitable choice of boundary feedback controls, that we give explicitly, the local exponential stability of the nonlinear Saint-Venant equations for the $H^2$-norm is guaranteed.

4.1 Introduction

Since discovered in 1871 by Barré de Saint-Venant [10], the shallow water equations (or Saint-Venant equations in unidimensional form) have been frequently used by hydraulic engineers in their practice. Their apparent simplicity and their ability to describe fairly well the behaviour of rivers and water channel make them a useful tool for many applications as for instance the regulation of navigable rivers and irrigation networks in agriculture. Among which, the problem of designing control tools to regularize the water level and the flow rate in the open hydraulic systems has been studied for a long time [71, 97, 135, 137, 139].

The Saint-Venant equations constitute a nonlinear $2 \times 2$ 1-D hyperbolic system. In the last decades, the boundary feedback stabilization problem for 1-D hyperbolic systems has been widely investigated, and many tools have been developed. To our knowledge, the first result for nonlinear $2 \times 2$ homogeneous systems was obtained by Greenberg and Li [93] in the framework of $C^1$ solutions by using the characteristic method. Later on, this result was generalized by Qin [170] to $n \times n$ homogeneous systems. In 1999, Coron et al. introduced another method: the quadratic Lyapunov function, firstly used to analyze the asymptotic behavior of linear hyperbolic equations in the $L^2$ norm but then generalized for nonlinear hyperbolic equations in the framework of $C^1$ and $H^2$ solutions [48, 50, 52, 53]. Both of these two methods guarantee the exponential stability of the nonlinear homogeneous hyperbolic systems when the boundary conditions satisfy an appropriate sufficient dissipativity property. Such boundary conditions are the so-called static boundary feedback control and lead to feedbacks that only depend on the measures at the boundaries. However, when inhomogenous systems are considered, it is usually difficult (or even impossible) to construct a quadratic
Lyapunov function with static boundary feedback [13, Chapter 5.6] or in [12]. The backstepping method introduced by Krstic et al. in [128] is a powerful tool to deal with the exponential stabilization of inhomogenous hyperbolic systems. Initially developed for parabolic equations [179], this method has been firstly applied to first order hyperbolic equations in [129], and then generalized to $(n+1) \times (n+1)$ linear hyperbolic systems with $n$ positive and one negative characteristic speed in [170, 195]. The case of general bidirectional linear systems was recently treated in [112]. For the nonlinear case, one can refer to [65], where the authors designed a full-state feedback control actuated on only one boundary and achieved exponential stability for linear systems was recently treated in [112].

For the nonlinear case, one can refer to [65], where the authors also been used in [80] to stabilize a linearized bilayered Saint-Venant model, a $4 \times 4$ system of two Saint-Venant systems interacting with each other. With the backstepping method, one can realize rapid decay (i.e., exponential decay with arbitrary rate) or even finite time stabilization for some linear case [4, 50, 112].

However, one requires a full-state feedback control rather than static boundary feedback control depending only on the values at the boundaries. Nevertheless, in some cases, it is possible to design an observer to tackle this issue [75, 77, 78, 195]. In this chapter, we will use a direct Lyapunov approach to study the exponential stability for nonhomogeneous Saint-Venant equations with arbitrary friction and space-varying slope. The advantage is that using this method, we only need to measure the value at the boundary, which is much easier for practical implementations.

The first result concerning this method applied to Saint-Venant equations was obtained by Coron et al. in 1999 for the homogeneous case, i.e., without any friction or slope [52]. There, they use an entropy of the system as a Lyapunov function. But this Lyapunov function has only a semi-negative definite time derivative. One has to conclude the stability result using LaSalle’s invariance principle which is usually difficult to apply due to the problem of precompactness of the trajectories. Later on, the authors introduce a strict Lyapunov function for conservation laws that can be diagonalized with Riemann invariants. The time derivative of the Lyapunov function can be made strictly negative definite by an appropriate choice of the boundary conditions. They apply the result to regulate the level and flow in an open channel without friction or slope. Under the assumption that the friction and the slope are sufficiently small in $C^1$-norm, with a bound depending on the length of the channel, the stability may be proved using the method of characteristics as in [83, 169]. Note that in [83], the results are applied to real data obtained from the river Sambre in Belgium. Semigroup approaches have also been developed in [50] but using proportional integral control rather than purely static feedback control. The stability with sufficiently small coefficients may also be proved using a Lyapunov approach as in [47, 50]. In the special case where the slope is a constant and “compensate” with the influence of the friction, thus resulting the uniform steady-states, the stability of the linearized system was considered in [18]. There, the authors use the same Lyapunov function as in [53]. More recently, in [116], the authors managed to study the stability in $H^2$-norm of the nonlinear and nonhomogeneous system when the friction is arbitrary but in the absence of the slope. It should be noted that this last result was proved using a basic quadratic Lyapunov function (see [12] for a proper definition) independent of the length of the water channel which is not trivial as, usually, the existence of a basic quadratic Lyapunov function for a nonhomogenous system depends on the size of the domain [199]. Of course for realistic description of the behaviour of rivers one can easily understand that adding a slope is essential not only because it is the prime mover of the flow but also because in some common cases the effect of the slope can be much larger than the effect of the friction, both being non-negligible: it is the steep-slope regime (see for instance [43, Chapter 5-3]). To our knowledge, no study so far takes into account arbitrary non-negligible friction, slope, and river length through static feedback control, except in the special case mentioned previously where the slope compensate exactly the friction and cancels the source term [18].

Our contribution in this chapter is that we managed to construct an explicit Lyapunov control function to analyze the local exponential stability in $H^2$-norm of the nonlinear Saint-Venant equations with static boundary feedbacks in the case where both the friction and the slope are arbitrary (not necessarily small), and where the river length can be arbitrary as well. This enables us to design robust static feedback con-
trollers to ensure the exponential stability of the steady-states of the system. Especially we deal with the case where the slope may vary with respect to the space variable. This is all the more important that the slope is likely to vary in a river, even sometimes on short distances. We first describe three regimes depending whether the influence of the slope is smaller, equal or greater to the influence of the friction and we show that the dynamics in two opposite regimes are inverted. Then we construct a quadratic Lyapunov function for the $H^2$-norm whatever the friction and the slope are.

The organization of the chapter is as follows. In Section 4.2, we give a description of the non-linear Saint-Venant equations together with some definitions and we state our main result (Theorem 4.2.2). In Section 4.3, the exponential stability of the linearized system is firstly studied by constructing a quadratic Lyapunov function. Based on the results of the linearized system, we then show that a similar expanded Lyapunov function enables us to get the exponential stability of the nonlinear system by properly choosing the boundary feedback controls. In Section 4.4, some numerical illustrations are given to support our theoretical result.

### 4.2 Description of the Saint-Venant equations and the main result

The non-linear Saint-Venant equations with slope and friction are given by the following system:

\[ \begin{align*}
\partial_t H + \partial_x (HV) &= 0, \\
\partial_t V + \partial_x \left( \frac{V^2}{2} + gH \right) + \left( \frac{kV^2}{H} - gC \right) &= 0,
\end{align*} \tag{4.2.1, 4.2.2} \]

where $H(t, x)$ is the water depth, $V(t, x)$ is the horizontal water velocity. The slope $C(x) \in C^2([0, L])$ is defined by $C(x) = -\frac{dB}{dx}$ with $B(x)$ the elevation of the bottom (bathymetry) which is therefore supposed to be $C^3$, $g$ is the constant gravity acceleration and $k$ is a constant friction coefficient.

Note from (4.2.1) and (4.2.2) that the equilibrium $H^*$ and $V^*$ verifies:

\[ \begin{align*}
H^*(x)V^*(x) &= Q^*, \\
V^*V_x^* + gH_x^* + \left( \frac{kV^*}{H^*} - gC \right) &= 0.
\end{align*} \tag{4.2.3, 4.2.4} \]

As we are interested in physical stationary states, we suppose that $H^* > 0$ and $V^* > 0$. Therefore $Q^* > 0$ is any given constant set point and corresponds to the flow rate. Substituting (4.2.3) to (4.2.4), we get that $V^*$ satisfies

\[ V_x^* = \frac{V^*}{gQ^* - V^*H^*}. \tag{4.2.5} \]

Observe that the steady-states are therefore non necessary uniform. As we are interested in navigable rivers we also suppose that the flow is in the fluvial regime, i.e.,

\[ gH^* > V^*H^2 \tag{4.2.6} \]

or equivalently $gQ^* - V^*H^3 > 0$. Then the system (4.2.1) and (4.2.2) has a positive and a negative eigenvalue and for any flow rate $Q^*$, equation (4.2.5) has a unique $C^3$ solution on $[0, L]$ with any given boundary data $V^*(0) = V_0^*$. Moreover, the steady-states have three possible dynamics depending on the slope as the following.

1. When $gC < \frac{kV^*H^2}{H^*}$, also known in hydraulic engineering as “mild slope regime”. This covers also the case without slope. Note from (4.2.3) and (4.2.5) that in this case, $H^*$ decreases while $V^*$ increases and consequently the system becomes closer to the critical point where $gH^* = V^*H^2$ which is the limit of the fluvial regime.
2. When \( gC = \frac{kV^2}{H^2} \), which means that the friction and the slope “compensate” each other. When the slope \( C \) is in addition constant, the steady-states are uniform. This special case has been studied in [18] and only for the linearized system.

3. When \( gC > \frac{kV^2}{H^2} \), also known in hydraulic engineering as “steep slope regime”. Then the dynamics of the steady-states are inverted: \( H^* \) tends to increase while \( V^* \) decreases and consequently the system moves away from the limit of the fluvial regime defined by the critical point where \( gH^* = V^{*2} \).

Our goal is to ensure the exponential stability of the steady-states of the nonlinear system (4.2.1) and (4.2.2) for all the above three cases under some boundary conditions of the form:

\[
V(t, 0) = B(H(t, 0)), \quad V(t, L) = B(H(t, L)),
\]

where \( B : \mathbb{R} \rightarrow \mathbb{R} \) is of class \( C^2 \). These kind of boundary conditions are imposed by physical devices located at the ends of the channel where the controls are implemented, as for instance mobile spillways or tunable hydraulic gates as in irrigation canals and navigable rivers.

Taking for instance a spillway outflow gate at \( x = L \) modelled by a simple linear model and imposing the inflow with a tunable hydraulic gate at \( x = 0 \), the boundary conditions are

\[
H(t, 0)V(t, 0) = U_0(t), \quad H(t, L)V(t, L) = \kappa(H(t, L) - U_1(t)),
\]

where \( \kappa \) is a constant depending on the gate, and where \( U_0 \) the inflow and \( U_1 \) the spillway gate elevation are our control functions with \( H(t, 0) \) and \( H(t, L) \) the observations as shown in (4.2.7). Some other explicit expressions of the boundary conditions are given in [13, Section 1.4].

We will first prove the exponential stability for the linearized system for the \( L^2 \)-norm. Note that for nonlinear systems, the stability depends on the topology considered as shown in [61]. In this chapter, we will consider the exponential stability in \( H^2 \)-norm.

For any given initial condition

\[
H(0, x) = H_0(x), \quad V(0, x) = V_0(x), \quad x \in [0, L],
\]

we suppose that the following compatibility conditions hold

\[
\begin{align*}
V_0(0) &= B(H_0(0)), \quad V_0(L) = B(H_0(L)), \\
\partial_x \left( \frac{V_0^2}{2} + gH_0 \right)(0) + \frac{kV_0^2}{H_0}(0) - gC(0) &= B'(H_0(0))\partial_x(H_0V_0)(0), \\
\partial_x \left( \frac{V_0^2}{2} + gH_0 \right)(L) + \frac{kV_0^2}{H_0}(L) - gC(L) &= B'(H_0(L))\partial_x(H_0V_0)(L).
\end{align*}
\]

These compatibility conditions guarantee the well-posedness of the system (4.2.1), (4.2.2), (4.2.7) and (4.2.8) for sufficiently small initial initial data. More precisely, we have (see [13, Appendix B])

**Theorem 4.2.1.** There exists \( \delta_0 > 0 \) such that for every \( (H_0, V_0)^T \in H^2((0, L); \mathbb{R}^2) \) satisfying

\[
\|(H_0 - H^*, V_0 - V^*)^T\|_{H^2((0, L); \mathbb{R}^2)} \leq \delta_0
\]

and compatibility conditions (4.2.9) to (4.2.11). The Cauchy problem (4.2.1), (4.2.2), (4.2.7) and (4.2.8) has a unique maximal classical solution

\[
(H, V)^T \in C^0([0, T); H^2((0, L); \mathbb{R}^2))
\]

with \( T \in (0, +\infty) \).
We recall the definition of the exponential stability in $H^2$-norm:

**Definition 4.2.1.** The steady-state $(H^*, V^*)^T$ of the system (4.2.1), (4.2.2) and (4.2.7) is exponentially stable for the $H^2$-norm if there exist $\gamma > 0$, $\delta > 0$ and $C > 0$ such that for every $(H_0, V_0)^T \in H^2((0, L); \mathbb{R}^2)$ satisfying $\|(H_0 - H^*, V_0 - V^*)^T\|_{H^2((0, L); \mathbb{R}^2)} \leq \delta$ and the compatibility conditions (4.2.9) to (4.2.11), the Cauchy problem (4.2.1), (4.2.2), (4.2.7) and (4.2.8) has a unique classical solution on $[0, +\infty) \times [0, L]$ and satisfies

$$\| (H(t, \cdot) - H^*, V(t, \cdot) - V^*)^T \|_{H^2((0, L); \mathbb{R}^2)} \leq Ce^{-\gamma t} \|(H_0 - H^*, V_0 - V^*)^T\|_{H^2((0, L); \mathbb{R}^2)},$$

for all $t \in [0, +\infty)$.

Based on this definition, our main result is the following:

**Theorem 4.2.2.** The nonlinear Saint-Venant system (4.2.1), (4.2.2) and (4.2.7) is exponentially stable for the $H^2$-norm provided that the boundary conditions satisfy

$$B'(H^*(0)) \in \left( -\frac{g}{V^*(0)}, \frac{V^*(0)}{H^*(0)} \right),$$

and

$$B'(H^*(L)) \in \mathbb{R} \setminus \left( -\frac{g}{V^*(L)}, \frac{V^*(L)}{H^*(L)} \right).$$

The proof of this theorem is given in Section 3. To that end, we will first prove the exponential stability result (Proposition 4.3.1) for the linearized system for the $L^2$-norm by finding a suitable Lyapunov function. Then we show that this Lyapunov function enables us to obtain the exponential stability for the $H^2$-norm for the nonlinear system under boundary control conditions (4.2.7) with properties (4.2.13).

### 4.3 Exponential stability for the $H^2$-norm with arbitrary friction and space-varying slope

#### 4.3.1 Exponential stability of the linearized system

In this section, we study the exponential stability of the linearized system about a steady-state $(H^*, V^*)^T$ for the $L^2$-norm. We define the perturbation functions $h$ and $v$ as

$$h(t, x) = H(t, x) - H^*(x),$$

$$v(t, x) = V(t, x) - V^*(x).$$

The linearization of the system (4.2.1) and (4.2.2) about the steady-state is

$$\begin{pmatrix} h \\ v \end{pmatrix}_t + \begin{pmatrix} V^* \\ H^* \end{pmatrix} g \begin{pmatrix} h \\ v \end{pmatrix}_x + \begin{pmatrix} V^*_x \\ H^*_x \end{pmatrix} f_H + \begin{pmatrix} V^* \\ V^* \end{pmatrix} f_V \begin{pmatrix} h \\ v \end{pmatrix} = 0,$$

where $f_H$ and $f_V$ are defined by

$$f_V = \frac{2kV^*}{H^*}, \quad f_H = \frac{-kV^{*2}}{H^{*2}}.$$

The corresponding linearization of the boundary conditions (4.2.7) are given by

$$v(t, 0) = b_0 h(t, 0), \quad v(t, L) = b_1 h(t, L),$$

where

$$b_0 = B'(H^*(0)), \quad b_1 = B'(H^*(L)).$$

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The initial condition is given as follows
\[ h(0, x) = h_0(x), \quad v(0, x) = v_0(x), \quad (4.3.7) \]
where \((h_0, v_0)^T \in L^2((0, L) ; \mathbb{R}^2)\). The Cauchy problem \((4.3.3), (4.3.5)\) and \((4.3.7)\) is well-posed (see \[13\] Appendix A). Note that the exponential stability of the linearized system is now a problem of null-stabilization for \(h\) and \(v\). We have the following result:

**Proposition 4.3.1.** For the linearized Saint-Venant system \((4.3.3), (4.3.5)\) and \((4.3.7)\), if the boundary conditions satisfy
\[ b_0 \in \left(-\frac{g}{V^*(0)} \frac{V^*(0)}{H^*(0)}, b_1 \in \mathbb{R} \right] \left(-\frac{g}{V^*(L)} \frac{V^*(L)}{H^*(L)}\right), \quad (4.3.8) \]

Then there exists a constant \(\mu > 0\), \(q_1 \in C^1([0, L] ; (0, +\infty))\), \(q_2 \in C^1([0, L] ; (0, +\infty))\) and \(\delta > 0\) such that the following control Lyapunov function candidate
\[ V(h, v) = \int_0^L \frac{q_1 + q_2}{H^*} \left(gh^2 + 2\frac{q_1 - q_2}{q_1 + q_2} \sqrt{gH^*hv + H^*v^2}\right) dx \quad (4.3.9) \]
verifies:
\[ V(h, v) \geq \delta \left(\|h\|_{L^2(0, L)}^2 + \|v\|_{L^2(0, L)}^2\right) \quad (4.3.10) \]
for any \((h, v) \in L^2((0, L) ; \mathbb{R}^2)\), where \(L^2(0, L)\) denotes \(L^2((0, L) ; \mathbb{R})\). If in addition, \((h, v)^T\) is a solution of the system \((4.3.3), (4.3.5)\) and \((4.3.7)\), we have
\[ \frac{d}{dt} (V(h(t, \cdot), v(t, \cdot))) \leq -\mu V(h(t, \cdot), v(t, \cdot)) \quad (4.3.11) \]
in the distribution sense which implies the exponential stability of the linearized system \((4.3.3), (4.3.5)\) and \((4.3.7)\) for the \(L^2\)-norm.

In order to prove Proposition 4.3.1 we introduce the following lemma, the proof of which is given in the Appendix.

**Lemma 4.3.1.** The function \(\eta\) defined by
\[ \eta = \frac{\lambda_2}{\lambda_1}\varphi \quad (4.3.12) \]
is a solution to the equation
\[ \eta' = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right| \quad (4.3.13) \]
where \(\lambda_1\) and \(\lambda_2\) are defined in \((4.3.14)\), \(\varphi\) is given by \((4.3.23)\), \(a\) and \(b\) are given by \((4.3.26)\) below.

**Proof of Proposition 4.3.1.** Let us denote
\[ A(x) = \begin{pmatrix} V^* & H^* \\ g & V^* \end{pmatrix} \quad (4.3.23) \]
Under the subcritical condition \((4.2.6)\), the matrix \(A(x)\) has two real distinct eigenvalues \(\lambda_1\) and \(-\lambda_2\) with
\[ \lambda_1(x) = \sqrt{gH^2 + V^*} > 0, \lambda_2(x) = \sqrt{gH^2 - V^*} > 0. \quad (4.3.14) \]
We define the characteristic coordinates as follows
\[ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{g}{H^2}} & 1 \\ -\sqrt{\frac{g}{H^2}} & 1 \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}. \quad (4.3.15) \]
With these definitions and notations, the linearized Saint-Venant equations \((4.3.3)\) are written in characteristic form:
\[ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0. \quad (4.3.16) \]
In (4.3.16),

\[
\begin{align*}
\gamma_1(x) &= -\frac{3f(H^*, V^*)}{4(\sqrt{gH^*} + V^*)} + \frac{kV^*}{H^*} - \frac{kV^{*2}}{2H^*\sqrt{gH^*}}, \\
\delta_1(x) &= -\frac{f(H^*, V^*)}{4(\sqrt{gH^*} + V^*)} + \frac{kV^*}{H^*} + \frac{kV^{*2}}{2H^*\sqrt{gH^*}}, \\
\gamma_2(x) &= \frac{f(H^*, V^*)}{4(\sqrt{gH^*} - V^*)} + \frac{kV^*}{H^*} - \frac{kV^{*2}}{2H^*\sqrt{gH^*}}, \\
\delta_2(x) &= -\frac{3f(H^*, V^*)}{4(\sqrt{gH^*} - V^*)} + \frac{kV^*}{H^*} + \frac{kV^{*2}}{2H^*\sqrt{gH^*}},
\end{align*}
\]  

(4.3.17) (4.3.18) (4.3.19) (4.3.20)

where \( f(H^*, V^*) = \frac{kV^{*2}}{H^*} - gC. \)

As the diagonal coefficients of the source term in (4.3.16) may bring complexity on the analysis of the stability, we then make a coordinate transformation inspired by [128] (see also [16]) to remove the diagonal coefficients. We introduce the notations

\[
\varphi_1(x) = \exp \left( \int_0^x \frac{\gamma_1(s)}{\lambda_1(s)} \, ds \right), \\
\varphi_2(x) = \exp \left( -\int_0^x \frac{\delta_2(s)}{\lambda_2(s)} \, ds \right), \\
\varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)},
\]

(4.3.21) (4.3.22) (4.3.23)

and the new coordinates

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}.
\]

(4.3.24)

Then system (4.3.16) is transformed into the following system expressed in the new coordinates

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_x + \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0
\]

(4.3.25)

with

\[
a(x) = \varphi(x) \delta_1(x), \quad b(x) = \varphi^{-1}(x) \gamma_2(x).
\]

(4.3.26)

From (4.3.5), (4.3.15) and (4.3.24), we obtain the following boundary conditions for system (4.3.25)

\[
y_1(t, 0) = k_0 \frac{\varphi_1(0)}{\varphi_2(0)} y_2(t, 0), \\
y_2(t, L) = k_1 \frac{\varphi_2(L)}{\varphi_1(L)} y_1(t, L),
\]

(4.3.27)

where

\[
k_0 = \frac{b_0 H^*(0) + \sqrt{gH^*(0)}}{b_0 H^*(0) - \sqrt{gH^*(0)}}, \quad k_1 = \frac{b_1 H^*(L) - \sqrt{gH^*(L)}}{b_1 H^*(L) + \sqrt{gH^*(L)}}.
\]

(4.3.28)

Note that from (4.3.28), it is easy to check that condition (4.3.8), using our notation (4.3.14), is equivalent to

\[
k_0^2 < \left( \frac{\lambda_2(0)}{\lambda_1(0)} \right)^2, \quad k_1^2 < \left( \frac{\lambda_1(L)}{\lambda_2(L)} \right)^2.
\]

(4.3.29)

Let us define

\[
V : \quad L^2(0, L) \times L^2(0, L) \to \mathbb{R}^+.
\]
\[ V(\psi_1, \psi_2) = \int_0^L \left( f_1(x)\psi_1^2(x)e^{-\frac{\mu}{2}x} + f_2(x)\psi_2^2(x)e^{\frac{\mu}{2}x} \right) dx \quad (4.3.30) \]

where the parameter \( \mu > 0 \) and two functions \( f_1 \in C^1([0, L]; (0, +\infty)) \) and \( f_2 \in C^1([0, L]; (0, +\infty)) \) are to be determined. Obviously, there exists \( \delta > 0 \) such that for any \( (\psi_1, \psi_2) \in L^2((0, L); \mathbb{R}^2) \)

\[ V(\psi_1, \psi_2) \geq \delta \left( \|\psi_1\|_{L^2(0, L)}^2 + \|\psi_2\|_{L^2(0, L)}^2 \right). \quad (4.3.31) \]

For any arbitrary \( C^1 \)-solution \( y_1 \) and \( y_2 \) to system (4.3.25) and (4.3.27), we denote \( V(t) \) by

\[ V(t) = V(y_1(t, \cdot), y_2(t, \cdot)). \quad (4.3.32) \]

From (4.3.30) and differentiating \( V \) with respect to time \( t \) we get

\[ \frac{dV}{dt} = -\mu V - \left[ \lambda_1 f_1 e^{-\frac{\mu}{2}x} y_1^2 - \lambda_2 f_2 e^{-\frac{\mu}{2}x} y_2^2 \right]_{0}^{L} \]

\[ - \int_0^L \left[ \left( -\lambda_1 f_1 - \frac{\mu x \lambda_1}{\lambda_1} f_1 \right) e^{-\frac{\mu}{2}x} y_1^2 \right. \]

\[ + \left( \lambda_2 f_2 - \frac{\mu x \lambda_2}{\lambda_2} f_2 \right) e^{\frac{\mu}{2}x} y_2^2 \]

\[ + 2( f_1 e^{\frac{\mu}{2}x} a(x) + f_2 e^{-\frac{\mu}{2}x} b(x) ) y_1 y_2 \] \( dx \). \quad (4.3.33)

We observe that in (4.3.33), there is a term relying on the boundary controls that will be chosen to make this term negative along the system trajectories. Moreover, there also appears to have an interior term which is intrinsic to the system. Let us deal firstly with the interior term, we denote by

\[ I_1 := \int_0^L \left[ \left( -\lambda_1 f_1 - \frac{\mu x \lambda_1}{\lambda_1} f_1 \right) e^{-\frac{\mu}{2}x} y_1^2 \right. \]

\[ + \left( \lambda_2 f_2 - \frac{\mu x \lambda_2}{\lambda_2} f_2 \right) e^{\frac{\mu}{2}x} y_2^2 \]

\[ + 2( f_1 e^{\frac{\mu}{2}x} a(x) + f_2 e^{-\frac{\mu}{2}x} b(x) ) y_1 y_2 \] \( dx \). \quad (4.3.34)

To ensure that for \( \mu > 0 \) sufficiently small, \( I_1 \) is positive for any \( t > 0 \) and any solution \( (y_1, y_2) \), one only needs to construct \( f_1 \) and \( f_2 \) to guarantee that for any \( x \in [0, L] \)

\[ -(\lambda_1 f_1)_x > 0, \quad (\lambda_2 f_2)_x > 0, \quad (4.3.35) \]

\[ - (\lambda_1 f_1)(\lambda_2 f_2) - (f_1(x)a(x) + f_2(x)b(x))^2 > 0. \quad (4.3.36) \]

Indeed in this case from the strict inequality in (4.3.35) and (4.3.36), by continuity, there exists \( \mu_1 > 0 \) such that for all \( \mu \in (0, \mu_1), I_1 \) is positive.

Let us point out that there exist \( f_1 \) and \( f_2 \) such that (4.3.35) and (4.3.36) hold as soon as there exists a positive function \( \eta \) well defined on \( [0, L] \) and satisfying the following equation (see [12])

\[ \eta' = \left[ \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right]. \quad (4.3.37) \]

Therefore, one of the key points to prove Proposition 4.3.1 is to find a positive solution to (4.3.37). And from Lemma 4.3.1 we know that such solution does exist. Hence, we can define a map

\[ f : (\eta, \varepsilon) \rightarrow \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right| + \varepsilon, \]

which is locally Lipschitz (and even \( C^1 \)) in \( \varepsilon \) around 0. From Lemma 4.3.1 we know that

\[
\begin{cases}
\eta' = f(\eta, 0), \\
\eta(0) = \frac{\lambda_2(0)}{\lambda_1(0)} \varphi(0)
\end{cases}
\quad (4.3.38)
\]
Thus, (4.3.35) and (4.3.36) hold. Now, let us consider the boundary term in (4.3.33), we denote by

\[ \varepsilon \]

\[ \eta \]

admits a unique solution on \([0, L]\) which is given by \((4.3.12)\). Therefore, there exists \(\varepsilon_0 > 0\) such that for all \(\varepsilon \in [0, \varepsilon_0]\), the Cauchy problem

\[
\left\{
\begin{array}{l}
\eta'_x = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right| + \varepsilon, \\
\eta(0) = \frac{\lambda_2(0)}{\lambda_1(0)} \varphi(0)
\end{array}
\right.
\]

(4.3.39)

admits a unique solution \(\eta_\varepsilon\) on \([0, L]\). Moreover as \(\eta_\varepsilon(0) > 0\), we have \(\eta_\varepsilon(x) > 0\) for all \(x \in [0, L]\). Now proceeding as in \([12]\), we choose \(f_1\) and \(f_2\) as

\[
f_1 = f_{1,\varepsilon} := \frac{1}{\lambda_1} \eta_\varepsilon, \quad f_2 = f_{2,\varepsilon} := \frac{\eta_\varepsilon}{\lambda_2},
\]

(4.3.40)

then we have for any \(\varepsilon \in (0, \varepsilon_0]\) that

\[
-(\lambda_1 f_1)_x > 0, \quad (\lambda_2 f_2)_x > 0, \quad (4.3.41)
\]

\[
-(\lambda_1 f_1)(\lambda_2 f_2)_x - (f_1 a + f_2 b)^2 = \varepsilon^2 + 2\varepsilon \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta_\varepsilon^2 > 0.
\]

(4.3.42)

Thus, (4.3.35) and (4.3.36) hold. Now, let us consider the boundary term in (4.3.33), we denote by

\[
I_2 := - \left[ \lambda_1 f_1 e^{-\frac{\mu}{\lambda_1} y_1^2} - \lambda_2 f_2 e^{\frac{\mu}{\lambda_2} y_2^2} \right]_0^L.
\]

(4.3.43)

Suppose that (4.3.8) is satisfied, from (4.3.12), (4.3.29), (4.3.39) and (4.3.40), we have

\[
k_0^2 < \left( \frac{\lambda_1(0)}{\lambda_2(0)} \right)^2 = \frac{\lambda_2(0) f_{2,0}(0)}{\lambda_1(0) f_{1,0}(0)} \varphi^{-2}(0) = \frac{\lambda_2(0) f_{2,\varepsilon}(0)}{\lambda_1(0) f_{1,\varepsilon}(0)} \varphi^{-2}(0),
\]

(4.3.44)

\[
k_1^2 < \left( \frac{\lambda_1(L)}{\lambda_2(L)} \right)^2 = \frac{\lambda_1(L) f_{1,0}(L)}{\lambda_2(L) f_{2,0}(L)} \varphi^2(L).
\]

(4.3.45)

By the continuity of \(f_{1,\varepsilon}(L)\) and \(f_{2,\varepsilon}(L)\) with \(\varepsilon\), there exists \(0 < \varepsilon_1 < \varepsilon_0\) such that for any \(\varepsilon \in (0, \varepsilon_1]\)

\[
k_1^2 < \frac{\lambda_1(L) f_{1,\varepsilon}(L)}{\lambda_2(L) f_{2,\varepsilon}(L)} \varphi^2(L),
\]

(4.3.46)

thus, there exists \(0 < \mu_2 < \mu_1\) such that for any \(\mu \in (0, \mu_2]\)

\[
k_1^2 < \frac{\lambda_1(L) f_{1,\varepsilon}(L) e^{-\frac{\mu}{\lambda_1} y_1^2 L}}{\lambda_2(L) f_{2,\varepsilon}(L) e^{\frac{\mu}{\lambda_2} y_2^2 L}} \varphi^2(L).
\]

(4.3.47)

Combining (4.3.44) and (4.3.47), we get

\[
I_2 = - \left[ \lambda_1 f_1 e^{-\frac{\mu}{\lambda_1} y_1^2} - \lambda_2 f_2 e^{\frac{\mu}{\lambda_2} y_2^2} \right]_0^L = \left( k_1^2 \lambda_2(L) f_{2,\varepsilon}(L) \varphi^{-2}(L) e^{\frac{\mu}{\lambda_2} y_2^2 L} - \lambda_1(L) f_{1,\varepsilon}(L) e^{-\frac{\mu}{\lambda_1} y_1^2 L} \right) y_1^2(t, L)
\]

(4.3.48)

\[
+ \left( k_0^2 \lambda_1(0) f_{1,\varepsilon}(0) \varphi^2(0) - \lambda_2(0) f_{2,\varepsilon}(0) \right) y_2^2(t, 0) < 0.
\]

From (4.3.33), (4.3.41), (4.3.42) and (4.3.48), we obtain

\[
\frac{dV}{dt} < -\mu V
\]

(4.3.49)
along the $C^1$-solutions of the system (4.3.25) and (4.3.27) for any $\mu \in (0, \mu_2]$. Since the $C^1$-solutions are dense in the set of $L^2$-solutions, inequality (4.3.49) also holds in the sense of distributions for the $L^2$-solutions (see [13, Section 2.1] for the details).

Let us define

$$q_1 := f_1\varphi_1^2 e^{-\frac{\mu}{\lambda_1}x} \quad \text{and} \quad q_2 := f_2\varphi_2^2 e^{\frac{\mu}{\lambda_2}x}. \quad (4.3.50)$$

For any $(h, v) \in L^2((0, L); \mathbb{R}^2)$, let $(\psi_1, \psi_2)$ be the result of the change of variable as in (4.3.15) and (4.3.24), we get immediately from (4.3.30) and (4.3.50) the expression of Lyapunov function candidate as in (4.3.9).

Moreover, from the positivity of $f_1$ and $f_2$, we had (4.3.31), which implies that there exists $\delta > 0$ such that (4.3.10) holds using the change of variable (4.3.15) and (4.3.24). From (4.3.49), we get (4.3.11) as well. The proof of Proposition 4.3.1 is completed.

4.3.2 Exponential stability of the steady-state of the nonlinear system in $H^2$-norm

We will now prove our main result, Theorem 4.2.2. Firstly, we recall the following theorem which gives sufficient conditions for the exponential stability of the steady-state of the nonlinear system (4.2.1), (4.2.2) and (4.2.7).

**Theorem 4.3.2.** The steady-state $(H^*, V^*)^T$ of the system (4.2.1), (4.2.2) and (4.2.7) is exponentially stable for the $H^2$-norm if

- There exists two functions $f_1, f_2 \in C^1([0, L]; (0, +\infty))$ such that
  $$-(\lambda_1 f_1)_x > 0, \quad (\lambda_2 f_2)_x > 0 \quad (4.3.51)$$

  and
  $$- (\lambda_1 f_1)_x (\lambda_2 f_2)_x > \left( \frac{a(x)}{\lambda_1(x)} f_1(x) + \frac{b(x)}{\lambda_2(x)} f_2(x) \right)^2 \quad (4.3.52)$$

  for any $x \in [0, L]$, where $a$ and $b$ are given by (4.3.26).
- The following inequalities are satisfied:
  $$\left( \frac{b_0 H^*(0) + \sqrt{gH^*(0)}}{b_0 H^*(0) - \sqrt{gH^*(0)}} \right)^2 < \frac{\lambda_2(0) f_2(0)}{\lambda_1(0) f_1(0)} \varphi^{-2}(0),$$
  $$\left( \frac{b_1 H^*(L) - \sqrt{gH^*(L)}}{b_1 H^*(L) + \sqrt{gH^*(L)}} \right)^2 < \frac{\lambda_1(L) f_1(L)}{\lambda_2(L) f_2(L)} \varphi^2(L), \quad (4.3.53)$$

  where $b_0, b_1$ and $\varphi$ are given by (4.3.6) and (4.3.23) respectively.

**Remark 4.3.1.** This theorem comes directly from [13, Theorem 6.6 and 6.10]. Note that finding such $f_1$ and $f_2$ corresponds to finding a quadratic Lyapunov function $V$ for the $H^2$-norm of the perturbations (4.3.1) and (4.3.2) such that:

$$\frac{1}{\beta} \|[h, v]^T\|_{H^2} \leq V \leq \beta\|[h, v]^T\|_{H^2} \quad \text{and} \quad \frac{dV}{dt} \leq -\alpha V \quad (4.3.54)$$

for some $\alpha > 0$ and $\beta > 0$. In particular, such Lyapunov function has some robustness with respect to small perturbations of the system dynamics. More details about the construction of such Lyapunov function as well as the proof of this theorem can be found in the Appendix.

Using Theorem 4.3.2 we shall finally prove Theorem 4.2.2 that is now straightforward.
**Proof of Theorem 4.2.2.** Note that the condition (4.3.51) and (4.3.52) are exactly the same with conditions (4.3.35) and (4.3.36). Therefore from the proof of Proposition 4.3.1 for all \( \varepsilon \in (0, \varepsilon_0] \), there exist \( f_1 = f_{1,\varepsilon} \) and \( f_2 = f_{2,\varepsilon} \) defined by (4.3.40), continuous with respect to \( \varepsilon \), such that (4.3.51) and (4.3.52) are verified and

\[
\frac{f_{1,\varepsilon}}{2b_n^2} \frac{\lambda_1}{\lambda_2} \varphi^2 = \left( \frac{\lambda_1}{\lambda_2} \right)^2,
\]

(4.3.55)

where \( \varphi \) is given by (4.3.23). Under hypothesis (4.2.13) of Theorem 4.2.2, we have (4.3.29), which together with (4.3.55) gives (4.3.44). Recall that by the continuity of \( f_{1,\varepsilon} \) and \( f_{2,\varepsilon} \) with respect to \( \varepsilon \), (4.3.46) holds for any \( \varepsilon \in (0, \varepsilon_1] \). Combining (4.3.44), (4.3.46) and noticing (4.3.28), we obtain that

\[
\begin{align*}
\left( b_0 H^*(0) + g H^*(0) \right)^2 &< \frac{\lambda_2(0)}{\lambda_1(0)} f_{2,\varepsilon}(0) \varphi^2(0), \\
\left( b_1 H^*(L) - g H^*(L) \right)^2 &< \frac{\lambda_1}{\lambda_2(0)} f_{1,\varepsilon}(L) \varphi^2(L).
\end{align*}
\]

(4.3.56)

Thus, we get from Theorem 4.3.2 that the steady-state \((H^*, V^*)^T\) of the system (4.2.1), (4.2.2) and (4.2.7) is exponentially stable for the \(H^2\)-norm. This ends the proof of Theorem 4.2.2 \( \square \)

**Remark 4.3.2.** We emphasize that the exponential stability in \(H^p\)-norm holds in fact for any \( p \in \mathbb{N} \setminus \{0, 1\} \) under the same condition (4.2.13) given in Theorem 4.2.2 when the map \( B \) is of class \( C^p \) and the definition of the exponential stability involves an appropriate extension of the compatibility conditions of order \( p - 1 \) (see [13, Page.153] for the definition). This is a consequence of [13, Theorem 6.10]. Roughly speaking, this can be obtained by considering the augmented systems and then using the same method as in the proof of Theorem 4.3.2.

### 4.4 Numerical simulations

In this section, we illustrate Theorem 4.2.2 by providing numerical simulations of the \(H^2\)-norm of the solutions of the nonlinear system (4.2.1), (4.2.2) and (4.2.7). We focus on the steep slope regime (i.e. \( gC > kV^2/H \)) as it is the most challenging situation to stabilize. The steady-state is chosen with initial condition prescribed as \( H^*(0) = 2 \, m, \quad V^*(0) = 0.5 \, m/s \). And the parameters are chosen as follows: \( k = 0.002 \), \( L = 3.10^3 \, m, \quad gC = 2kV^2(0)/H^*(0) \), the boundary controls are chosen such that \( B^*(H^*(0)) = 1.1b_{0,lim} \), \( B^*(H^*(L)) = 0.9b_{1,lim} \) where \( b_{0,lim} = -V^*(0)/H^*(0) \) and \( b_{1,lim} = -V^*(L)/H^*(L) \) are the critical upper bounds given in Theorem 4.2.2. The initial perturbations are chosen to be compatible at the boundaries and sinusoidal in the domain with an amplitude of 0.2 \( m \) in height and 0.05 \( m/s \) in velocity. In Fig. 4.1, the blue curve represents the variation of the \(H^2\)-norm of the perturbations for the water level with time, while in Fig. 4.2, the red curve represents the behavior of the downstream controller \( V(t, L) \), which converges to the value \( V^*(L) \) (in green).

### 4.5 Conclusion

In this chapter we addressed the problem of the exponential stability of the Saint-Venant equations with arbitrary friction and space-varying slope. An explicit boundary condition was given which guarantees the exponential stability of the nonlinear system in \(H^2\)-norm. To that end, we first studied a corresponding linearized system and proved the exponential stability result in \(L^2\)-norm by constructing a quadratic Lyapunov function. Then by expanding the Lyapunov function, we obtained the exponential stability of the nonlinear system in \(H^2\)-norm by requiring proper conditions on the boundaries. These boundary conditions are related to physical devices located at the ends of the channel where the controls acting as feedback are implemented. Finally, some numerical simulations are given to support our main result, Theorem 4.2.2.
4.6 Appendix

4.6.1 Proof of Lemma 4.3.1

Proof. From (4.2.3), (4.2.5), (4.3.17) to (4.3.23) and (4.3.26), we get that

\[
\left(\frac{\lambda_2}{\lambda_1} \varphi\right)' = \frac{\lambda_2^2 \lambda_1 - \lambda_1^2 \lambda_2}{\lambda_1^2} \varphi + \frac{\lambda_2}{\lambda_1} \left(\frac{\gamma_1}{\lambda_1} + \frac{\delta_2}{\lambda_2}\right) \varphi
\]

\[
= \left(3\sqrt{gH^*V^*} \left(gC - \frac{kV^{*2}}{H^*}\right) + \frac{\lambda_2 \gamma_1 + \delta_2 \lambda_1}{\lambda_1^2}\right) \varphi
\]

\[
+ \frac{1}{\lambda_1^2} \left[\frac{3}{4} \left(gC - \frac{kV^{*2}}{H^*}\right) \left[\frac{\sqrt{gH^*} - V}{\sqrt{gH^*} + V} - \frac{\sqrt{gH^*} + V}{\sqrt{gH^*} - V}\right]
\]

\[
+ \frac{kV^{*2}}{H^*} \left(\frac{2\sqrt{gH^*}}{V^*} + \frac{V^*}{\sqrt{gH^*}}\right)\right) \varphi
\]
\[
\frac{kV^*}{H^*} \left( \frac{2\sqrt{gH^*}}{V^*} + \frac{V^*}{\sqrt{gH^*}} \right) \frac{\varphi}{\lambda_1^2}.
\]

Besides, we have
\[
\frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 = \left( \frac{\delta_1 \lambda_1 + \gamma_2 \lambda_2}{\lambda_1^2} \right) \varphi
\]
\[
= \frac{kV^*}{H^*} \left( \frac{2\sqrt{gH^*}}{V^*} + \frac{V^*}{\sqrt{gH^*}} \right) \frac{\varphi}{\lambda_1^2} > 0.
\]

Therefore
\[
\eta' = \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right|.
\]

This ends the proof of Lemma 4.3.1.

### 4.6.2 Proof of Theorem 4.3.2

**Proof.** This theorem is a particular case of [13, Theorem 6.10]. One just need to check that the system (4.2.1), (4.2.2) and (4.2.8) with boundary conditions (4.2.7) satisfying the dissipative conditions (4.2.13) can be written in the form of [13, (6.54)-(6.57)]. Note that this also implies the well-posedness of the system as well. Indeed, we perform the change of variable
\[
\left( \begin{array}{c} z_1 \\ z_2 \\ \lambda_1 \\ \lambda_2 \\ \eta \\
 \end{array} \right) = \left( \begin{array}{c} \varphi_1 \sqrt{\frac{g}{H^*}} \\ -\varphi_2 \sqrt{\frac{g}{H^*}} \\ \varphi_1 \\ \varphi_2 \\ h \\
 \end{array} \right) \left( \begin{array}{c} v \\ 0 \\
 \end{array} \right),
\]

where \( h \) and \( v \) are the perturbations given by (4.3.1) and (4.3.2) and \( \varphi_1 \) and \( \varphi_2 \) are given by (4.3.21) and (4.3.22) respectively. If we denote by \( z = (z_1, z_2) \), the nonlinear system (4.2.1), (4.2.2) and (4.2.7) is equivalent to:
\[
\begin{aligned}
\dot{z}_1 + A(z, x)z_x + B(z, x) &= 0 \\
\left( \begin{array}{c} z_1(t, 0) \\ z_2(t, L) \\
 \end{array} \right) &= G \left( \begin{array}{c} z_1(t, L) \\ z_2(t, 0) \\
 \end{array} \right),
\end{aligned}
\]

where
\[
A(0, x) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \quad B(0, x) = 0
\]
and
\[
\frac{\partial B}{\partial z}(0, x) = \begin{pmatrix} 0 & a(x) \\ b(x) & 0 \end{pmatrix}
\]
and
\[
G(0) = 0, \quad G'(0) = \begin{pmatrix} 0 & k_0 \varphi(0) \\ k_1 \varphi^{-1}(L) & 0 \end{pmatrix}
\]
with \( k_0 \) and \( k_1 \) defined in (4.3.28). Note that the boundary condition (4.6.3) obtained from (4.2.7) is true at least locally, thus is in the form used in [13 Theorem 6.10]. To be more precise, noticing \( \varphi_1(0) = 1 \), we get from (4.6.1) that
\[
\begin{aligned}
z_1(t, 0) &= v(t, 0) + \sqrt{\frac{g}{H^*(0)}} h(t, 0) \\
&= V(t, 0) - V^*(0) + \sqrt{\frac{g}{H^*(0)}} h(t, 0) \\
&= B(H(t, 0)) - V^*(0) + \sqrt{g H^*(0)} h(t, 0)
\end{aligned}
\]
\[ B(h(t,0) + H^*(0)) - V^*(0) + \sqrt{\frac{g}{H^*(0)}} h(t,0) \]
\[ := l_1(h(t,0)). \quad (4.6.5) \]

Similarly note that \( \phi_2(0) = 1 \), we obtain
\[ z_2(t,0) = v(t,0) - \sqrt{\frac{g}{H^*(0)}} h(t,0) \]
\[ = B(h(t,0) + H^*(0)) - V^*(0) - \sqrt{\frac{g}{H^*(0)}} h(t,0) \]
\[ := l_2(h(t,0)). \quad (4.6.6) \]

From (4.3.6) and (4.6.6), we have
\[ l'_1(0) = B'(H^*(0)) - \sqrt{\frac{g}{H^*(0)}} = b_0 - \sqrt{\frac{g}{H^*(0)}}, \quad (4.6.7) \]

which together with the definition of \( b_0 \) in (4.3.8) gives that
\[ l'_2(0) < 0. \quad (4.6.8) \]

Thanks to the implicit function theorem, we get from (4.6.5), (4.6.6) and (4.6.8) that in a neighborhood of 0
\[ z_1(t,0) = m_1(z_2(t,0)). \quad (4.6.9) \]

Similarly, we can obtain in a neighborhood of 0 that
\[ z_2(t,L) = m_2(z_1(t,L)). \quad (4.6.10) \]

Note that (4.6.9) and (4.6.10) are indeed in the form of (4.6.3). Then [13, Theorem 6.6 and 6.10] can be directly applied to this particular case and gives the sufficient conditions (4.3.51), (4.3.52) and (4.3.53). We remark here that the essential element of the proof for [13, Theorem 6.6 and 6.10] is that finding such \( f_1 \) and \( f_2 \) corresponds to finding a quadratic Lyapunov function for the \( H^2 \)-norm of the form:
\[
V = \int_0^L \left( f_1(x)z_1^2(t,x) + f_2(x)z_2^2(t,x) \right) dx \\
+ \int_0^L \left( f_1(x)z_{1t}^2(t,x) + f_2(x)z_{2t}^2(t,x) \right) dx \\
+ \int_0^L \left( f_1(x)z_{1tt}^2(t,x) + f_2(x)z_{2tt}^2(t,x) \right) dx.
\]

One can look at in particular Lemma 6.8 and (6.19) to (6.22) in [13]. This completes the statement of Remark 4.3.1.

\[
\square
\]

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Chapter 5

Exponential stability of density-velocity systems with boundary conditions and source term for the $H^2$ norm

This chapter takes most of the following article (also referred to as [109]):


Abstract.
In this chapter, we address the problem of the exponential stability of density-velocity systems with boundary conditions. Density-velocity systems are omnipresent in physics as they encompass all systems that consist in a flux conservation and a momentum equation. In this chapter we show that any such system can be stabilized exponentially quickly in the $H^2$ norm using simple local feedbacks, provided a condition on the source term which holds for most physical systems. This is true even when the source term is not dissipative. Besides, the feedback laws obtained only depends on the target values at the boundaries, which implies that they do not depend on the expression of the source term or the force applied on the system and makes them very easy to implement in practice and robust to model errors. For instance, for a river modelled by the Saint-Venant equations this means that the feedback laws do not require any information on the friction model, the slope or the shape of the channel considered. This feat is obtained by showing the existence of a basic $H^2$ Lyapunov functions and we apply it to numerous systems: the general Saint-Venant equations, the isentropic Euler equations, the motion of water in rigid-pipe, the osmosis phenomenon, etc.

5.1 Introduction
Density-velocity systems are important hyperbolic systems as they represent the physical phenomena where the flux is conserved, while the energy can be either increased or decreased. In physics they are found in fluid mechanics, electromagnetism, etc. The increase or decrease of the energy leads to nonuniform steady-states with sometimes large variations in space. In this chapter, we address the exponential stability of such nonlinear systems for the $H^2$ norm, although the result is also true for the $H^p$ norm for any $p \geq 2$. Mentioning the norm is not superfluous as, for nonlinear systems, the stability for different norms are not equivalent [62]. In particular it has been shown in [12] that the basic quadratic Lyapunov functions fail to ensure the stabilization in the $L^2$ norm for nonlinear hyperbolic systems systems and that one has to study the $H^2$ norm instead. Other attempt of basic Lyapunov functions have been constructed to ensure the stability of
hyperbolic systems in the $C^1$ norm, for instance \cite{19, 106, 107}.

Physical density-velocity systems often have well-known conservative or dissipative energy or entropy functions when no source term occurs \cite{68}. These dissipative energy or entropy functions are quite useful for the analysis of such system and enable to prove stability results (see for instance \cite{52, 55} for the use of entropy as control Lyapunov function for Saint-Venant equations and \cite{16} for the Euler equations). When source terms appear, however, no such function is usually known, especially when the source term is not dissipative. In the previous contribution \cite{16}, the authors also studied the stabilization of hyperbolic density-velocity equations, but with dissipative source terms only depending on the unknown functions. This is the case for Saint-Venant equations with no slope and with a constant friction, or for the isentropic Euler equations when the gas pressure is simply assumed to be a function of a gas density and a friction proportional to the square of the velocity. However, the source terms may also depend on the space variable in practice and may not be dissipative. This is the case for example for Saint-Venant equations with both slope and arbitrary friction, or Euler equations with arbitrary friction slope, and general gas pressure, which are more realistic.

For general density-velocity systems, we find that for any $H^2$ steady-state, there always exists a basic quadratic Lyapunov function for the $H^2$ norm (or basic $H^2$ Lyapunov function) that guarantees the exponential stability of the steady-state for the $H^2$ norm provided suitable boundary conditions and physical condition on the source term. For the link between dissipative entropies and basic quadratic Lyapunov functions, one can look at Section 1.4.4 in the Introduction. Our result in this chapter is quite generic and can be widely used in applications, we illustrate it by applying it to several physical systems: the general nonlinear Saint-Venant equations, the general isentropic Euler equations, the motion of water in a rigid pipe, and a flow model under osmosis phenomenon. Moreover, our method has many advantages when applying it in the real world. For example, to stabilize the Saint-Venant equations, we require some knowledge on the section and the velocities only at the boundaries. No information on the internal section profile, on the slope or on the friction model is required. This is very convenient in practice, as this feedback law can be applied without a clear information of the inner state of the channel (bathymetry, material, profile, etc.) since there may be no way to know properly the precise shape or material of the channel. Besides, while many friction models exist (see e.g. \cite{42, Section 4.5}), it also completes the debate about which friction model to use as this feedback law works for any of them, without requiring to know it.

The organization of the chapter is as follows. In Section 5.2, we present the main results: the exponential stabilization of general density-velocity systems with two boundary controls. Moreover in Theorem 5.2.2, the exponential stabilization result with single boundary control is presented. Then we apply the result to several physical models. In Section 5.3, we give the proof of our main results, namely Theorems 5.2.1 and Theorem 5.2.2. Finally, some detailed computations are provided in the appendix.

5.2 Model considered and main result

A nonlinear hyperbolic density-velocity system is composed of a mass conservation law and a balance of momentum \cite{16} and is thus given by

\[
\begin{align*}
\partial_t H + \partial_x (HV) &= 0, \\
\partial_t V + V \partial_x V + \partial_x (P(H, x)) + S(H, V, x) &= 0,
\end{align*}
\]  

(5.2.1)

(5.2.2)

where $t \in [0, +\infty)$, $x \in [0, L]$ with $L > 0$ any arbitrary constant. In many applications, $H : [0, +\infty) \times [0, L] \to (0, +\infty)$ denotes the density, $V : [0, +\infty) \times [0, L] \to (0, +\infty)$ denotes the propagation velocity, $HV$ is the flow density and and $S(H, V, x)$ is a source term resulting of non-conservative forces acting on the system, such as slope or friction. The first equation expresses the flux conservation and is often known as continuity equation, while the second equation is usually referred as dynamical or momentum equation. In this second equation, $V \partial_x V$ represents the variation of the kinetic energy, while $\partial_x (P(H, x))$ represents the variation of the potential energy and corresponds to a conservative force (e.g. pressure, gravitation, etc.). As we are interested in physical systems, we assume that $S \in C^2((0, +\infty)^3; \mathbb{R})$, $P \in C^2((0, +\infty)^2; \mathbb{R})$ and here and
hereafter, we also assume that
\[ H > 0, \quad V > 0, \quad \partial_H P(H, x) > 0. \] (5.2.3)
The steady-states \((H^*, V^*)\) of \((5.2.1)-(5.2.2)\) are the solutions of
\[ \begin{align*}
H^*_x &= V^*_x, \\
V^*_x &= V^* S(H^*, V^*; \cdot) + \partial_x P(H^*, \cdot). 
\end{align*} \] (5.2.4) (5.2.5)
For each initial condition \((H^*(0), V^*(0)) \in (0, +\infty)^2\) satisfying \(\partial_H P(H^*(0), 0) H^*(0) - V(0)^2 > 0\), there exists a unique maximal solution to \((5.2.4)-(5.2.5)\) and this maximal solution exists as soon as the condition \(\partial_H P(H^*, \cdot) H^* > V^*\) is satisfied. Besides, as hyperbolic systems with propagation velocities of the same sign can always be stabilized by the means of proportional boundary feedback (see e.g. [107]), we assume in the following that the propagation velocities of this system have opposite signs, which, from \((5.2.1)-(5.2.2)\), means that \(\partial_H P(H^*, \cdot) H^* > V^*\). This holds for example in the case of the fluvial regime for Saint-Venant equations.

In the following, we give two strategies of boundary controls. As a first strategy, Theorem \((5.2.1)\) relies on two boundary controls, i.e. the number of controls is equal to the number of the unknown functions. While in practice, one may control only one boundary. In the regulation of navigable rivers, for instance, one usually apply only one control at the downstream of the channel. Theorem \((5.2.2)\) is thus concerned with the stabilization of general density-velocity systems with a single boundary control.

**Two boundary controls** We aim at stabilizing the steady-states of \((5.2.1)-(5.2.2)\) with boundary feedback controls. We suppose that the boundary conditions have the form
\[ \begin{align*}
V(t, 0) &= B_1(H(t, 0)), \\
V(t, L) &= B_2(H(t, L)), 
\end{align*} \] (5.2.6)
where the control function \(B = (B_1, B_2) : \mathbb{R}^2 \to \mathbb{R}^2\) is of class \(C^2\). These kind of boundary conditions are imposed by physical devices in engineering system (e.g. sluice gates, feeding valves, pumps, etc.). This control function is one of the most simple possible feedback law as one does not need to know the full state of the system. Moreover, this control is local, which means that one only needs to measure the value at the same end where the control acts.

As we study the stabilization in the \(H^2\) norm, for any given initial condition
\[ H(0, x) = H_0(x), \quad V(0, x) = V_0(x), \quad x \in [0, L], \] (5.2.7)
with \((H_0, V_0) \in H^2((0, L); \mathbb{R}^2)\), we need to add the following first-order compatibility condition for any given initial condition sufficiently regular
\[ \begin{align*}
V_0(0) &= B_1(H_0(0)), \\
V_0(L) &= B_2(H_0(L)), \\
(V_0 \partial_x V_0 + \partial_H P(H_0, \cdot) \partial_x H_0 + \partial_x P(H_0, \cdot) + S(H_0, V_0, \cdot))(0) &= B'_1(H_0(0)) \partial_x (H_0 V_0)(0), \\
(V_0 \partial_x V_0 + \partial_H P(H_0, \cdot) \partial_x H_0 + \partial_x P(H_0, \cdot) + S(H_0, V_0, \cdot))(L) &= B'_2(H_0(L)) \partial_x (H_0 V_0)(L)). 
\end{align*} \] (5.2.8)
We recall now the definition of the exponential stability for the \(H^2\) norm.

**Definition 5.2.1.** A steady-state \((H^*, V^*)\) of the system \((5.2.1)-(5.2.2)\) is exponentially stable for the \(H^2\) norm if there exist \(\delta > 0, \gamma > 0\) and \(C > 0\) such that, for any \((H_0, V_0) \in H^2((0, L); \mathbb{R}^2)\) satisfying
\[ |H_0 - H^*|_{H^2} + |V_0 - V^*|_{H^2} < \delta, \] (5.2.9)
and the compatibility conditions \((5.2.8)\), and for any \(T > 0\), the Cauchy problem \((5.2.1)-(5.2.2)\) and \((5.2.6)\) has a unique solution \((H(t, \cdot), V(t, \cdot)) \in H^2((0, L); \mathbb{R}^2)\) satisfying
\[ |H(t, \cdot) - H^*|_{H^2} + |V(t, \cdot) - V^*|_{H^2} \leq C e^{-\gamma t} \left( |H_0 - H^*|_{H^2} + |V_0 - V^*|_{H^2} \right), \quad \forall t \in [0, T). \] (5.2.10)
Our main result is the following

**Theorem 5.2.1.** Assume that \( S \in C^2(\mathbb{R}^3; \mathbb{R}) \) and \( P \in C^2(\mathbb{R}^3; \mathbb{R}) \), let \((H^*, V^*)\) be a steady-state of the nonlinear hyperbolic density-velocity system \((5.2.1), (5.2.2), (5.2.6)\) satisfying

\[
\frac{\partial V S(H^*, V^*, \cdot)}{V^*} - \frac{\partial H S(H^*, V^*, \cdot)}{H^*} \geq 0.
\]  
(5.2.11)

If the boundary conditions satisfy:

\[
B'_1(H^*(0)) \in \left\{ -\frac{\partial H P(H^*(0), 0)}{V^*(0)}, -\frac{V^*(0)}{H^*(0)} \right\},
\]

\[
B'_2(H^*(L)) \in \mathbb{R} \setminus \left\{ -\frac{\partial H P(H^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)} \right\},
\]

then the steady-state \((H^*, V^*)\) is exponentially stable for the \(H^2\) norm.

**Remark 5.2.1.** Note that condition \((5.2.11)\) holds naturally for most physical systems with source terms, (e.g. friction, slope, electric field, external forces, etc.), as illustrated in Section 5.2.1. Besides, note also that the source term does not have to be dissipative, as \(S\) could be negative.

The proof of this result is given in Section 5.3. As announced in the introduction, this is done by showing the existence of a basic quadratic Lyapunov function for the \(H^2\) norm (or basic \(H^2\) Lyapunov function).

**Single boundary control** Suppose now that we have only a single feedback control, the other boundary condition being imposed, for instance by a constant but unknown upstream flow rate on which we cannot act. The boundary conditions are now

\[
H(t, 0)V(t, 0) = Q_0, \\
V(t, L) = B_2(H(t, L)),
\]  
(5.2.13)

where \(Q_0\) is the unknown constant inflow upstream, while \(B_2 : \mathbb{R} \to \mathbb{R}\) of class \(C^2\) is still the control function. Using the same basic quadratic Lyapunov function, we can still achieve the exponential stability which is a direct application of Theorem 5.2.1 by noticing now that \(B_1(H(t, 0)) = Q_0/H(t, 0)\) and that the steady-state satisfies \(H^*(x)V^*(x) = Q_0\).

**Theorem 5.2.2.** Assume that \( S \in C^2([0, +\infty[; \mathbb{R}) \) and \( P \in C^2([0, +\infty[; \mathbb{R}) \), let \((H^*, V^*)\) be a steady-state of the nonlinear hyperbolic density-velocity system \((5.2.1), (5.2.2), (5.2.13)\) satisfying \((5.2.11)\). If the boundary control satisfies:

\[
B'_2(H^*(L)) \in \mathbb{R} \setminus \left\{ -\frac{\partial H P(H^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)} \right\},
\]

then the steady-state \((H^*, V^*)\) is exponentially stable for the \(H^2\) norm.

**Remark 5.2.2.** Note that \(Q_0\) is assumed to be constant otherwise no steady-state \((H^*, V^*)\) exists. However, the stabilization of slowly-varying target-state when \(Q_0\) can vary, possibly a lot, but slowly would hold under the same condition, adapting the control as in \((77)\).

**Remark 5.2.3.** Theorem 5.2.2 still holds if the control is located in \(x = 0\) instead, while the imposed flow is located in \(x = L\).

### 5.2.1 Applications to physical systems

The system \((5.2.1), (5.2.2)\) with boundary conditions \((5.2.6)\) covers many well-known systems and we give now a few examples.
General Saint-Venant equations  The Saint-Venant equations are the basis model for the regulation of navigable rivers and irrigation networks in agriculture. The stabilization of the Saint-Venant equations by means of local boundary feedbacks has been widely studied [16, 18, 52, 53, 54, 69]. Recently in [110], the authors obtained the stabilization of the Saint-Venant equations with non-negligible friction and arbitrary slope. However, this result is obtained under the assumption of a rectangular cross section with a constant width and a known friction model. In the following, we show that our result applies to the most general 1D Saint-Venant equations with arbitrary varying slope, section profile and friction model [10]:

\[
\partial_t A + \partial_x (AV) = 0, \\
\partial_t (AV) + \partial_x (AV^2) + gA(\partial_x H - S_b(x) + S_f(A, V, x)) = 0,
\]

where \(A\) is the cross-sectional area of the section, \(V\) is the velocity, \(AV\) is consequently the flux, \(H\) is the height of the water, \(S_b\) is the slope, \(S_f\) is the friction and \(g\) is the gravity acceleration. Note that the friction logically depends on \(H\) and \(V\) but can also depend on \(x\) for external reasons, for instance if the material of the channel changes. Whatever is the section profile, \(A\) is strictly increasing with \(H\), thus there exists a function \(G\) strictly increasing with \(A\) such that \(H = G(A, x)\) and consequently, using (5.2.15), (5.2.16) can be written as

\[
\partial_t V + V \partial_x V + g \partial_A G(A, x) \partial_x A + g \partial_x S(A, V, x) + g(S_f(A, V, x) - S_b(x)) = 0.
\]

Thus, system (5.2.15)-(5.2.16) has the form (5.2.1)-(5.2.2) with \(P = gG(A, x)\) and \(S = g(S_f - S_b)\). Besides, to be physically acceptable, the friction term has to be increasing with \(V\) and decreasing with \(A\). Hence \(\partial_V S = g \partial_V S_f > 0\) and \(\partial_A S = g \partial_A S_f < 0\). Noticing that \(\partial_A P = g \partial_A G(A, x) > 0\), condition (5.2.11) is satisfied and we have the following theorem

**Theorem 5.2.3.** Any steady-state \((A^*, V^*)\) of the general Saint-Venant equations (5.2.15), (5.2.17) with boundary conditions (5.2.6) with \(A\) instead of \(H\), is exponentially stable for the \(H^2\) norm provided that

\[
B'(A^*(0)) \in \left[ \begin{array}{cc} -g \partial_A G'(A^*(0), 0) & V^*(0) \\ V^*(0) & -A^*(0) \end{array} \right], \\
B'(A^*(L)) \in \mathbb{R} \setminus \left[ \begin{array}{cc} -g \partial_A G'(A^*(L), L) & V^*(L) \\ V^*(L) & -A^*(L) \end{array} \right].
\]

Water motion in a rigid pipe  The water motion in a rigid pipe is a common example for engineering system, whose equations are given in [15] as follows

\[
\begin{align*}
\partial_t \left( \exp \left( \frac{gP}{c^2} \right) \right) + \partial_x \left( V \exp \left( \frac{gP}{c^2} \right) \right) &= 0, \\
\partial_t V + V \partial_x V + \partial_x (gP) + S_f(V, x) &= 0,
\end{align*}
\]

where \(P\) is the piezometric head, \(V > 0\) is the speed of the water, \(c\) is the sound velocity in water, \(g\) is the gravity acceleration, and \(S_f\) is the friction term. As previously, to be physically acceptable, the friction term has to be nondecreasing with \(V\), thus (5.2.11) holds. Denoting \(H = \exp \left( \frac{gP}{c^2} \right)\), this system has the form of (5.2.1)-(5.2.2) with \(P = gH\). And obviously \(\partial_H P > 0\) thus Theorem 5.2.1 applies again.

The isentropic Euler equations  The isentropic Euler equations are used to model the gas transportation in pipelines. There are many literatures on the stabilization of the isentropic Euler equations [16, 51, 52, 94, 95, 99, 101]. But all those results are obtained without considering the pipeline slope and using the polytropic gas assumption or the isothermal assumption. The isentropic Euler equations with slope and friction have exactly the form (5.2.1)-(5.2.2) as (see e.g. [14] [1.8.1] or [102])

\[
\begin{align*}
\partial_t \rho + \partial_x (gV) &= 0, \\
\partial_t V + V \partial_x V + \frac{\partial_x \left( P(V) \right)}{g} + \frac{1}{2} g |V|^2 + g \sin \alpha(x) &= 0,
\end{align*}
\]

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The proofs of Theorems 5.2.4 and 5.2.5 are given in Section 5.3.

Theorem 5.2.5. Assume that

be significantly improved. The situation is even clearer in the case of a single boundary control. This theorem shows therefore that the condition (5.2.12) given in Theorem 5.2.1 is quite sharp and cannot

\[ \lambda > \frac{1}{20}V|\dot{V}| + g \sin \alpha. \]

Thus, \( \partial_s P(\gamma) > 0, \partial_s S = 0, \partial_v S > 0 \) as long as \( V > 0 \), which implies that [5.2.11] holds and that Theorem [5.2.1] applies. Note that this holds in particular in the case where the gas is polytropic, i.e. \( P = a^2 \rho^\gamma \) with \( \gamma > 1 \) (as in [16]), and in the case of the isothermal Euler equation, i.e. \( P = a^2 \rho \) (as in [102]).

Flow under osmosis Osmosis is a spontaneous movement of solvant or solute through a semipermeable membrane in a solute/solvent mix. This phenomenon is extremely important in chemistry and biology as it is the main way by which water is transported out of cells in living organisms. Besides, biological membranes allow much faster filtration than any artificial mechanical membrane, thus attempts have been recently made to design active membranes that would mimic this behavior and a mechanical model for this phenomenon can be found in [152].

Osmosis phenomenon through a membrane permeable to the solute but not to the solvent can be modeled by a potential barrier which acts on the solute. This creates, from Newton’s law, a volume force on the fluid\(-c(x)\partial_s U\), where \( U \) is the profile of the potential barrier, compactly supported, \( c \) is the concentration and \( x \) is the space variable [152]. In an inviscid fluid modeled by the isentropic Euler equations (5.2.20), this reduces to adding a pressure term. Therefore, we still have \( \partial_s P(\gamma) > 0, \partial_s S = 0, \partial_v S > 0 \) as long as \( V > 0 \), and Theorem [5.2.1] applies. Note that any external potential acting on a fluid modeled by the isentropic Euler equations would fit in our framework, osmosis is only an example.

5.2.2 Optimality of the conditions on the control

In this section, we will show the optimality of the conditions on the control in the sense that no basic quadratic Lyapunov function that would always exist for density-velocity systems satisfying (5.2.11) can give strictly less restrictive boundary conditions than (5.2.12) and (5.2.14), making these conditions quite sharp. These results are given by Theorem 5.2.4 and Theorem 5.2.5.

Theorem 5.2.4. Assume that \( S \in C^2((0, +\infty)^3; \mathbb{R}) \) and \( P \in C^2((0, +\infty)^3; \mathbb{R}) \). Let a steady-state \((H^*, V^*) \in C^3([0, L])\) of the nonlinear hyperbolic density-velocity system (5.2.1), (5.2.2), (5.2.6) satisfying (5.2.11) with \( S(H^*, V^*, \cdot) \leq 0 \). For any \( \varepsilon > 0 \) there exists \( L > 0 \) such that, if there exists a basic quadratic Lyapunov function for the the \( H^2 \) norm, then

\[
B'(H^*(0)) \in \left( -\varepsilon - \frac{\partial_s P(H^*(0), 0)}{V^*(0)}, \varepsilon - \frac{V^*(0)}{H^*(0)} \right),
\]

\[
B'(H^*(L)) \in \mathbb{R} \setminus \left[ -\varepsilon - \frac{\partial_s P(H^*(L), L)}{V^*(L)}, -\varepsilon - \frac{V^*(L)}{H^*(L)} \right]. \tag{5.2.21}
\]

This theorem shows therefore that the condition (5.2.12) given in Theorem [5.2.1] is quite sharp and cannot be significantly improved. The situation is even clearer in the case of a single boundary control.

Theorem 5.2.5. Assume that \( S \in C^2((0, +\infty)^3; \mathbb{R}) \) and \( P \in C^2((0, +\infty)^3; \mathbb{R}) \). Let a steady-state \((H^*, V^*) \in C^3([0, L])\) of the nonlinear hyperbolic density-velocity system (5.2.1), (5.2.2), (5.2.13) satisfying (5.2.11). There exists a basic quadratic Lyapunov function for the the \( H^2 \) norm if and only if

\[
B'(H^*(L)) \in \mathbb{R} \setminus \left[ -\varepsilon - \frac{\partial_s P(H^*(L), L)}{V^*(L)}, -\varepsilon - \frac{V^*(L)}{H^*(L)} \right]. \tag{5.2.22}
\]

The proofs of Theorems 5.2.4 and 5.2.5 are given in Section 5.3.
5.3 Exponential stability of density-velocity hyperbolic systems

In this section we prove Theorem 5.2.1. Let \( (H^*, V^*) \) be a steady-state of \([5.2.1]-[5.2.2]\). We start by proving the exponential stability of the linearized system around this steady-state for the \( L^2 \) norm to give an idea of how the proof works and then, we show that the same type of Lyapunov function can be applied to ensure the exponential stability of the nonlinear system for the \( H^2 \) norm.

5.3.1 Exponential stability of the linearized system

Around the steady-state \( (H^*, V^*) \), the linearized system of \([5.2.1]-[5.2.2] \) and \([5.2.6] \) is given by:

\[
\begin{align*}
\partial_t h + V^* \partial_x h + H^* \partial_x v + V_x^* h + H_x^* v &= 0, \\
\partial_t v + V^* \partial_x v + V_x^* v + \partial_H P(H^*, x) \partial_x h + \partial^2_H P(H^*, x) H_x^* h + \partial^2_x H P(H^*, x) h \\
+ \partial_H S(H^*, V^*, x) h + \partial_V S(H^*, V^*, x) v &= 0.
\end{align*}
\]  

(5.3.1)

and

\[
\begin{align*}
v(t, 0) &= c_1 h(t, 0), \\
v(t, L) &= c_2 h(t, L),
\end{align*}
\]  

(5.3.2)

where \( h = H - H^* \) and \( v = V - V^* \) are the perturbations and \( c_1 = \mathcal{B}'(H^*(0)) \) and \( c_2 = \mathcal{B}'(H^*(L)) \). To simplify the notations, we denote from now on

\[
\partial_H P(H^*, x) := f(H^*, x), \quad S_{H^*} := \partial_H S(H^*, V^*, x), \quad S_{V^*} := \partial_V S(H^*, V^*, x),
\]

(5.3.3)

Thus, the linearized system of \([5.2.1]-[5.2.2] \) and \([5.2.6] \) around the steady-state \( (H^*, V^*) \) given by \([5.3.1] \) becomes

\[
\begin{pmatrix}
h \\
v
\end{pmatrix}_t + \begin{pmatrix}
V^* & H^* \\
f(H^*, x) & V^*
\end{pmatrix}
\begin{pmatrix}
h \\
v
\end{pmatrix}_x + \begin{pmatrix}
V_x^* & H_x^* \\
\mathcal{I}_{H^*} + \partial_H f(H^*, x) H_x^* & S_{V^*} + V_x^*
\end{pmatrix}
\begin{pmatrix}
h \\
v
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]  

(5.3.4)

We prove the following proposition

**Proposition 5.3.1.** Let \( (H^*, V^*) \) be any given steady-state such that \([5.2.1] \) holds, if the boundary conditions satisfy:

\[
c_1 \in \left[-\frac{f(H^*(0), 0)}{V^*(0)}, \frac{V^*(0)}{H^*(0)}\right], \quad c_2 \in \mathbb{R} \setminus \left[-\frac{f(H^*(L), L)}{V^*(L)}, \frac{V^*(L)}{H^*(L)}\right],
\]

(5.3.5)

then the null steady-state \( h = 0, v = 0 \) of the system \([5.3.4]-[5.3.2] \) is exponentially stable for the \( L^2 \) norm.

**Proof.** Observe that the matrix \( \begin{pmatrix}
V^* & H^* \\
f(H^*, \cdot) & V^*
\end{pmatrix} \) can be diagonalized, therefore the system can be put under the Riemann invariant form by the following change of variables

\[
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{f(H^*, x)}{H^*}} & 1 \\
-\sqrt{\frac{f(H^*, x)}{H^*}} & 1
\end{pmatrix}
\begin{pmatrix}
h \\
v
\end{pmatrix}.
\]

(5.3.6)

Then \([5.3.4] \) becomes (see Appendix \([5.4.1] \))

\[
\begin{align*}
\partial_t z_1 + \lambda_1 \partial_x z_1 + \gamma_1 z_1 + \delta_1 z_2 &= 0, \\
\partial_t z_2 - \lambda_2 \partial_x z_2 + \gamma_2 z_1 + \delta_2 z_2 &= 0.
\end{align*}
\]

(5.3.7)
The linear change of variables (5.3.6). Therefore, it suffices to show the exponential decay of \( V \), which means that \( V \) \( \phi \)  

Lemme 5.3.1. For any constant friction coefficient.

(\text{5.3.7}) \text{with any given initial condition } z(t,0) = (z_1, z_2) \in L^2((0, L); \mathbb{R}^2) \text{ is well-posed (see [13] Appendix A), which implies that the original system in physical coordinates is also well-posed.}

We define the function \( \phi \) by

\[
\phi(x) = \exp \left( \int_0^x \frac{\gamma_1(s)}{\lambda_1(s)} + \frac{\delta_2(s)}{\lambda_2(s)} \, ds \right)
\]

(5.3.12)

and we introduce the following lemma, that can also be found in [10] in the particular case of the Saint-Venant equations with constant rectangular section and friction given by \( S_f = kV^2A^{-1} \) where \( k > 0 \) is a constant friction coefficient.

**Lemme 5.3.1.** For any \( x \in [0, L] \),

\[
\left( \begin{array}{c} \lambda_2 \\ \lambda_1 \end{array} \right) \phi(x) = \phi(x) \frac{\lambda_2}{\lambda_1} + \phi(x) \frac{\lambda_2^2}{\lambda_1^2} \left( \begin{array}{c} \lambda_2 \\ \lambda_1 \end{array} \right) \left( \begin{array}{c} \lambda_2 \\ \lambda_1 \end{array} \right).
\]

(5.3.13)

The proof of this lemma is given in the Appendix 5.4.2. We introduce now the following Lyapunov function candidate for the \( L^2 \) norm

\[
V = \int_0^L \left( f_1(x)e^{2\int_0^x \frac{\lambda_2(s)}{\lambda_1(s)} \, ds} e^{-\frac{\lambda_2}{\lambda_1}z_1^2(t,x)} + f_2(x)e^{-2\int_0^x \frac{\lambda_2(s)}{\lambda_1(s)} \, ds} e^{\frac{\lambda_2}{\lambda_1}z_2^2(t,x)} \right) \, dx,
\]

(5.3.14)

where \( \mu > 0 \) is a constant and \( f_1, f_2 \) are positive \( C^1 \) functions to be chosen later on. From the positivity of \( f_1 \) and \( f_2 \), there exist \( a_1 \) and \( a_2 \) positive constants such that

\[
a_2 \|(z_1, z_2)\|_{L^2((0,L),\mathbb{R}^2)} \leq V \leq a_1 \|(z_1, z_2)\|_{L^2((0,L),\mathbb{R}^2)}
\]

(5.3.15)

which means that \( V \) is equivalent to the \( L^2 \) norm of \( (z_1, z_2) \), thus is equivalent to the \( L^2 \) norm of \( (h, v) \) from the linear change of variables (5.3.6). Therefore, it suffices to show the exponential decay of \( V \) to obtain the
Using the boundary conditions (5.3.10), we get exponential stability of (5.3.4) and (5.3.2) for the $L^2$ norm. Differentiating (5.3.14) with time along the $C^1$ solutions, one has

\[
\frac{dV}{dt} = - \left[ \lambda_1 f_1 e^{2 \int_0^t \frac{s_1(t, t')}{x(t')}} dt' - \frac{s_1(t, t')}{x(t')} - \lambda_2 f_2 e^{-2 \int_0^t \frac{s_1(t, t')}{x(t')}} dt' \right] L \frac{s_1(t, t')}{x(t')} z_1^2 \\
- \int_0^L \left[ \left( 2 f_1 \gamma_1 e^{2 \int_0^t \frac{s_1(t, t')}{x(t')}} dt' - \frac{s_1(t, t')}{x(t')} - \lambda_1 f_1 e^{2 \int_0^t \frac{s_1(t, t')}{x(t')}} dt' \right) \frac{s_1(t, t')}{x(t')} z_1^2 \\
+ \left( 2 f_2 \gamma_2 e^{-2 \int_0^t \frac{s_1(t, t')}{x(t')}} dt' \right) \frac{s_1(t, t')}{x(t')} z_1^2 \\
+ \left( 2 f_1 \delta_1 e^{2 \int_0^t \frac{s_1(t, t')}{x(t')}} dt' \frac{s_1(t, t')}{x(t')} z_1^2 \right) dx \right].
\] (5.3.16)

Using the boundary conditions (5.3.10), we get

\[
\frac{dV}{dt} = - \left( \lambda_1 (L) f_1 (L) e^{2 \int_0^L \frac{s_1(t, t')}{x(t')}} dt' - \frac{s_1(L, t)}{x(L)} - k_1^2 \lambda_2 (L) f_2 (L) e^{-2 \int_0^L \frac{s_1(t, t')}{x(t')}} dt' \right) z_1^2 \]

\[= \left( \lambda_1 f_1 e^{2 \int_0^L \frac{s_1(t, t')}{x(t')}} dt' - \frac{s_1(L, t)}{x(L)} - k_1^2 \lambda_2 f_2 e^{-2 \int_0^L \frac{s_1(t, t')}{x(t')}} dt' \right) z_1^2 \]

\[- \int_0^L \left( e^{2 \int_0^L \frac{s_1(t, t')}{x(t')}} dt' \right)^T I \left( e^{-2 \int_0^L \frac{s_1(t, t')}{x(t')}} dt' \right) dx,
\] (5.3.17)

where

\[
I = \begin{pmatrix}
-(\lambda_1 f_1)' - \frac{\mu s_1}{\lambda_1} f_1 e^{-s_1 x} & f_1 \delta_1 \phi(x) e^{-s_1 x} + f_2 \gamma_2 \phi^{-1}(x) e^{s_1 x} \\
-f_1 \delta_1 \phi(x) e^{-s_1 x} & (\lambda_2 f_2)' - \frac{\mu s_1}{\lambda_2} f_2 e^{s_1 x}
\end{pmatrix}. \] (5.3.18)

Therefore, by continuity and using the definition of $\phi$ given in (5.3.12), it suffices to show that there exist $f_1$ and $f_2$, such that the following matrix

\[
J_0 = \begin{pmatrix}
-(\lambda_1 f_1)' & f_1 \delta_1 \phi(x) + f_2 \gamma_2 \phi^{-1}(x) \\
(f_1 \delta_1 \phi(x) + f_2 \gamma_2 \phi^{-1}(x)) & (\lambda_2 f_2)'
\end{pmatrix}
\] (5.3.19)

is positive definite and that

\[
\lambda_1 (L) f_1 (L) \phi^2 (L) - k_1^2 \lambda_2 (L) f_2 (L) > 0,
\]

\[
\lambda_2 f_2 (0) - k_1^2 \lambda_1 (0) f_1 (0) > 0
\] (5.3.20)

to prove the exponential decay of $V$.

Before going any further, observe that under the assumption (5.2.11), from (5.3.9), (5.3.8) and noticing the notations (5.3.3), one has

\[
\frac{\phi \delta_1}{\lambda_1} + \frac{\phi}{\lambda_2} \left( \frac{\lambda_2}{\lambda_1} \phi \right)^2 = \frac{\phi}{\lambda_1^2} \left( \frac{\lambda_1 \delta_1 + \lambda_2 \gamma_2}{\lambda_1} \right)
\]

\[
= \frac{\phi}{\lambda_1^2} \left( \frac{\lambda_1 + \lambda_2}{2} S_\phi + \frac{(\lambda_2 - \lambda_1)}{2} \sqrt{H^*} \int \frac{H^*}{f(H^*, x)} + \frac{(\lambda_1^2 - \lambda_2^2)}{4} \partial_x f(H^*, x) \right)
\]
\[\lambda_2 \phi / \lambda_1 \text{ is a solution to the differential equation }\]
\[\eta' = \left[ \frac{\delta_1}{\lambda_1} \phi + \frac{\gamma_2}{\lambda_2} \phi^{-1} \eta^2 \right], \quad \eta(0) = \frac{\lambda_2(0)}{\lambda_1(0)}\]
(5.3.22)
on \([0, L]\). Thus, there exists \(\varepsilon_1 > 0\) such that for any \(\varepsilon \in [0, \varepsilon_1]\), there exists a solution \(\eta_\varepsilon\) on \([0, L]\) to
\[\eta_\varepsilon' = \left[ \frac{\delta_1}{\lambda_1} \phi + \frac{\gamma_2}{\lambda_2} \phi^{-1} \eta_\varepsilon^2 \right] + \varepsilon, \quad \eta_\varepsilon(0) = \frac{\lambda_2(0)}{\lambda_1(0)}\]
(5.3.23)
and such that we can define a map \(\varepsilon \to \eta_\varepsilon\) which is \(C^0\) on \([0, \varepsilon_1]\). Let us define
\[f_1 = (\lambda_1 \eta_\varepsilon)^{-1} \text{ and } f_2 = \lambda_2^{-1} \eta_\varepsilon,\]
(5.3.24)
where \(\varepsilon \in (0, \varepsilon_1)\) can be chosen later on. One has from (5.3.5) and (5.3.11) that
\[k_1^2 \leq \left( \frac{\lambda_2(0)}{\lambda_1(0)} \right)^2, \quad k_2^2 < \left( \frac{\lambda_1(L)}{\lambda_2(L)} \right)^2.\]
(5.3.25)
Therefore from the continuity of \(\varepsilon \to \eta_\varepsilon\), there exists \(0 < \varepsilon_2 < \varepsilon_1\) such that for all \(\varepsilon \in (0, \varepsilon_2)\)
\[k_1^2 < \frac{\lambda_2(0)f_2(0)}{\lambda_1(0)f_1(0)}, \quad k_2^2 < \frac{\lambda_1(L)f_1(L)}{\lambda_2(L)f_2(L)} \phi^2(L),\]
(5.3.26)
which is exactly the same as condition (5.3.20) from the definition of \(\phi\) in (5.3.12). We choose such \(\varepsilon \in (0, \varepsilon_2)\), and we are left to prove that \(I_0\) defined by (5.3.19) is positive definite. We have from (5.3.19), (5.3.24) and (5.3.23) that
\[\det(I_0) = - (\lambda_1 f_1)'(\lambda_2 f_2)' - (f_1 \delta_1 \phi + f_2 \gamma_2 \phi^{-1})^2\]
\[= \frac{1}{\eta_\varepsilon^2} \left( \eta_\varepsilon' \right)^2 \left( \frac{\delta_1}{\lambda_1} \phi + \frac{\gamma_2}{\lambda_2} \phi^{-1} \eta_\varepsilon^2 \right)^2 > 0.\]
(5.3.27)
Besides, from (5.3.23) and (5.3.24), one has \(- (\lambda_1 f_1)' > 0\) and \((\lambda_2 f_2)' > 0\), hence \(I\) is positive definite. By continuity, there exists \(\mu > 0\) such that
\[\frac{dV}{dt} \leq - \mu V\]
(5.3.28)
along the \(C^1\)-solutions of the system (5.3.7) and (5.3.10) for any \(\mu \in (0, \mu_1)\). Since the \(C^1\)-solutions are dense in the set of \(L^2\)-solutions, inequality (5.3.28) also holds in the sense of distributions for the \(L^2\)-solutions (see [15 Section 2.1]) for the details). Thus, the exponential stability of (5.3.4)–(5.3.10) in the \(L^2\) norm is also guaranteed thanks to the linear change of variables (5.3.6). This ends the proof of Proposition 5.3.1.  

### 5.3.2 Exponential stability of the nonlinear system

For the exponential stability of nonlinear system, the proof will be similar to the linearized case. For a given steady-state \((H^*, V^*)\) defined on \([0, L]\), we can still define \(h = H - H^*\) and \(v = V - V^*\) as previously and \((z_1, z_2)\) using the same change of variables (5.3.6). Then, for \((z_1, z_2)\) small enough, the system (5.2.1)–(5.2.2), (5.2.6) is equivalent to
\[z_t + A(z, x)z_x + M(z, x)z = 0,\]
(5.3.29)
where
\[ A(0, x) = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & -\lambda_2(x) \end{pmatrix}, \quad M(0, x) = \begin{pmatrix} \gamma_1(x) & \delta_1(x) \\ \gamma_2(x) & \delta_2(x) \end{pmatrix}, \] (5.3.30)
and
\[ z_1(t, 0) = m_1(z_2(t, 0)), \quad z_2(t, L) = m_2(z_1(t, L)), \] (5.3.31)
with
\[ m'_1(0) = k_1, \quad m'_2(0) = k_2, \] (5.3.32)
here, \( k_1 \) and \( k_2 \) are defined as (5.3.11). In (5.3.31), \( m_1 \) and \( m_2 \) are found by the implicit function theorem around \( 0 \), for \( z_1 \) and \( z_2 \) small enough (see [110, A.2] for more details in a similar case). Noticing that the exponential stability of the steady-state (\( H^*, V^* \)) of system (5.2.1)–(5.2.2) and (5.2.6) is therefore equivalent to the exponential stability of the null steady-state \((z_1 = 0, z_2 = 0)\) of system (5.3.29)–(5.3.32), we use the following theorem, which is a direct application of [15, Theorem 6.10].

**Theorem 5.3.2.** If there exists \( C^1 \) functions \( g_1(x) > 0 \) and \( g_2(x) > 0 \) such that, with \( Q = \text{diag}(g_1(x), g_2(x)) \), one has
\[ -(QA(0, \cdot)' + QM(0, x) + MT(0, x)Q) \] is positive definite on \([0, L] \) and the following inequalities hold
\[ k_1^2 \leq \frac{\lambda_2(0)g_2(0)}{\lambda_1(0)g_1(0)}, \quad k_2^2 < \frac{\lambda_1(L)g_1(L)}{\lambda_2(L)g_2(L)}, \] (5.3.34)
then the null steady-state of the system (5.3.29)–(5.3.32) is exponentially stable for the \( H^2 \) norm.

**Remark 5.3.1.** This theorem actually shows the existence of a Lyapunov function for the \( H^2 \) norm of the form
\[ V = \int_0^L (f_1(E(z, x)z_x)^2 + f_2(E(z, x)z_x)^2)dx + \int_0^L (f_1(E(z, x)z_x)^2 + f_2(E(z, x)z_x)^2)dx \]
where \( E(0, \cdot) = 1d \) (see [13, Chapter 6] for more details). This is the reason why we claim that this proof is actually the same as the proof of the exponential stability in the linearized case, and we will now see that we can use a similar Lyapunov function but for the \( H^2 \) norm.

**Proof of Theorem 5.2.1.** Let
\[ g_1 := e^{\int_0^x \frac{\gamma_1(s)}{\delta_1(s)} ds} f_1, \quad g_2 := e^{-\int_0^x \frac{\gamma_2(s)}{\delta_2(s)} ds} f_2, \]
where \( f_1 \) and \( f_2 \) are defined in (5.3.21). One can directly check that
\[ -(QA(0, \cdot)' + QM(0, x) + MT(0, x)Q) = \begin{pmatrix} e^{\int_0^L \frac{\gamma_1(s)}{\delta_1(s)} ds} 0 \\ 0 e^{-\int_0^L \frac{\gamma_2(s)}{\delta_2(s)} ds} \end{pmatrix} I_0 \begin{pmatrix} e^{\int_0^L \frac{\gamma_1(s)}{\delta_1(s)} ds} 0 \\ 0 e^{-\int_0^L \frac{\gamma_2(s)}{\delta_2(s)} ds} \end{pmatrix} \]
with \( I_0 \) defined as (5.3.19), as \( I_0 \) is positive definite from (5.3.27), condition (5.3.33) is thus satisfied. Condition (5.3.34) is satisfied from (5.3.26) by noticing the definition of \( \phi \) given in (5.3.12). Thus, Theorem 5.3.2 applies and Theorem 5.2.1 holds. \( \square \)
5.3.3 Optimality of the control

In this subsection we show Theorem 5.2.4 and 5.2.5.

Proof of Theorem 5.2.4 Let assume that along \( H^*, V^* \), \( S(H^*, V^*, x) + \partial_x P(H^*, V^*, x) \leq 0 \). Then from 5.2.1, 5.2.2, the steady-state \( (H^*, V^*) \) exists and is \( C^1 \) for any length \( L > 0 \). Suppose that there exists \( \varepsilon_1 > 0 \) such that for any length \( L > 0 \), there exists a basic quadratic Lyapunov function for the \( H^2 \) norm with

\[
B'(H^*(0)) \in \mathbb{R} \setminus \left( -\varepsilon - \frac{\partial_x P(H^*(0), 0)}{V^*(0)}, \varepsilon - \frac{V^*(0)}{H^*(0)} \right),
\]

\[
B'(H^*(L)) \in \left[ \varepsilon - \frac{\partial_x P(H^*(L), L)}{V^*(L)}, -\varepsilon - \frac{V^*(L)}{H^*(L)} \right].
\]

We can then use the same change of variables 5.3.6, as in Section 5.3. The system (5.2.1)–(5.2.2), (5.2.6) becomes (5.3.29) with boundary conditions (5.3.31). From (5.3.36), we have

\[
k_1^2 > \frac{\phi^2(L)}{\eta^2(L)}
\]

or

\[
k_2^2 > \eta^2(0),
\]

where \( k_1, k_2 \) are defined by 5.3.32 and \( \eta = \lambda_2 \phi/\lambda_1 \). We define now

\[
a = \delta_1 \phi, \quad b = \gamma_2 \phi^{-1}.
\]

As there exists a basic quadratic Lyapunov function for the \( H^2 \) norm, thus from [12] (see also [107, Theorem 3.5], and [107, (24),(40)–(43)]), there exists a function \( \eta_2 \in C^1([0, L]) \) such that

\[
\eta' = \left| \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right|
\]

on \([0, L]\) and there exists \( \varepsilon_1 > 0 \) depending only on \( \varepsilon \) such that

\[
\eta_2(L) \leq \eta(L) - \varepsilon_1,
\]

or \( \eta_2(0) \geq \eta(0) + \varepsilon_1 \).

Now, as \( L \) can be taken arbitrarily, \( \eta_2 \) exists for any \( L \), and thus on \([0, +\infty)\). We claim now that

\[
\lim_{x \to +\infty} \eta_2(x) \in \mathbb{R}_+.
\]

Indeed, let assume that \( \lim_{x \to +\infty} \eta_2(x) = +\infty \), then when \( x \) is large enough we have (see [107, Section 4])

\[
\eta'_2 = \frac{|b|}{\lambda_2} \eta_2^2 - \frac{\eta_2}{\lambda_1},
\]

thus using the estimates of [107, Section 4], there would exists \( C > 0 \) and \( x_1 > 0 \) such that for any \( x \geq x_1 \),

\[
\eta'_2 \geq \frac{C}{x} \eta_2^2,
\]

hence

\[
\eta_2 \geq \frac{1}{\eta_2(x_1)} - C \ln(x/x_1).
\]
And $\eta_2$ exist and is positive and $[x_1, +\infty)$, hence the contradiction. Thus $\eta_2$ converges when $x$ goes to $+\infty$ to a limit $\eta_{2, \infty}$. Note that $\phi$ converges to $\phi_\infty > 0$ [107] Section 4. Besides,

$$
\eta'_2 = \frac{|\gamma_2|}{\lambda_1 \phi(x)} \left[ \phi^2(x) - \frac{\lambda_1 |\delta_1|}{\lambda_2 \gamma_2} \eta^2 \right].
$$

(5.3.45)

As $\frac{\lambda_1 |\delta_1|}{\lambda_2 \gamma_2}$ goes to 1 when $x$ goes to infinity, assume by contradiction that $\eta_{2, \infty} \neq \phi_\infty$, there exists $C_3$ and $x_3$ such that for all $x > x_3$,

$$
\eta'_2 \geq \frac{C_3}{x},
$$

(5.3.46)

which implies that $\lim_{x \to +\infty} \eta_2(x) = +\infty$, hence contradiction. Thus $\eta_2$ converges to $\phi_\infty$, just as $\eta(L)$, which implies that in any cases the condition at $x = L$ become arbitrarily close to the one we obtain with $\eta$ when $L$ goes to infinity and prove that the first inequality of (5.3.40) is impossible.

Now let assume by contradiction that the second inequality of (5.3.40) is satisfied. Then $\eta_2(0) > \eta(0)$ and from (5.3.39),

$$
\eta'_2 \geq \left( \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \right) \eta^2,
$$

(5.3.47)

which implies that

$$
(\eta_2 - \eta)' \geq -2 \frac{|b|}{\lambda_2} (\eta_2 - \eta) \phi_\infty.
$$

(5.3.48)

Thus

$$
(\eta_{2, \infty} - \eta_\infty) \geq (\eta_2(0) - \eta(0)) \exp \left( -\phi_\infty \int_0^{+\infty} 2 \frac{|b|}{\lambda_2} \right).
$$

(5.3.49)

But, as seen in [107] Section 4], $\int_0^{+\infty} 2 \frac{|b|}{\lambda_2} < +\infty$, which implies, using that $\eta_2(0) > \eta(0)$,

$$
\eta_{2, \infty} > \eta_\infty,
$$

(5.3.50)

while we know that $\eta_{2, \infty} = \eta_\infty$, hence the contradiction. This ends the proof of Theorem 5.2.4.

We can now prove Theorem 5.2.5 in a very similar fashion.

**Proof of Theorem 5.2.5** Let $(H^*, V^*) \in C^1([0, L])$ be a steady-state of (5.2.1)–(5.2.2). Let assume by contradiction that there exists a basic quadratic Lyapunov function for the $H^2$ norm and that

$$
B'(H^*(L)) \in \left[ -\frac{\partial H^P(H^*(L), L)}{V^*(L)}, -\frac{V^*(L)}{H^*(L)} \right].
$$

(5.3.51)

Then using again the change of variables (5.3.6) the system (5.2.1)–(5.2.2), (5.2.6) is again equivalent to (5.3.29) with boundary conditions (5.3.31) From (5.3.51), one has

$$
k_1^2 := (D'_1(0))^2 = \eta^2(0).
$$

(5.3.52)

where $k_1$ is again given by (5.3.32) and $\eta = \lambda_2 \phi / \lambda_1$. As previously, as there exists a basic quadratic Lyapunov function for the $H^2$ norm, from [12] (see also [107] Theorem 3.5) there exists a function $\eta_2 \in C^1([0, L])$ such that

$$
\eta' = \left[ \frac{a}{\lambda_1} + \frac{b}{\lambda_2} \eta^2 \right]
$$

(5.3.53)

on $[0, L]$, where $a$ and $b$ are defined by (5.3.38), and there exists $\varepsilon_1 > 0$ such that

$$
\eta_2(L) \leq \eta(L) - \varepsilon_1, \forall \ L > 0,
$$

(5.3.54)

$$
\eta_2(0) \geq \eta(0).
$$

(5.3.55)

Using (5.3.54) and the same argument as (5.3.47)–(5.3.50), we get that $\eta_2(L) \geq \eta(L)$ thus

$$
\eta_2(L) \geq \lambda_2 \phi(L) / \lambda_1 (L)
$$

(5.3.55)

which is in contradiction with (5.3.54). This ends the proof of Theorem 5.2.5.
5.4 Appendix

5.4.1 Derivation of $\gamma_1$, $\gamma_2$, $\delta_1$ and $\delta_2$

Looking at [5.3.6] we denote by

$$\Delta = \begin{pmatrix} \sqrt{\frac{f(H^*, x)}{H^*}} & 1 \\ \sqrt{\frac{f(H^*, x)}{H^*}} & 1 \end{pmatrix}$$

$$\Delta^{-1} = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{H^*}{f(H^*, x)}} & -\sqrt{\frac{H^*}{f(H^*, x)}} \\ \sqrt{\frac{H^*}{f(H^*, x)}} & -\sqrt{\frac{H^*}{f(H^*, x)}} \end{pmatrix}$$

Then, using the notations [5.3.3], [5.3.4] becomes

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left( \begin{array}{cc} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{array} \right) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \left( \begin{array}{cc} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{array} \right) \Delta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \Delta \left( \mathcal{S}_{H^*} + \partial_H f(H^*, x) H_x^* \right) S_{V^*, + V_x^*} \Delta^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

(5.4.1)

where $\lambda_1$ and $\lambda_2$ are given by (5.3.8). Let us compute the coefficient of the first part of the source term,

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \Delta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} \lambda_1 \left( H_x^* f_H^* - \frac{\sqrt{f(H^*, x)}}{f} \right) - \lambda_2 \left( H_x^* f_H^* - \frac{\sqrt{f(H^*, x)}}{f} \right) \\ \lambda_2 \left( H_x^* f_H^* - \frac{\sqrt{f(H^*, x)}}{f} \right) - \lambda_1 \left( H_x^* f_H^* - \frac{\sqrt{f(H^*, x)}}{f} \right) \end{pmatrix}.$$  

(5.4.2)

The coefficient of the second part of the source term is

$$\Delta \left( \mathcal{S}_{H^*} + \partial_H f(H^*, x) H_x^* \right) S_{V^*, + V_x^*} \Delta^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \partial_H f H_x^* \sqrt{\frac{H^*}{f}} + \mathcal{S}_{H^*} \sqrt{\frac{H^*}{f}} + H_x^* \sqrt{\frac{H^*}{f}} + 2 V_x^* + S_{V^*} - \partial_H f H_x^* \sqrt{\frac{H^*}{f}} - \mathcal{S}_{H^*} \sqrt{\frac{H^*}{f}} - H_x^* \sqrt{\frac{H^*}{f}} + S_{V^*} \end{pmatrix}.$$  

(5.4.3)

Thus,

$$\gamma_1 = \frac{1}{4} \left[ \lambda_1 \left( \frac{H_x^* f_H^*}{f} \right) - \lambda_2 \frac{\sqrt{f(H^*, x)}}{f} \right] + 2 \left( \partial_H f H_x^* \sqrt{\frac{H^*}{f}} + \mathcal{S}_{H^*} \sqrt{\frac{H^*}{f}} + H_x^* \sqrt{\frac{f}{H^*}} + 2 V_x^* + S_{V^*} \right)$$

$$= \frac{1}{4} \left[ 2 S_{V^*} + 2 \mathcal{S}_{H^*} \sqrt{\frac{H^*}{f(H^*, x)}} - \lambda_1 \frac{\sqrt{f(H^*, x)}}{f} \right] + \left( (V^* + \sqrt{f H^*}) \left( \frac{H_x^*}{H^*} - \frac{\partial_H f(H^*, x) H_x^*}{f} \right) + 2 \partial_H f H_x^* \sqrt{\frac{H^*}{f}} + 2 H_x^* \sqrt{\frac{f}{H^*}} + 4 V_x^* \right)$$

$$= \frac{1}{4} \left( 2 S_{V^*} + 2 \mathcal{S}_{H^*} \sqrt{\frac{H^*}{f(H^*, x)}} - 3 \lambda_2 \frac{V_x^*}{V^*} - \lambda_1 \frac{\sqrt{f(H^*, x)}}{f} + \lambda_2 \frac{\partial_H f(H^*, x) H_x^*}{f} \right).$$

and $\gamma_2$, $\delta_1$ and $\delta_2$ can be found similarly.
5.4.2 Proof of Lemma 5.3.1

Differentiating $\lambda_2\phi/\lambda_1$ using (5.3.8), (5.3.9) and (5.3.12), we have

\[
\left( \frac{\lambda_2}{\lambda_1} \phi \right)' = \frac{\phi}{\lambda_1^2} (\lambda_1 \lambda_2' - \lambda_1' \lambda_2 + (\lambda_2 \gamma_1 + \lambda_1 \delta_2))
\]

\[
= \frac{\phi}{\lambda_1^2} \left( (V^* + \sqrt{f(H^*,x)H^*}) - V^* + \frac{(f(H^*,x)H^*)'}{2\sqrt{f(H^*,x)H^*}} \right)
\]

\[
- (\sqrt{f(H^*,x)H^*} - V^*) \left( V^* + \frac{(f(H^*,x)H^*)'}{2\sqrt{f(H^*,x)H^*}} \right)
\]

\[
+ \frac{1}{4} \left[ (\lambda_1^2 - \lambda_2^2) \left( 3 \frac{V^*}{V^*} - \frac{\partial_H f(H^*,x) H^*_x}{f(H^*,x)} \right) + 2(\lambda_2 - \lambda_1) \mathcal{J}_{H^*} \sqrt{\frac{H^*}{f(H^*,x)}} + 2(\lambda_2 + \lambda_1) S_{V^*} \right]
\]

\[
= \frac{\phi}{\lambda_1^2} \left( \sqrt{\frac{H^*}{f(H^*,x)}} \partial_x f(H^*,x) + H^*_x V^* \sqrt{\frac{f(H^*,x)}{H^*}} + V^* \sqrt{\frac{f(H^*,x)}{H^*}} \right)
\]

\[
- V^* \mathcal{J}_{H^*} \sqrt{\frac{H^*}{f(H^*,x)}} + \sqrt{f(H^*,x)H^*} S_{V^*} \right). \tag{5.4.4}
\]

Noticing the notations (5.3.3) and from (5.2.4), (5.4.4) becomes

\[
\left( \frac{\lambda_2}{\lambda_1} \phi \right)' = \frac{\phi}{\lambda_1^2} \left( \sqrt{f(H^*,x)H^*} S_{V^*} - V^* \sqrt{\frac{H^*}{f(H^*,x)}} S_{H^*} \right), \tag{5.4.5}
\]

which, together with (5.3.21) gives

\[
\left( \frac{\lambda_2}{\lambda_1} \phi \right)' = \frac{\phi \delta_1}{\lambda_1} + \phi^{-1} \gamma_2 \left( \frac{\lambda_2}{\lambda_1} \right)^2. \tag{5.4.6}
\]
Chapter 6

Exponential boundary feedback stabilization of a shock steady state for the inviscid Burgers equation

This chapter is taken from the following article (also referred to as [20]):


Abstract. In this chapter, we study the exponential stabilization of a shock steady state for the inviscid Burgers equation on a bounded interval. Our analysis relies on the construction of an explicit strict control Lyapunov function. We prove that by appropriately choosing the feedback boundary conditions, we can stabilize the state as well as the shock location to the desired steady state in $H^2$-norm, with an arbitrary decay rate.

6.1 Introduction

The problem of asymptotic stabilization for hyperbolic systems using boundary feedback control has been studied for a long time. We refer to the pioneer work due to Rauch and Taylor [171] and Russell [175] for linear coupled hyperbolic systems. The first important result of asymptotic stability concerning quasilinear hyperbolic equations was obtained by Slemrod [178] and Greenberg and Li [93]. These two works dealt with local dissipative boundary conditions. The result was established by using the method of characteristics, which allows to estimate the related bounds along the characteristic curves in the framework of $C^1$ solutions. Another approach to analyze the dissipative boundary conditions is based on the use of Lyapunov functions. Especially, Coron, Bastin and Andrea-Novel [53] used this method to study the asymptotic behavior of the nonlinear hyperbolic equations in the framework of $H^2$ solutions. In particular, the Lyapunov function they constructed is an extension of the entropy and can be made strictly negative definite by properly choosing the boundary conditions. This method has been later on widely used for hyperbolic conservation laws in the framework of $C^1$ solutions [48, 106, 107] or $H^2$ solutions [12, 16, 18, 47, 50, 66, 110] (see [13] for an overview of this method).

But all of these results concerning the asymptotic stability of nonlinear hyperbolic equations focus on the convergence to regular solutions, i.e., on the stabilization of regular solutions to a desired regular steady state. It is well known, however, that for quasilinear hyperbolic partial differential equations, solutions
may break down in finite time when their first derivatives break up even if the initial condition is smooth. They give rise to the phenomena of shock waves with numerous important applications in physics and fluid mechanics. Compared to classical case, very few results exist on the stabilization of less regular solutions, which requires new techniques. This is also true for related fields, as the optimal control problem [132]. For the problem of control and asymptotic stabilization of less regular solutions, we refer to [33] for the controllability of a general hyperbolic system of conservation laws, [27,162] for the stabilization in the scalar case and [33,67] for the stabilization of a hyperbolic system of conservation laws. In [27,67,162], by using suitable feedback laws on both side of the interval, one can steer asymptotically any initial data with sufficiently small total variations to any close constant steady states. All those results concern the boundary stabilization of constant steady states. In particular, as the target state is regular there is no need to stabilize any shock location. In this work, we will study the boundary stabilization of steady states with jump discontinuities for a scalar equation. We believe that our method can be applied to nonlinear hyperbolic systems as well. While preparing the revised version, our attention was drawn to a very recent work [163] studying a similar problem in the BV norm. The method and the results are quite different and complementary to this work.

Hyperbolic systems have a wide application in fluid dynamics, and hydraulic jump is one of the best known examples of shock waves as it is frequently observed in open channel flow such as rivers and spillways. Other physical examples of shock waves can be found in road traffic or in gas transportation, with the water hammer phenomenon. In the literature, Burgers equation often appears as a simplification of the dynamical model of flows, as well as the most studied scalar model for transportation. Burgers turbulence has been investigated both analytically and numerically by many authors either as a preliminary approach to turbulence prior to an occurrence of the Navier-Stokes turbulence or for its own sake since the Burgers equation describes the formation and decay of weak shock waves in a compressible fluid [120,143,185]. From a mathematical point of view, it turns out that the study of Burgers equation leads to many of the ideas that arise in the field of nonlinear hyperbolic equations. It is therefore a natural first step to develop methods for the control of this equation. For the boundary stabilization problem of viscous Burgers equation, we refer to works by Krstic et al. [125,180] for the stabilization of regular shock-like profile steady states and [33,124] for the stabilization of null-steady-state. In [180], the authors proved that the shock-like profile steady states of the linearized unit viscous Burgers equation is exponentially stable when using high-gain “radiation” boundary feedback (i.e. static boundary feedback only depending on output measurements). However, they showed that there is a limitation in the decay rate achievable by radiation feedback, i.e., the decay rate goes to zero exponentially as the shock becomes sharper. Thus, they have to use another strategy (namely backstepping method) to achieve arbitrarily fast local convergence to arbitrarily sharp shock profiles. However, this strategy requires a kind of full-state feedback control, rather than measuring only the boundary data.

In this chapter, we study the exponential asymptotic stability of a shock steady state of the Burgers equation in $H^2$-norm, which has been commonly used as a proper norm for studying the stability of hyperbolic systems (see e.g. [65,114,182]), as it enables to deal with Lyapunov functions that are integrals on the domain of quadratic quantities, which is relatively easy to handle. To that end, we construct an explicit Lyapunov function with a strict negative definite time derivative by properly choosing the boundary conditions. Though it has been shown in [61] that exponential stability in $H^2$-norm is not equivalent to $C^1$-norm, our result could probably be generalized to the $C^1$-norm for conservation laws by transforming the Lyapunov functions as in [48,109].

The first problem is to deal with the well-posedness of the corresponding initial boundary value problem (IBVP) on a bounded domain. The existence of the weak solution to the initial value problem (IVP) of Burgers equation was first studied by Hopf by using vanishing viscosity [111]. The uniqueness of the entropy solution was then studied by Oleinik [160]. One can refer to [132] for a comprehensive study of the well-posedness of hyperbolic conservation laws in piecewise continuous entropy solution case and also to [68] in the class of entropy BV functions. Although there are many results for the well-posedness of the (IVP) for hyperbolic conservation laws, the problem of (IBVP) is less studied due to the difficulty of handling the boundary condition. In [3], the authors studied (IBVP) but in the quarter plane, i.e., $x > 0, t > 0$. By requiring that the boundary condition at $x = 0$ is satisfied in a weak sense, they can apply the method introduced by LeFloch [133] and obtain the explicit formula of the solution. However, our case is more complicated since we consider the Burgers equation defined on a bounded interval.
The organization of the chapter is the following. In Section 6.2, we formulate the problem and state our main results. In Section 6.3, we prove the well-posedness of the Burgers equation in the framework of piecewise continuously differentiable entropy solutions, which is one of the main results in this chapter. Based on this well-posedness result, we then prove in Section 6.4 by a Lyapunov approach that for appropriately chosen boundary conditions, we can achieve the exponential stability in $H^2$-norm of a shock steady state with any given arbitrary decay rate and with an exact exponential stabilization of the desired shock location. This result also holds for the $H^k$-norm for any $k \geq 2$. In Section 6.5, we extend the result to a more general convex flux by requiring some additional conditions on the flux. Conclusion and some open problems are provided in Section 6.6. Finally, some technical proof are given in the Appendix.

6.2 Problem statement and main result

We consider the following nonlinear inviscid Burgers equation on a bounded domain

$$y_t(t, x) + \left( \frac{y^2}{2} \right)_x(t, x) = 0 \quad (6.2.1)$$

with initial condition

$$y(0, x) = y_0(x), \quad x \in (0, L), \quad (6.2.2)$$

where $L > 0$ and boundary controls

$$y(t, 0^+) = u_0(t), \quad y(t, L^-) = u_L(t). \quad (6.2.3)$$

In this article, we will be exclusively concerned with the case where the controls $u_0(t) > 0$, $u_L(t) < 0$ have opposite signs and the state $y(t, \cdot)$ at each time $t$ has a jump discontinuity as illustrated in Figure 6.1. The discontinuity is a shock wave that occurs at position $x_s(t) \in (0, L)$. According to the Rankine-Hugoniot condition, the shock wave moves with the speed

$$\dot{x}_s(t) = \frac{y(t, x_s(t)^+) + y(t, x_s(t)^-)}{2} \quad (6.2.4)$$

which satisfies the Lax entropy condition [132]

$$y(t, x_s(t)^+) < \dot{x}_s(t) < y(t, x_s(t)^-), \quad (6.2.5)$$

Figure 6.1: Entropy solution to the Burgers equation with a shock wave.

<table>
<thead>
<tr>
<th>$y(t, x)$</th>
<th>$u_0(t)$</th>
<th>+1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$x_s(t)$</td>
<td>$\dot{x}_s(t)$</td>
</tr>
<tr>
<td>$L$</td>
<td>$u_L(t)$</td>
<td>-1</td>
</tr>
</tbody>
</table>
For all stated in the following theorem. The first result of this chapter deals with the well-posedness of system (6.2.1)–(6.2.4), (6.2.6), (6.2.8) and is together with the initial condition

\[ x_s(0) = x_{s0}. \]  

(6.2.6)

Under a constant control \( u_0(t) = -u_L(t) = 1 \) for all \( t \), for any \( x_0 \in (0, L) \), the system \( 6.2.1, 6.2.3, 6.2.4 \) has a steady state \((y^*, x^*_s)\) defined as follows:

\[
y^*(x) = \begin{cases} 
1, & x \in [0, x_0), \\
-1, & x \in (x_0, L], 
\end{cases} 
\]

(6.2.7)

\[ x^*_s = x_0. \]

These equilibria are clearly not isolated and, consequently, not asymptotically stable. Indeed, one can see that for any given equilibrium \( y^* \) satisfying \( 6.2.7 \), we can find initial data arbitrarily close to \( y^* \) which is also an equilibrium of the form \( 6.2.7 \). As the solution cannot be approaching the given equilibrium when \( t \) tends to infinity as long as the initial data is another equilibrium, this feature prevents any stability no matter how close the initial data is around \( y^* \). With such open-loop constant control another problem could appear: any small mistake on the boundary control could result in a non-stationary shock moving far away from \( x_0 \). It is therefore relevant to study the boundary feedback stabilization of the control system \( 6.2.1, 6.2.3, 6.2.4 \).

In this chapter, our main contribution is precisely to show how we can exponentially stabilize any of the steady states defined by \( 6.2.7 \) with boundary feedback controls of the following form:

\[
u_0(t) = k_1 y(t, x_s(t)^-) + (1 - k_1) + b_1(x_0 - x_s(t)),
\]

\[
u_L(t) = k_2 y(t, x_s(t)^+) - (1 - k_2) + b_2(x_0 - x_s(t)).
\]

(6.2.8)

Here, it is important to emphasize that, with these controls, we are able not only to guarantee the exponential convergence of the solution \( y(t, x) \) to the steady state \( y^* \) but also to exponentially stabilize the location of the shock discontinuity at the exact desired position \( x_0 \). In practice, if the system was used for instance to model gas transportation, the measures of the state around the shock could be obtained using sensors in the pipe. Note that if the control is applied properly, sensors would be only needed on a small region as the shock would remain located in a small region.

Before addressing the exponential stability issue, we first show that there exists a unique piecewise continuously differentiable entropy solution with \( x_s(t) \) as its single shock for system \( 6.2.1, 6.2.3, 6.2.4, 6.2.6, 6.2.8 \) provided that \( y_0 \) and \( x_{s0} \) are in a small neighborhood of \( y^* \) and \( x_0 \) respectively.

For any given initial condition \( 6.2.2, 6.2.5 \), we define the following zero order compatibility conditions

\[
y_0(0^+) = k_1 y_0(x_{s0}^-) + (1 - k_1) + b_1(x_0 - x_{s0}),
\]

\[
y_0(L^-) = k_2 y_0(x_{s0}^+) - (1 - k_2) + b_2(x_0 - x_{s0}).
\]

(6.2.9)

Differentiating \( 6.2.9 \) with respect to time \( t \) and using \( 6.2.4 \), we get the following first order compatibility conditions

\[
y_0(0^+)y_{0x}(0^+) = k_1 y_0(x_{s0}^-)y_{0x}(x_{s0}^-) - k_1 y_0(x_{s0}^-)y_0(x_{s0}^+) + y_0(x_{s0}^+) + b_1 y_0(x_{s0}^-) + y_0(x_{s0}^+),
\]

\[
y_0(L^-)y_{0x}(L^-) = k_2 y_0(x_{s0}^+)y_{0x}(x_{s0}^+) - k_2 y_0(x_{s0}^+)y_0(x_{s0}^-) + y_0(x_{s0}^-) + b_2 y_0(x_{s0}^+) + y_0(x_{s0}^-). 
\]

(6.2.10)

The first result of this chapter deals with the well-posedness of system \( 6.2.1, 6.2.3, 6.2.6, 6.2.8 \) and is stated in the following theorem.

**Theorem 6.2.1.** For all \( T > 0 \), there exists \( \delta(T) > 0 \) such that, for every \( x_{s0} \in (0, L) \) and \( y_0 \in H^2((0, x_{s0}); \mathbb{R}) \cap H^2((x_{s0}, L); \mathbb{R}) \) satisfying the compatibility conditions \( 6.2.9, 6.2.10 \) and

\[
|y_0 - 1|_{H^2((0, x_{s0}); \mathbb{R})} + |y_0 + 1|_{H^2((x_{s0}, L); \mathbb{R})} \leq \delta(T), \]

\[
|x_{s0} - x_0| \leq \delta(T),
\]

(6.2.11)
The steady state (Definition 6.2.1) has a unique piecewise continuously differentiable entropy solution $y \in C^0([0,T]; H^2((0, x_s(t)); \mathbb{R})) \cap H^2((x_s(t), L); \mathbb{R}))$ with $x_s \in C^1([0,T]; (0, L))$ as its single shock. Moreover, there exists $C(T)$ such that the following estimate holds for all $t \in [0,T]$,

$$
|y(t, \cdot) - 1|_{H^2((0, x_s(t)); \mathbb{R})} + |y(t, \cdot) + 1|_{H^2((x_s(t), L); \mathbb{R})} + |x_s(t) - x_0| \leq C(T) \left( |y_0 - 1|_{H^2((0, x_0); \mathbb{R})} + |y_0 + 1|_{H^2((x_0, L); \mathbb{R})} + |x_0 - x_0| \right). \tag{6.2.12}
$$

The proof of this result is given in Section 6.3.

Our next result deals with the exponential stability of the steady state (6.2.7) for the $H^2$-norm according to the following definition.

**Definition 6.2.1.** The steady state $(y^*, x_0) \in (H^2((0, x_0); \mathbb{R}) \cap H^2((x_0, L); \mathbb{R})) \times (0, L)$ of the system (6.2.1), (6.2.3), (6.2.4), (6.2.8) is exponentially stable for the $H^2$-norm with decay rate $\gamma$, if there exists $\delta^* > 0$ and $C > 0$ such that for any $y_0 \in H^2((0, x_0); \mathbb{R}) \cap H^2((x_0, L); \mathbb{R})$ and $x_0 \in (0, L)$ satisfying

$$
|y_0 - y_1^*(0, \cdot)|_{H^2((0, x_0); \mathbb{R})} + |y_0 - y_2^*(0, \cdot)|_{H^2((x_0, L); \mathbb{R})} \leq \delta^*,
$$

and the compatibility conditions (6.2.9), (6.2.10), and for any $T > 0$ the system (6.2.1), (6.2.4), (6.2.6), (6.2.8) has a unique solution $(y, x_s) \in C^0([0,T]; H^2((x_0, L); \mathbb{R})) \cap H^2((x_s(t), L); \mathbb{R})) \times C^1([0,T]; \mathbb{R})$ and

$$
|y(t, \cdot) - y_1^*(t, \cdot)|_{H^2((0, x_s(t)); \mathbb{R})} + |y(t, \cdot) - y_2^*(t, \cdot)|_{H^2((x_s(t), L); \mathbb{R})} + |x_s(t) - x_0| \leq C e^{-\gamma t} \left( |y_0 - y_1^*(0, \cdot)|_{H^2((0, x_0); \mathbb{R})} + |y_0 - y_2^*(0, \cdot)|_{H^2((x_0, L); \mathbb{R})} + |x_0 - x_0| \right), \forall t \in [0,T). \tag{6.2.14}
$$

In (6.2.13) and (6.2.14),

$$
y_1^*(t, x) = \begin{cases} y^* \left( \frac{x - x_0}{x_s(t)} \right), \\ \end{cases} \tag{6.2.15}
$$

$$
y_2^*(t, x) = \begin{cases} y^* \left( \frac{x - L}{x_s(t) - L} \right), \\ \end{cases}
$$

**Remark 6.2.1.** At first glance it could seem peculiar to define $y_1^*$ and $y_2^*$ and to compare $y(t, \cdot)$ with these functions. However the steady state $y^*$ is piecewise $H^2$ with discontinuity at $x_0$, while the solution $y(t, x)$ is piecewise $H^2$ with discontinuity at the shock $x_s(t)$, which may be moving around $x_0$. Thus, to compare the solution $y$ with the steady state $y^*$ on the same space interval, it is necessary to define such functions $y_1^*$ and $y_2^*$.

**Remark 6.2.2.** We emphasize here that the “exponential stability for the $H^2$-norm” is not the usual convergence of the $H^2$-norm of $y - y^*$ taken on $(0, L)$ as $y$ and $y^*$ do not belong to $H^2(0, L)$. This definition enables to define an exponential stability in $H^2$-norm for a function that has a discontinuity at some point and is regular elsewhere. Note that, the convergence to 0 of the $H^2$-norm in the usual sense does not ensure the convergence of the shock location $x_s$ to $x_0$. Thus, to guarantee that the state converges to the shock steady state, we have to take account of the shock location, which is explained in Definition 2.1.

**Remark 6.2.3.** Note that this definition of exponential stability only deals a priori with $t \in [0,T)$ for any $T > 0$. However this, together with Theorem (6.2.7) implies the global existence in time of the solution $(y, x_s)$ and the exponential stability on $[0, +\infty)$. This is shown at the end of the proof of Theorem 6.4.1.

We can now state the main result of this chapter.
Theorem 6.2.2. Let $\gamma > 0$. If the following conditions hold:

\[ b_1 \in \left( e^{-\gamma x_0}, \frac{\gamma e^{-\gamma x_0}}{1 - e^{-\gamma x_0}} \right), \quad b_2 \in \left( e^{-\gamma(L-x_0)}, \frac{\gamma e^{-\gamma(L-x_0)}}{1 - e^{-\gamma(L-x_0)}} \right), \]

(6.2.16a)

\[ k_1^2 < e^{-\gamma x_0} \left( 1 - \frac{b_1}{\gamma} \left( \frac{1}{e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{e^{-\gamma(L-x_0)}} \right) \right), \]

(6.2.16b)

\[ k_2^2 < e^{-\gamma(L-x_0)} \left( 1 - \frac{b_2}{\gamma} \left( \frac{1}{e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{e^{-\gamma(L-x_0)}} \right) \right), \]

(6.2.16c)

then the steady state $(y^*, x_0)$ of the system (6.2.1), (6.2.3), (6.2.4), (6.2.8) is exponentially stable for the $H^2$-norm with decay rate $\gamma/4$.

The proof of this theorem is given in Section 6.4.

Remark 6.2.4. One can actually check that for any $\gamma > 0$ there exist parameters $b_1$, $b_2$ and $k_1$, $k_2$ satisfying (6.2.16) as, for $b_1 = e^{-\gamma x_0}$ and $b_2 = e^{-\gamma(L-x_0)}$, one has

\[ 1 - \frac{b_1}{\gamma} \left( \frac{1}{e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{e^{-\gamma(L-x_0)}} \right) = \frac{1 - e^{-\gamma x_0}}{\gamma} \left( 2 - e^{-\gamma x_0} - e^{-\gamma(L-x_0)} \right) \]

\[ = e^{-2\gamma x_0} \left( e^{\gamma x_0} - 1 \right)^2 + e^{-\gamma} \L > 0. \]

Similarly, we get

\[ 1 - \frac{b_2}{\gamma} \left( \frac{1}{e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{e^{-\gamma(L-x_0)}} \right) = e^{-2\gamma(L-x_0)} \left( e^{\gamma(L-x_0)} - 1 \right)^2 + e^{-\gamma} \L > 0. \]

Therefore, by continuity, there exist $b_1$ and $b_2$, satisfying condition (6.2.16a) such that there exist $k_1$ and $k_2$ satisfying (6.2.16b) and (6.2.16c). This implies that $\gamma$ can be made arbitrarily large. And, from (6.2.16a) - (6.2.16c), we can note that for large $\gamma$ the conditions on the $k_i$ tend to

\[ k_1^2 < e^{-\gamma x_0}, \quad k_2^2 < e^{-\gamma(L-x_0)}. \]

Remark 6.2.5. The result can also be generalized to $H^2$-norm for any integer $k \geq 2$ in the sense of Definition 6.2.1 by replacing $H^2$ with $H^k$. This can be easily done by just adapting the Lyapunov function defined below by (6.4.3) - (6.4.9), as was done in [13, Sections 4.5 and 6.2].

Remark 6.2.6. If we set $k_1 = k_2 = b_1 = b_2 = 0$, then from (6.2.8), $u_0(t) \equiv 1$ and $u_L(t) \equiv -1$. Thus it seems logical that the larger $\gamma$ is, the smaller $k_1$ and $k_2$ are. However, it could seem counter-intuitive that $b_1$ and $b_2$ have to tend to 0 when $\gamma$ tends to $+\infty$, as if one sets $b_1 = 0$ and $b_2 = 0$, one cannot stabilize the location of the system just like in the constant open-loop control case. In other words, for any $\gamma > 0$ the prescribed feedback works while the limit feedback we obtain by letting $\gamma \to +\infty$ cannot even ensure the asymptotic stability of the system. The explanation behind this apparent paradox is that when $\gamma$ tends to infinity, the Lyapunov function candidate used to prove Theorem 6.4.7 is not equivalent to the norm of the solution and cannot guarantee anymore the exponential decay of the solution in the $H^2$-norm. More precisely, one can see, looking at (6.4.69) and (6.4.71), that the hypothesis (6.4.16) of Lemma 6.4.1 does not hold anymore.
6.3 An equivalent system with shock-free solutions

Our strategy to analyze the existence and the exponential stability of the shock wave solutions to the scalar Burgers equation \((6.2.1)\) is to use an equivalent \(2 \times 2\) quasilinear hyperbolic system having shock-free solutions. In order to set up this equivalent system, we define the two following functions

\[
y_1(t,x) = y(t, x \frac{x_s(t)}{x_0}), \quad y_2(t,x) = y(t, L + x \frac{x_s(t) - L}{x_0})
\]

(6.3.1)

and the new state variables as follows:

\[
z(t,x) = \begin{pmatrix} z_1(t,x) \\ z_2(t,x) \end{pmatrix} = \begin{pmatrix} y_1(t,x) - 1 \\ y_2(t,x) + 1 \end{pmatrix}, \quad x \in (0, x_0).
\]

(6.3.2)

The idea behind the definition of \(y_1, y_2\) is to describe the behavior of the solution \(y(t,x)\) before and after the moving shock, while studying functions on a time invariant interval. Observe indeed that the functions \(y_1\) and \(y_2\) in \((6.3.1)\) correspond to the solution \(y(t,x)\) on the time varying intervals \((0, x_s(t))\) and \((x_s(t), L)\) respectively, albeit with a time varying scaling of the space coordinate \(x\) which is driven by \(x_s(t)\) and allows to define the new state variables \((z_1, z_2)\) on the fixed time invariant interval \((0, x_0)\). The reason to rescale \(y_2\) on \((0, x_0)\) instead of \((x_0, L)\) is to simplify the analysis by defining state variables on the same space interval with the same direction of propagation.

Besides, from \((6.3.2)\), the former steady state \((y^*, x_0)\) corresponds now to the steady state \((z = 0, x_s = x_0)\) in the new variables. With these new variables, the dynamics of \((y, x_s)\) can now be expressed as follows:

\[
z_{1t} + \left(1 + z_1 - x \frac{\hat{x}_s}{x_0}\right) z_{1x} \frac{x_0}{x_s} = 0,
\]

(6.3.3)

\[
z_{2t} + \left(1 - z_2 + x \frac{\hat{x}_s}{x_0}\right) z_{2x} \frac{x_0}{L - x_s} = 0,
\]

\[
\hat{x}_s(t) = \frac{z_1(t,x_0) + z_2(t,x_0)}{2},
\]

with the boundary conditions:

\[
z_1(t,0) = k_1 z_1(t,x_0) + b_1(x_0 - x_s(t)),
\]

(6.3.4)

\[
z_2(t,0) = k_2 z_2(t,x_0) + b_2(x_0 - x_s(t)),
\]

and initial condition

\[
z(0,x) = z^0(x), \quad x_s(0) = x_{s0},
\]

(6.3.5)

where \(z^0 = (z^0_1, z^0_2)^T\) and

\[
z^0_1(x) = y_0 \left(\frac{x_{s0}}{x_0}\right) - 1,
\]

(6.3.6)

\[
z^0_2(x) = y_0 \left(\frac{L + x_{s0} - L}{x_0}\right) + 1.
\]

Furthermore, in the new variables, the compatibility conditions \((6.2.9)-(6.2.10)\) are expressed as follows:

\[
z^0_1(0) = k_1 z^0_1(x_0) + b_1(x_0 - x_{s0}),
\]

(6.3.7)

\[
z^0_2(0) = k_2 z^0_2(x_0) + b_2(x_0 - x_{s0}),
\]

and

\[
(1 + z^0_1(0)) z^0_{1x}(0) \frac{x_0}{x_{s0}} = k_1 \left(1 + z^0_1(x_0) - \frac{z^0_1(x_0) + z^0_2(x_0)}{2}\right) z^0_{1x}(x_0) \frac{x_0}{x_{s0}} + b_1 \frac{z^0_1(x_0) + z^0_2(x_0)}{2},
\]

\[
(1 - z^0_2(0)) z^0_{2x}(0) \frac{x_0}{L - x_{s0}} = k_2 \left(1 - z^0_2(x_0) + \frac{z^0_1(x_0) + z^0_2(x_0)}{2}\right) z^0_{2x}(x_0) \frac{x_0}{L - x_{s0}} + b_2 \frac{z^0_1(x_0) + z^0_2(x_0)}{2}.
\]

(6.3.8)
Concerning the existence and uniqueness of the solution to the system (6.3.3)–(6.3.5), we have the following lemma.

**Lemma 6.3.1.** For all $T > 0$, there exists $\delta(T) > 0$ such that, for every $x_0 \in (0, L)$ and $z^0 \in H^2((0, x_0); \mathbb{R}^2)$ satisfying the compatibility conditions (6.3.7)–(6.3.8) and

$$|z^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta(T), \quad |x_{s0} - x_0| \leq \delta(T),$$

the system (6.3.3)–(6.3.5) has a unique classical solution $(z, x_s) \in C^0([0,T]; H^2((0,x_0); \mathbb{R}^2)) \times C^1([0,T]; (0,L))$. Moreover, there exists $C(T)$ such that the following estimate holds for all $t \in [0,T]$

$$|z(t,.)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s(t) - x_0| \leq C(T) \left(|z^0|_{H^2((0,x_0);\mathbb{R}^2)} + |x_{s0} - x_0| \right).$$

**(Proof.** The proof of Lemma 6.3.1 is given in Appendix 6.7.1.)

**Proof of Theorem 6.2.1.** The change of variables (6.3.1), (6.3.2) induces an equivalence between the classical solutions $(z, x_s)$ of the system (6.3.3)–(6.3.5) and the entropy solutions with a single shock $(y, x_s)$ of the system (6.2.1)–(6.2.6), (6.2.8). Consequently, from (6.3.2) and provided $|z^0|_{H^2((0,x_0);\mathbb{R}^2)}$ and $|x_{s0} - x_0|$ are sufficiently small, the existence and uniqueness of a solution with a single shock $(y, x_s)$ to the system (6.2.1)–(6.2.6), (6.2.8) satisfying the entropy condition (6.2.5) when $(y_0, x_0)$ is in a sufficiently small neighborhood of $(y^*, x_0)$, follows directly from the existence and uniqueness of the classical solution $(z, x_s)$ to the system (6.3.3)–(6.3.5) which is guaranteed by Lemma 6.3.1.)

**Remark 6.3.1.** Under the assumption in Lemma 6.3.1, if we assume furthermore that $z^0 \in H^k((0, x_0); \mathbb{R}^2)$ with $k \geq 2$ satisfying the $k$-th order compatibility conditions (see the definition in [23], p.143]), then $(z, x_s) \in C^0([0,T]; H^k((0,x_0); \mathbb{R}^2)) \times C^k([0,T]; \mathbb{R})$ and (6.3.10) still holds. This is a straightforward extension of the proof in Appendix 6.7.1 thus we will not give the details of this proof here.

### 6.4 Exponential stability for the $H^2$-norm

This section is devoted to the proof of Theorem 6.2.2 concerning the exponential stability of the steady state of system (6.2.1)–(6.2.4), (6.2.6). Actually, on the basis of the change of variables introduced in the previous section, we know that we only have to prove the exponential stability of the steady state of the auxiliary system (6.3.3)–(6.3.4) according to the following theorem which is equivalent to Theorem 6.2.2.

**Theorem 6.4.1.** For any $\gamma > 0$, if condition (6.2.16) on the parameters of the feedback holds, then there exist $\delta^* > 0$ and $C > 0$ such that for any $z^0 \in H^2((0,x_0); \mathbb{R}^2)$ and $x_{s0} \in (0, L)$ satisfying

$$|z^0|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta^*, \quad |x_{s0} - x_0| \leq \delta^*$$

and the compatibility conditions (6.3.7)–(6.3.8), and for any $T > 0$ the system (6.3.3)–(6.3.5) has a unique classical solution $(z, x_s) \in C^0([0,T]; H^2((0,x_0); \mathbb{R}^2)) \times C^1([0,T]; \mathbb{R})$ such that

$$|z(t,.)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s(t) - x_0| \leq Ce^{-\gamma T/4} \left(|z^0|_{H^2((0,x_0);\mathbb{R}^2)} + |x_{s0} - x_0| \right), \quad \forall t \in [0,T).$$

When this theorem holds, we say that the steady state $(z = 0, x_s = x_0)$ of the system (6.3.3)–(6.3.4) is exponentially stable for the $H^2$-norm with convergence rate $\gamma/4$. Recall that, from Remark 6.2.4 there always exist parameters such that (6.2.16) holds.

Before proving Theorem 6.4.1, let us give an overview of our strategy. We first introduce a Lyapunov function candidate $V$ with parameters to be chosen. Then, in Lemma 6.4.1 we give a condition on the parameters such that $V$ is equivalent to the square of the $H^2$-norm of $z$ plus the absolute value of $x_s - x_0$, which implies that proving the exponential decay of $V$ with rate $\gamma/2$ is enough to show the exponential stability of the system with decay rate $\gamma/4$ for the $H^2$-norm. In Lemma 6.4.2 we show that in order to obtain Theorem
it is enough to prove that $V$ decays along any solutions $(z, x_s) \in C^3([0, T] \times [0, x_0]; \mathbb{R}^2) \times C^3([0, T]; \mathbb{R})$ with a density argument. Then in Lemma 6.4.3 we compute the time derivative of $V$ along any $C^3$ solutions of the system and we give a sufficient condition on the parameters such that $V$ satisfies a useful estimate along these solutions. Finally, we show that there exist parameters satisfying the sufficient condition of Lemma 4.3. This, together with Lemma 4.2, ends the proof of Theorem 4.1.

We now introduce the following candidate Lyapunov function which is defined for all $z = (z_1, z_2)^T \in H^2((0, x_0); \mathbb{R}^2)$ and $x_s \in (0, L)$:

$$V(z, x_s) = V_1(z) + V_2(z, x_s) + V_3(z, x_s) + V_4(z, x_s) + V_5(z, x_s) + V_6(z, x_s)$$ (6.4.3)

with

$$V_1(z) = \int_0^{x_0} p_1 e^{-\frac{\mu t}{2}} z_1^2 + p_2 e^{-\frac{\mu t}{2}} z_2^2 dx,$$ (6.4.4)

$$V_2(z, x_s) = \int_0^{x_0} p_1 e^{-\frac{\mu t}{2}} z_1^2 + p_2 e^{-\frac{\mu t}{2}} z_2^2 dx, \quad V_3(z, x_s) = \int_0^{x_0} p_1 e^{-\frac{\mu t}{2}} z_1^2 + p_2 e^{-\frac{\mu t}{2}} z_2^2 dx,$$ (6.4.5)

$$V_4(z, x_s) = \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu t}{2}} z_1(x_s - x_0) dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu t}{2}} z_2(x_s - x_0) dx + \kappa(x_s - x_0)^2,$$ (6.4.6)

$$V_5(z, x_s) = \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu t}{2}} z_1(x) dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu t}{2}} x_s dx + \kappa(x_s)^2,$$ (6.4.7)

$$V_6(z, x_s) = \int_0^{x_0} \bar{p}_1 e^{-\frac{\mu t}{2}} z_1 dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{\mu t}{2}} x_s dx + \kappa(x_s)^2.$$ (6.4.8)

In (6.4.4)-(6.4.9), $\mu, p_1, p_2, \bar{p}_1, \bar{p}_2$ are positive constants. Moreover

$$\eta_1 = 1, \quad \eta_2 = \frac{x_0}{L - x_0}$$ (6.4.10)

and

$$\kappa > 1.$$ (6.4.11)

Actually, in this section, we will need to evaluate $V(z, x_s)$ only along the system solutions for which the variables $z_t = (z_1t, z_2t), z_{tt} = (z_1tt, z_2tt), \bar{x}_t$ and $\bar{x}_s$ that appear in the definition of $V$ can be well defined as functions of $(z, x_s) \in H^2((0, x_0); \mathbb{R}^2) \times (0, L)$ from the system (6.3.1)-(6.3.4) and their space derivatives. For example, $z_{1t}$ and $z_{2t}$ are defined as functions of $(z, x_s)$ by

$$z_{1t} := -\left(1 + z_1 - \frac{z_1(x_0) + z_2(x_0)}{2x_0}\right) \frac{x_0}{x_s},$$ (6.4.12)

$$z_{2t} := -\left(1 - z_2 + \frac{z_1(x_0) + z_2(x_0)}{2x_0}\right) \frac{x_0}{L - x_s},$$ (6.4.13)

and $z_{1tt}$ and $z_{2tt}$ as functions of $(z, x_s)$ by

$$z_{1tt} := -\left(1 + z_1 - \frac{z_1(x_0) + z_2(x_0)}{2x_0}\right) \frac{x_0}{x_s} \frac{x_0}{x_s} - \left(1 - z_1 - \frac{z_1(x_0) + z_2(x_0)}{2x_0}\right) \frac{x_0}{x_s} \frac{2x_0}{L - x_s},$$ (6.4.14)

$$z_{2tt} := -\left(1 - z_2 + \frac{z_1(x_0) + z_2(x_0)}{2x_0}\right) \frac{x_0}{L - x_s}.$$
Proof of Lemma 6.4.1. Let us start with
\begin{equation}
\left(2t - x \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2x_0}\right) \frac{x_0}{L - x} + \frac{z_1(x_0) + z_2(x_0)}{2(L - x)}.
\end{equation}
\tag{6.4.15}

The functions $z_{1t}$ and $z_{2t}$ which appear in (6.4.14) and (6.4.15) are supposed to be defined by (6.4.12) and (6.4.13) respectively.

\textbf{Remark 6.4.1.} When looking for a Lyapunov function to stabilize the state $(z_1, z_2)$ in $H^2$-norm, the component $(V_1 + V_2 + V_3)$ can be seen as the most natural and easiest choice, as it is equivalent to a weighted $H^2$-norm by properly choosing the parameters. This kind of Lyapunov function, sometimes called basic quadratic Lyapunov function, is used for instance in [12] or [13] Section 4.4]. However, in the present case one needs to stabilize both the state $z$ and the shock location $x_s$, which requires to add additional terms to the Lyapunov function in order to deal with $x_s$. Besides, as we have no direct control on $x_s$ (observe that none of the terms of the right-hand side of (6.2.4), or equivalently of the third equation of (6.3.3), is a control), we need to add some coupling terms between the state $z$ on which we have a control and the shock location $x_s$ in the Lyapunov function. Thus, $V_4$ is designed to provide such coupling with the product of the component of $z$ and $x_s$, while $V_5$ and $V_6$ are its analogues for the time derivatives terms (as $V_2$ and $V_3$ are the analogues of $V_1$ respectively for the first and second time derivative of $z$).

We now state the following lemma, providing a condition on $\mu$, $p_1$, $p_2$, $\bar{p}_1$ and $\bar{p}_2$ such that $V(z, x_s)$ is equivalent to \(|z|^2_{H^2((0, x_0); \mathbb{R}^2)} + |x_s - x_0|^2\).

\textbf{Lemma 6.4.1.} If
\begin{equation}
\max (\Theta_1, \Theta_2) < 2,
\end{equation}

where
\begin{equation}
\Theta_1 := \frac{\bar{p}_1}{p_1} \frac{\eta_1}{\mu} \left(1 - e^{-\frac{x_0}{\eta_1}}\right), \quad \Theta_2 := \frac{\bar{p}_2}{p_2} \frac{\eta_2}{\mu} \left(1 - e^{-\frac{x_0}{\eta_2}}\right),
\end{equation}

there exists $\beta > 0$ such that
\begin{equation}
\beta \left(|z|^2_{H^2((0, x_0); \mathbb{R}^2)} + |x_s - x_0|^2\right) \leq V \leq \frac{1}{\beta} \left(|z|^2_{H^2((0, x_0); \mathbb{R}^2)} + |x_s - x_0|^2\right)
\end{equation}

for any $(z, x_s) \in H^2((0, x_0); \mathbb{R}^2) \times (0, L)$ satisfying
\begin{equation}
|z|^2_{H^2((0, x_0); \mathbb{R}^2)} + |x_s - x_0|^2 < \beta^2.
\end{equation}

\textbf{Proof of Lemma 6.4.1.} Let us start with
\begin{equation}
V_4 = \int_0^{x_0} \bar{p}_1 e^{-\frac{x_0}{\eta_1}} z_1(x_s - x_0) \, dx + \int_0^{x_0} \bar{p}_2 e^{-\frac{x_0}{\eta_2}} z_2(x_s - x_0) \, dx + \kappa(x_s - x_0)^2.
\end{equation}
\tag{6.4.20}

Using Young’s inequality we get
\begin{equation}
-\frac{1}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{x_0}{\eta_1}} z_1 \, dx\right)^2 - \frac{(x_s - x_0)^2}{2} - \frac{1}{2} \left(\int_0^{x_0} \bar{p}_2 e^{-\frac{x_0}{\eta_2}} z_2 \, dx\right)^2 - \frac{(x_s - x_0)^2}{2}
+ \kappa(x_s - x_0)^2 \leq V_4 \leq \frac{1}{2} \left(\int_0^{x_0} \bar{p}_1 e^{-\frac{x_0}{\eta_1}} z_1 \, dx\right)^2 + \frac{(x_s - x_0)^2}{2} + \kappa(x_s - x_0)^2.
\end{equation}
\tag{6.4.21}

Hence, using the Cauchy-Schwarz inequality and the expression of $V_1$ given in (6.4.4),
\begin{equation}
\begin{aligned}
p_1 \left(1 - \frac{1}{2} \Theta_1\right) \int_0^{x_0} e^{-\frac{\eta_1}{\mu}} z_1^2 \, dx + p_2 \left(1 - \frac{1}{2} \Theta_2\right) \int_0^{x_0} e^{-\frac{\eta_2}{\mu}} z_2^2 \, dx \\
+ (x_s - x_0)^2(\kappa - 1) \leq V_1 + V_4 \leq p_1 \left(1 + \frac{1}{2} \Theta_1\right) \int_0^{x_0} e^{-\frac{\eta_1}{\mu}} z_1^2 \, dx
+ p_2 \left(1 + \frac{1}{2} \Theta_2\right) \int_0^{x_0} e^{-\frac{\eta_2}{\mu}} z_2^2 \, dx + (x_s - x_0)^2(\kappa + 1),
\end{aligned}
\end{equation}
\tag{6.4.22}
and similarly
\[
p_1(1 - \frac{1}{2} \Theta_1) \int_0^{x_0} e^{-\frac{\mu}{2}} z_{t1}^2 dx + p_2(1 - \frac{1}{2} \Theta_2) \int_0^{x_0} e^{-\frac{\mu}{2}} z_{t2}^2 dx
+ (\dot{x}_s)^2 (\kappa - 1) \leq V_2 + V_3 \leq p_1(1 + \frac{1}{2} \Theta_1) \int_0^{x_0} e^{-\frac{\mu}{2}} z_{t1}^2 dx
+ p_2(1 + \frac{1}{2} \Theta_2) \int_0^{x_0} e^{-\frac{\mu}{2}} z_{t2}^2 dx + (\dot{x}_s)^2 (\kappa + 1),
\] (6.4.23)
and also
\[
p_1(1 - \frac{1}{2} \Theta_1) \int_0^{x_0} e^{-\frac{\mu}{2}} z_{t1}^2 dx + p_2(1 - \frac{1}{2} \Theta_2) \int_0^{x_0} e^{-\frac{\mu}{2}} z_{t2}^2 dx
+ (\dot{x}_s)^2 (\kappa - 1) \leq V_3 + V_4 \leq p_1(1 + \frac{1}{2} \Theta_1) \int_0^{x_0} e^{-\frac{\mu}{2}} z_{t1}^2 dx
+ p_2(1 + \frac{1}{2} \Theta_2) \int_0^{x_0} e^{-\frac{\mu}{2}} z_{t2}^2 dx + (\dot{x}_s)^2 (\kappa + 1).
\] (6.4.24)

Hence, from (6.4.11), \( \kappa > 1 \) and (6.4.16) is satisfied, there exists \( \sigma > 0 \) such that
\[
\sigma \left( |z|^2_{H^1((0, x_0), \mathbb{R}^2)} + |x_s - x_0|^2 \right) \leq V \leq \frac{1}{\sigma} \left( |z|^2_{H^2((0, x_0), \mathbb{R}^2)} + |x_s - x_0|^2 \right),
\] (6.4.25)
where, for a function \( z \in H^2((0, x_0); \mathbb{R}^2) \), \( |z|_{H^2((0, x_0), \mathbb{R}^2)} \) is defined by
\[
|z|_{H^2((0, x_0), \mathbb{R}^2)} = \left( |z|^2_{L^2((0, x_0), \mathbb{R}^2)} + |z_t|^2_{L^2((0, x_0), \mathbb{R}^2)} + |z_{tt}|^2_{L^2((0, x_0), \mathbb{R}^2)} \right)^{1/2},
\] (6.4.26)
with \( z_t \) and \( z_{tt} \) defined as (6.4.12) - (6.4.15). Let us point out that from (6.4.12) - (6.4.15), there exists \( C > 0 \) such that
\[
\frac{1}{C} |z|_{H^2((0, x_0), \mathbb{R}^2)} \leq |z|_{H^2((0, x_0), \mathbb{R}^2)} \leq C |z|_{H^2((0, x_0), \mathbb{R}^2)},
\] (6.4.27)
if \( \beta > 0 \) can be taken sufficiently small such that inequality (6.4.18) holds provided (6.4.19) is satisfied. This concludes the proof of Lemma 6.4.1.

Before proving Theorem 6.4.1, we introduce the following density argument, which shows that it is enough to prove the exponential decay of \( V \) along any \( C^1 \) solutions of the system.

**Lemma 6.4.2.** Let \( V \) be a \( C^1 \) and nonnegative functional on \( C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R}) \). If there exist \( \delta > 0 \) and \( \gamma > 0 \) such that for any \( (z, x_s) \in C^0([0, T] \times [0, x_0]; \mathbb{R}^2) \) \( \times C^1([0, T]; \mathbb{R}) \) solution of (6.3.3) - (6.3.4) with associated initial condition \( (z^0, x_{s0}) \) satisfying \( |z^0|_{H^2((0, x_0), \mathbb{R}^2)} \leq \delta \) and \( |x_{s0} - x_0| \leq \delta \), one has
\[
\frac{dV(z(t, \cdot), x_s(t))}{dt} \leq -\frac{\gamma}{2} V(z(t, \cdot), x_s(t)),
\] (6.4.28)
then (6.4.28) also holds in a distribution sense for any \( (z, x_s) \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R}) \) solution of (6.3.3) - (6.3.4) such that the associated initial condition \( (z^0, x_{s0}) \) satisfies \( |z^0|_{H^2((0, x_0), \mathbb{R}^2)} < \delta \) and \( |x_{s0} - x_0| < \delta \).

**Proof of Lemma 6.4.2.** Let \( V \) be a \( C^1 \) and nonnegative functional on \( C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R}) \) and let \( (z, x_s) \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \times C^1([0, T]; \mathbb{R}) \) be solution of (6.3.3) - (6.3.4) with associated initial condition \( |z^0|_{H^2((0, x_0), \mathbb{R}^2)} \leq \delta \) and \( |x_{s0} - x_0| \leq \delta \). Let \( (z_n^0, x_{s0}^n) \in H^2((0, x_0); \mathbb{R}^2) \times (0, L), n \in \mathbb{N} \) be a sequence of functions that satisfy the fourth order compatibility conditions and
\[
|z_n^0|_{H^2((0, x_0), \mathbb{R}^2)} \leq \delta, \quad |x_{s0}^n - x_0| \leq \delta,
\] (6.4.29)
such that $z^{n}$ converges to $z^{0}$ in $H^{2}((0,x_{0});\mathbb{R}^{2})$ and $x_{n}^{0}$ converges to $x_{0}$. From Remark 6.3.1, there exists a unique solution $(z^{n},x_{n}^{0}) \in C(0,T;H^{1}((0,x_{0});\mathbb{R}^{2})) \times C^{1}(0,T;\mathbb{R})$ to (6.3.3)–(6.3.4) corresponding to the initial condition $(z^{0n},x_{n}^{0})$ and for any $t \in [0,T]$, we have

$$|z^{n}(t,\cdot)|_{H^{2}((0,x_{0});\mathbb{R}^{2})} + |x_{n}^{0}(t) - x_{0}| \leq C(T) \left( |z^{0n}|_{H^{2}((0,x_{0});\mathbb{R}^{2})} + |x_{n}^{0} - x_{0}| \right). \tag{6.4.30}$$

Hence, from (6.4.29) and the third equation of (6.3.3), the sequence $(z^{n},x_{n}^{0})$ is bounded in $C^{0}([0,T];H^{2}((0,x_{0});\mathbb{R}^{2})) \times C^{1}([0,T];\mathbb{R})$. By [176] Corollary 4, we can extract a subsequence, which we still denote by $(z^{n},x_{n}^{0})$ that converges to $(u,y_{s})$ in $C^{0}([0,T];C^{1}([0,x_{0};\mathbb{R}^{2}]) \cap C^{1}([0,T];C^{0}([0,x_{0};\mathbb{R}^{2}])) \times C^{1}([0,T];\mathbb{R})$, which is a classical solution of (6.3.3)–(6.3.5). If we define

$$J(u) = \begin{cases} +\infty, & \text{if } u \notin L^{\infty}((0,T);H^{2}((0,x_{0});\mathbb{R}^{2})) , \\ |u|_{L^{\infty}((0,T);H^{2}((0,x_{0});\mathbb{R}^{2}))}, & \text{if } u \in L^{\infty}((0,T);H^{2}((0,x_{0});\mathbb{R}^{2})), \end{cases} \tag{6.4.31}$$

then $J$ is lower semi-continuous and we have

$$J(u) \leq \lim_{n \to +\infty} |z^{n}|_{C^{0}([0,T];H^{2}((0,x_{0});\mathbb{R}^{2}))}. \tag{6.4.32}$$

thus from (6.4.30) and the convergence of $(z^{n},x_{n}^{0})$ in $H^{2}((0,x_{0});\mathbb{R}^{2}) \times \mathbb{R}$, we have $J(u) \in \mathbb{R}$ and $u \in L^{\infty}((0,T);H^{2}((0,x_{0});\mathbb{R}^{2}))$. Moreover, as $(u,y_{s})$ is a solution to (6.3.3)–(6.3.5), we get the extra regularity $u \in C^{0}([0,T];H^{2}((0,x_{0});\mathbb{R}^{2}))$. Hence, from the uniqueness of the solution given by Lemma 6.3.1 $u = z$ and consequently $y_{s} = x_{s}$, which implies that $(z^{n},x_{n}^{0})$ converges to $(z,x_{s})$ in $(C^{0}([0,T];C^{1}([0,x_{0};\mathbb{R}^{2}]) \cap C^{1}([0,T];C^{0}([0,x_{0};\mathbb{R}^{2}])) \times C^{1}([0,T];\mathbb{R})$. Now, we define $V^{n}(t) := V(z^{n}(t,\cdot),x_{s}(t))$. Note that $V(t) = V(z(t,\cdot),x(t))$ is continuous with time $t$ and well-defined as, from Lemma 6.3.1 $z \in C^{1}([0,T];H^{2}((0,x_{0};\mathbb{R}^{2})))$. As $(z^{n},x_{s}^{0})$ belongs to $C^{0}([0,T];H^{1}((0,x_{0};\mathbb{R}^{2})) \times C^{1}([0,T];\mathbb{R})$ and is thus $C^{1}$, and as it is a solution of (6.3.3)–(6.3.4) with initial condition satisfying (6.4.29), we have from (6.4.28)

$$\frac{dV^{n}}{dt} \leq -\frac{\gamma}{2} V^{n}, \tag{6.4.33}$$

thus $V^{n}$ is decreasing on $[0,T]$. Therefore

$$V^{n}(t) - V^{n}(0) \leq -\frac{\gamma t}{2} V^{n}(t), \forall t \in [0,T], \tag{6.4.34}$$

which implies that

$$\left(1 + \frac{\gamma t}{2}\right) V^{n}(t) \leq V^{n}(0), \forall t \in [0,T]. \tag{6.4.35}$$

Using the lower semi-continuity of $J$, by the continuity of $V$ and the convergence of $(z^{0n},x_{n}^{0})$ in $H^{2}((0,x_{0});\mathbb{R}^{2}) \times \mathbb{R}$, we have

$$\left(1 + \frac{\gamma t}{2}\right) V(t) \leq V(0), \forall t \in [0,T]. \tag{6.4.36}$$

Note that instead of approximating $(z^{0},x_{0})$, we could have approximated $(z(s,\cdot),x_{s}(s))$ where $s \in [0,T]$ and follow the same procedure as above. Therefore we have in fact for any $s \in [0,T)$

$$\left(1 + \frac{\gamma(t-s)}{2}\right) V(t) \leq V(s), \forall t \in [s,T], \tag{6.4.37}$$

thus for any $0 \leq s < t \leq T$

$$\frac{V(t) - V(s)}{t-s} \leq -\frac{\gamma}{2} V(t), \tag{6.4.38}$$

which implies that (6.4.28) holds in the distribution sense. This ends the proof of Lemma 6.4.2

We now state our final lemma, which gives a sufficient condition so that $V$ defined by (6.4.3)–(6.4.9) satisfies a useful estimate along any $C^{3}$ solutions.
Lemma 6.4.3. Let \( V \) be defined by (6.4.3)–(6.4.9). If the matrix \( A \) defined by (6.4.59)–(6.4.64) is positive definite, then for any \( T > 0 \), there exists \( \delta_1(T) > 0 \) such that for any \( (z, x_s) \in C([0, T] \times (0, x_0; \mathbb{R}^2)) \times C^3([0, T]; \mathbb{R}) \) solution of (6.3.3)–(6.3.5) satisfying \( |z|^0_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_1(T) \) and \( |x_{s0} - x_0| \leq \delta_1(T) \),
\[
\frac{dV(z(t,:), x_s(t))}{dt} \leq -\frac{\mu}{2} V(z(t,:), x_s(t)) + O\left( (|z(t,:)|_{H^2((0,x_0);\mathbb{R}^2)} + |x_s - x_0|)^3 \right), \quad \forall t \in [0, T].
\] (6.439)

Here and hereafter, \( O(s) \) means that there exist \( \epsilon > 0 \) and \( C_1 > 0 \), both independent of \( z, x_s, T \) and \( t \in [0, T] \), such that
\[
(s \leq \epsilon) \implies (|O(s)| \leq C_1s).
\]

To prove this lemma, we differentiate \( V \) with respect to time along any \( C^3 \) solutions and perform several estimates on the different components of \( V \). For the sake of simplicity, for any \( z \in C^0([0, T]; H^2((0, x_0); \mathbb{R}^2)) \), we denote from now on \( |z(t,:)|_{H^2((0,x_0);\mathbb{R}^2)} \) by \( |z|^2_{H^2} \).

Proof of Lemma [6.4.3] Let \( V \) be given by (6.4.3)–(6.4.9) and \( T > 0 \). Let us assume that \( (z, x_s) \) is a \( C^3 \) solution to the system (6.3.3)–(6.3.5), with initial condition \( |z|^0_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_1(T) \) and \( |x_{s0} - x_0| \leq \delta_1(T) \) respectively with \( \delta_1(T) > 0 \) to be chosen later on. Let us examine the different components of the Lyapunov function. We start by studying \( V_1, V_2 \) and \( V_3 \) which can be treated similarly as in [3] Section 4.4. Differentiating \( V_1 \) along the solution \( (z, x_s) \) and integrating by parts, noticing (6.4.10), we have
\[
\frac{dV_1}{dt} = -2 \int_0^{x_0} \left( p_1 e^{-\frac{\alpha_1}{s_1} x_0} \frac{z_1}{x_s} \left| z_1 + \frac{\dot{x}_s}{x_0} \right| \right) dx.
\] (6.440)

From (6.3.3), we have
\[
z_{1tt} + (1 + z - \frac{\dot{x}_s}{x_0}) z_{1tx} \frac{x_0}{x_s} + (z_{1t} - \frac{\dot{x}_s}{x_0}) z_{1sx} + z_{1tx} \frac{x_0}{x_s} + z_{1sx} = 0,
\]
\[
z_{2tt} + (1 - z_2 + \frac{\dot{x}_s}{x_0}) z_{2tx} \frac{x_0}{L-x_s} - (z_{2t} - \frac{\dot{x}_s}{x_0}) z_{2sx} + z_{2tx} \frac{x_0}{L-x_s} + z_{2sx} = 0.
\] (6.441)

Therefore, similarly to (6.440), we can obtain
\[
\frac{dV_2}{dt} = -\mu V_2 - \left[ p_1 e^{-\frac{\alpha_1}{s_1} x_0} \frac{x_0}{x_s} z_{1t} + p_2 e^{-\frac{\alpha_2}{s_2} x_0} \frac{x_0}{L-x_s} z_{2t} \right]_{x_0} + O\left( (|z|^2_{H^2} + |x_s - x_0|)^3 \right).
\] (6.442)

From (6.441) and using (6.3.3), we get
\[
z_{1tt} + (1 + z - \frac{\dot{x}_s}{x_0}) z_{1tx} \frac{x_0}{x_s} + 2(z_{1t} - \frac{\dot{x}_s}{x_0}) z_{1sx} + \frac{\dot{x}_s}{x_s} (z_{1tt} + z_{1tx} \frac{x_0}{x_s}) + (z_{1t} - \frac{\dot{x}_s}{x_0}) z_{1sx} + z_{1tx} \frac{x_0}{x_s} + z_{1sx} - (\frac{\dot{x}_s}{x_s})^2 = 0,
\]
\[
z_{2tt} + (1 - z_2 + \frac{\dot{x}_s}{x_0}) z_{2tx} \frac{x_0}{L-x_s} - 2(z_{2t} - \frac{\dot{x}_s}{x_0}) z_{2sx} + \frac{\dot{x}_s}{L-x_s} (z_{2tt} + z_{2tx} \frac{x_0}{L-x_s}) - (z_{2t} - \frac{\dot{x}_s}{x_0}) z_{2sx} + z_{2tx} \frac{x_0}{L-x_s} - z_{2sx} \frac{x_0}{L-x_s} \left( \frac{L-x_s}{2} \right) = 0.
\] (6.443)

Then differentiating \( V_3 \) along the system solutions and using (6.4.43), we have
\[
\frac{dV_3}{dt} \leq - \left[ p_1 e^{-\frac{\alpha_1}{s_1} x_0} \frac{x_0}{x_s} (z_{1tt} + z_{1tx} \frac{x_0}{x_s}) \right]_{x_0} - \left[ p_2 e^{-\frac{\alpha_2}{s_2} x_0} \frac{x_0}{L-x_s} z_{2tt} (1 - z_2 + \frac{\dot{x}_s}{x_0}) \right]_{x_0} - \mu \min \left( \frac{x_0}{x_s} \frac{L-x_s}{L-x_s} \right) V_3 - \mu \int_0^{x_0} \left( \frac{x_0}{x_s} p_1 e^{-\frac{\alpha_1}{s_1} x_0} \frac{x_0}{x_s} z_{1tt} z_{1tx} \frac{x_0}{x_s} + \frac{\dot{x}_s}{L-x_s} p_2 e^{-\frac{\alpha_2}{s_2} x_0} \frac{x_0}{L-x_s} \right) dx
\]
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\[+ \mu \int_0^{x_0} \left( \frac{x_0}{x_0} p_1 e^{-\frac{\mu}{2} x_0^2} \frac{x_0}{x_0} - \frac{L - x_0}{L - x_0} p_2 e^{-\frac{\mu}{2} x_0^2 \frac{\partial}{\partial t}} \right) dx \]

\[-3 \int_0^{x_0} \left( p_1 e^{-\frac{\mu}{2} z_1^0 x_0^2} \frac{x_0}{x_0} - p_2 e^{-\frac{\mu}{2} z_2^0 (\frac{\partial}{\partial t})} \right) dx \]

\[-\int_0^{\infty} \left( p_1 e^{-\frac{\mu}{2} z_1^0 x_0^2} \frac{x_0}{x_0} - p_2 e^{-\frac{\mu}{2} z_2^0 (\frac{\partial}{\partial t})} \right) dx \]

\[\left(6.4.44\right)\]

\[-4 \int_0^{x_0} \left( p_1 e^{-\frac{\mu}{2} z_1^0 (\frac{\partial}{\partial t})} \frac{x_0}{x_0} - p_2 e^{-\frac{\mu}{2} z_2^0 (\frac{\partial}{\partial t})} \right) dx \]

\[-2 \int_0^{x_0} \left( p_1 e^{-\frac{\mu}{2} z_1^0 (\frac{\partial}{\partial t})} \frac{x_0}{x_0} - p_2 e^{-\frac{\mu}{2} z_2^0 (\frac{\partial}{\partial t})} \right) dx \]

\[-2 \int_0^{x_0} \left( p_1 e^{-\frac{\mu}{2} z_1^0 (\frac{\partial}{\partial t})} \frac{x_0}{x_0} - p_2 e^{-\frac{\mu}{2} z_2^0 (\frac{\partial}{\partial t})} \right) dx \]

\[\left(6.4.45\right)\]

Observe that, while previously all the cubic terms in \(z\) could be bounded by \(|z|_H^3\), here in the last line in \[6.4.45\] we have \(\tilde{x}_s\), which is proportional to \(z_1(t,x_0)\) and cannot be roughly bounded by the \(|z|_H^2\) norm. To overcome this difficulty, we transform these terms using Young’s inequality and we get

\[2 \int_0^{x_0} \left( p_1 e^{-\frac{\mu}{2} z_1^0 (\frac{\partial}{\partial t})} \frac{x_0}{x_0} - p_2 e^{-\frac{\mu}{2} z_2^0 (\frac{\partial}{\partial t})} \frac{x_0}{x_0} \right) dx \]

\[\leq C \left| \frac{z(t, \cdot)}{C} \right|_{C^1([0,x_0];\mathbb{R}^2)} (z_1^0(t,x_0) + z_2^0(t,x_0))^2 + O \left( \left| \frac{z(t, \cdot)}{C} \right|_{C^1([0,x_0];\mathbb{R}^2)} |z|_H^2 \right),\]

where \(C\) denotes a constant, independent of \(x, x_s, T\) and \(t \in [0,T]\). Note that the first term on the right is now proportional to \(z_1(t,x_0)\) with a proportionality coefficient \(C \left| \frac{z(t, \cdot)}{C} \right|_{C^1([0,x_0];\mathbb{R}^2)}\) that, by Sobolev inequality, can be made sufficiently small provided that \(|z|_H^2\) is sufficiently small and thus can be dominated by the boundary terms. More precisely, from \[6.4.44\] and \[6.4.45\], we have

\[\frac{dV_3}{dt} \leq - \mu V_3 - \left[ p_1 e^{-\frac{\mu}{2} z_1^0 (\frac{\partial}{\partial t})} \frac{x_0}{x_0} - p_2 e^{-\frac{\mu}{2} z_2^0 (\frac{\partial}{\partial t})} \frac{x_0}{x_0} \right]_0^{x_0} + O \left( \left| \frac{z(t, \cdot)}{C} \right|_{C^1([0,x_0];\mathbb{R}^2)} (z_1^0(t,x_0) + z_2^0(t,x_0))^2 + O \left( (|z|_H^2 + |x_s - x_0|)^3 \right) \right).\]

Let us now deal with the term \(V_4\) that takes into account the position of the jump. In the following, we use notations \(z(0)\) and \(z(x_s)\) instead of \(z(t,0)\) and \(z(t,x_s)\) for simplicity. We have

\[\frac{dV_4}{dt} = - \int_0^{x_0} \left[ p_1 e^{-\frac{\mu}{2} x_0^2} (1 + z_1 - \frac{x_0}{x_s}) z_1^0 (x_s - x_0) \right]_0^{x_0} dx + \int_0^{x_0} \left[ p_1 e^{-\frac{\mu}{2} x_0^2} z_1 \right]_0^{x_0} \]

\[\left(6.4.47\right)\]

\[\left(6.4.46\right)\]
According to Young’s inequality, for any positive ε₁ and ε₂, we have
\[
\frac{z_1(x_0) + z_2(x_0)}{2} \left( \int_0^{x_0} \tilde{p}_1 e^{-\frac{\mu x}{\kappa_1}} z_1 \, dx \right) \leq \frac{\varepsilon_1}{4} \left( \frac{z_1(x_0) + z_2(x_0)}{2} \right)^2 + \frac{1}{\varepsilon_1} \left( \int_0^{x_0} \tilde{p}_1 e^{-\frac{\mu x}{\kappa_1}} z_1 \, dx \right)^2,
\]
and therefore using Cauchy-Schwarz and Young’s inequalities, we get
\[
\frac{z_1(x_0) + z_2(x_0)}{2} \left( \int_0^{x_0} \tilde{p}_2 e^{-\frac{\mu x}{\kappa_2}} z_2 \, dx \right) \leq \frac{\varepsilon_2}{4} \left( \frac{z_1(x_0) + z_2(x_0)}{2} \right)^2 + \frac{1}{\varepsilon_2} \left( \int_0^{x_0} \tilde{p}_2 e^{-\frac{\mu x}{\kappa_2}} z_2 \, dx \right)^2.
\]

Furthermore, by differentiating (6.4.50) with respect to time, we have
\[
d\frac{V_4}{dt} \leq -\mu V_4 - \tilde{p}_1 \left( x_0 - x_s \right) \left( e^{-\frac{\mu x_0}{\kappa_1}} - k_1 \right) z_1(x_0) + b_1(x_s - x_0) \]
\[
- \tilde{p}_2 \left( x_0 - x_s \right) \frac{x_0}{L - x_s} \left( e^{-\frac{\mu x_0}{\kappa_2}} - k_2 \right) z_2(x_0) + b_2(x_s - x_0) \]
\[
+ \left( \varepsilon_1 + \varepsilon_2 \right) \frac{z_1^2(x_0) + z_2^2(x_0)}{8} + \max \left\{ \frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2} \right\} V_1
\]
\[
+ \kappa(x_s - x_0)(z_1(x_0) + z_2(x_0)) + \mu \kappa(x_s - x_0)^2 + O \left( \left| z \right|_{H^2} + |x_s - x_0|^3 \right).
\]

Let us now consider \( V_5 \). From (6.4.8) and (6.4.41), one has similarly
\[
\frac{dV_5}{dt} = -\int_0^{x_0} \tilde{p}_1 e^{-\frac{\mu x}{\kappa_1}} z_{1t} \hat{x_s} \frac{x_0}{x_s} \, dx + \int_0^{x_0} \tilde{p}_2 e^{-\frac{\mu x}{\kappa_2}} z_{2t} \hat{x_s} \frac{x_0}{x_s} \, dx
\]
\[
- \int_0^{x_0} \tilde{p}_2 e^{-\frac{\mu x}{\kappa_2}} z_{2t} \hat{x_s} \frac{x_0}{L - x_s} \, dx + \int_0^{x_0} \tilde{p}_2 e^{-\frac{\mu x}{\kappa_2}} Z_{2t} \hat{x_s} \, dx + 2\kappa \hat{x_s} \hat{x_s} + O \left( \left| z \right|_{H^2} + |x_s - x_0|^3 \right)
\]
\[
= -\hat{x_s} \left[ \tilde{p}_1 e^{-\frac{\mu x}{\kappa_1}} \frac{x_0}{x_s} z_{1t} + \tilde{p}_2 e^{-\frac{\mu x}{\kappa_2}} \frac{x_0}{L - x_s} z_{2t} \right]_{x_0}^{x_0} - \mu (V_5 - \kappa \hat{x_s}^2)
\]
\[
+ \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2} \left( \int_0^{x_0} \tilde{p}_1 e^{-\frac{\mu x}{\kappa_1}} z_{1t} \, dx \right) + \frac{z_{1t}(x_0) + z_{2t}(x_0)}{2} \left( \int_0^{x_0} \tilde{p}_2 e^{-\frac{\mu x}{\kappa_2}} z_{2t} \, dx \right)
\]
\[
+ \kappa(z_{1t}(x_0) + z_{2t}(x_0)) \hat{x_s} + O \left( \left| z \right|_{H^2} + |x_s - x_0|^3 \right).
\]

By differentiating (6.3.4) with respect to time, we have
\[
z_{1t}(0) = k_1 z_1(x_0) - b_1 \hat{x_s},
\]
\[
z_{2t}(0) = k_2 z_2(x_0) - b_2 \hat{x_s},
\]
and therefore using Cauchy-Schwarz and Young’s inequalities, we get
\[
\frac{dV_5}{dt} \leq -\mu V_5 - \tilde{p}_1 \hat{x_s} \frac{x_0}{x_s} \left( e^{-\frac{\mu x_0}{\kappa_1}} - k_1 \right) z_{1t}(x_0) + b_1 \hat{x_s}
\]
\[
- \tilde{p}_2 \hat{x_s} \frac{x_0}{L - x_s} \left( e^{-\frac{\mu x_0}{\kappa_2}} - k_2 \right) z_{2t}(x_0) + b_2 \hat{x_s}
\]
\[
+ \left( \varepsilon_1 + \varepsilon_2 \right) \frac{z_{1t}^2(x_0) + z_{2t}^2(x_0)}{8} + \max \left\{ \frac{\Theta_1}{\varepsilon_1}, \frac{\Theta_2}{\varepsilon_2} \right\} V_2
\]
\[
+ \kappa(z_{1t}(x_0) + z_{2t}(x_0)) + \mu \kappa \hat{x_s}^2 + O \left( \left| z \right|_{H^2} + |x_s - x_0|^3 \right).
\]

Furthermore, by differentiating (6.4.50) with respect to time, we have
\[
z_{1tt}(0) = k_1 z_{1tt}(x_0) - b_1 \hat{x_s},
\]
\[
z_{2tt}(0) = k_2 z_{2tt}(x_0) - b_2 \hat{x_s},
\]
and therefore using also (6.4.43), one has

\[
\frac{dV_6}{dt} = -\int_0^x \frac{p_1 e^{-\frac{w x}{w_t}} z_{1tt} x_s}{x_s} dx + \int_0^x \frac{p_1 e^{-\frac{w x}{w_t}} z_{1tt} \bar{x}_s}{x_s} dx \\
- \int_0^x \frac{p_2 e^{-\frac{w x}{w_t}} z_{2tt} x_s}{L-x_s} dx + \int_0^x \frac{p_2 e^{-\frac{w x}{w_t}} z_{2tt} \bar{x}_s}{x_s} dx + 2\bar{\epsilon}_s \bar{x}_s + \int_0^x \frac{p_1 e^{-\frac{w x}{w_t}} \bar{x}_s(x \bar{x}_s) z_{1tt} x_s}{x_s} dx \\
- \int_0^x \frac{p_2 e^{-\frac{w x}{w_t}} \bar{x}_s(x \bar{x}_s) z_{2tt} \bar{x}_s}{x_s} dx + O \left((|z|_{H^2} + |x_s - x_0|)^3\right) \\
= -\bar{x}_s \left[ p_1 e^{-\frac{w x}{w_t}} \frac{z_{1tt}}{x_s} + p_2 e^{-\frac{w x}{w_t}} \frac{x_s}{L-x_s} z_{2tt} \right]_{0}^{x_0} - \mu (V_6 - \kappa (\bar{x}_s)^2) \\
+ \frac{\bar{x}_s}{2} \left[ \left( \frac{p_1 e^{-\frac{w x}{w_t}}}{x_s} \right)^{z_{1tt}} \right]_{0}^{x_0} + \frac{\bar{x}_s}{2} \left[ \left( \frac{p_1 e^{-\frac{w x}{w_t}}}{x_s} \right)^{z_{2tt}} \right]_{0}^{x_0} + \kappa (z_{1tt}(x_0) + z_{2tt}(x_0)) \bar{x}_s + \int_0^x \frac{p_1 e^{-\frac{w x}{w_t}} \bar{x}_s(x \bar{x}_s) z_{1tt} x_s}{x_s} dx \\
- \int_0^x \frac{p_2 e^{-\frac{w x}{w_t}} \bar{x}_s(x \bar{x}_s) z_{2tt} \bar{x}_s}{x_s} dx + O \left((|z|_{H^2} + |x_s - x_0|)^3\right).
\]

Note that, as above for $V_3$, here appears again $\bar{x}_s$ which is proportional to $z_{tt}(t, x_0)$ and cannot be bounded by $|z|_{H^2}$. We therefore use Cauchy-Schwarz and Young’s inequalities as previously and the boundary condition (6.4.52), to get

\[
\frac{dV_6}{dt} \leq -\mu V_6 - \bar{p}_1 \bar{x}_s x_s \left( e^{-\frac{w x}{w_t}} - k_1 \right) z_{1tt}(x_0) + b_1 \bar{x}_s \\
- \bar{p}_2 \bar{x}_s x_s \left( e^{-\frac{w x}{w_t}} - k_2 \right) z_{2tt}(x_0) + b_2 \bar{x}_s \\
+ (\epsilon_1 + \epsilon_2) \frac{\bar{z}_s^2}{8} + \max \left\{ \frac{\Theta_1}{\epsilon_1}, \frac{\Theta_2}{\epsilon_2} \right\} V_2 \\
+ \frac{\bar{x}_s}{8} \left( \frac{b_1 k_1}{x_s} + \frac{x_0}{x_s} \frac{x_s}{x_s} \right) + \frac{\bar{x}_s}{8} \left( \frac{b_2 k_2}{x_s} + \frac{x_0}{x_s} \frac{x_s}{x_s} \right) \\
+ O \left((|z|_{H^2} + |x_s - x_0|)^3\right).
\]

Hence, from (6.4.40), (6.4.49) and the boundary conditions (6.3.4), we have

\[
\frac{dV_1}{dt} + \frac{dV_4}{dt} \leq -\mu (V_1 + V_4) \\
+ \max \left\{ \frac{\Theta_1}{\epsilon_1}, \frac{\Theta_2}{\epsilon_2} \right\} V_1 \\
+ \frac{\bar{x}_s}{8} \left( \frac{b_1 k_1}{x_s} + \frac{x_0}{x_s} \frac{x_s}{x_s} \right) + \frac{\bar{x}_s}{8} \left( \frac{b_2 k_2}{x_s} + \frac{x_0}{x_s} \frac{x_s}{x_s} \right) + O \left((|z|_{H^2} + |x_s - x_0|)^3\right).
\]

Let us now select $\epsilon_1$ and $\epsilon_2$ as follows:

\[
\epsilon_1 = 2 \frac{\Theta_1}{\mu}, \quad \epsilon_2 = 2 \frac{\Theta_2}{\mu}.
\]
where $\Theta_1$ and $\Theta_2$ are defined in (6.4.17). Then (6.4.54) can be rewritten in the following compact form:

$$\frac{dV_1}{dt} + \frac{dV_4}{dt} \leq -\frac{\mu}{2}V_1 - \mu V_4 - Z^T A_0 Z + O \left( (|z|_{H^2} + |x_s - x_0|)^3 \right).$$

(6.4.56)

This expression involves the quadratic form $Z^T A_0 Z$ with the vector $Z$ defined as

$$Z = (z_1(x_0) - z_2(x_0) - (x_s - x_0))^T.$$

(6.4.57)

and the matrix $A_0$ satisfies

$$A_0 = A + O(|x_s - x_0|),$$

(6.4.58)

where $A$ is given by

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

with

$$a_{11} = p_1 (e^{-\frac{a x_0}{2}} - k_1^2) - \frac{\varepsilon_1 + \varepsilon_2}{8},$$

(6.4.60)

$$a_{13} = a_{31} = p_1 b_1 k_1 + \frac{\bar{p}_1}{2} (e^{-\frac{a x_0}{2}} - k_1) - \frac{\kappa}{2},$$

(6.4.61)

$$a_{22} = \frac{x_0}{L-x_0} p_2 (e^{-\frac{a x_0}{2}} - k_2^2) - \frac{\varepsilon_1 + \varepsilon_2}{8},$$

(6.4.62)

$$a_{23} = a_{32} = \frac{x_0}{L-x_0} p_2 b_2 k_2 + \frac{x_0}{L-x_0} \frac{\bar{p}_2}{2} (e^{-\frac{a x_0}{2}} - k_2) - \frac{\kappa}{2},$$

(6.4.63)

$$a_{33} = -p_1 b_1^2 - \frac{x_0}{L-x_0} p_2 b_2^2 + \bar{p}_1 b_1 + \frac{x_0}{L-x_0} \bar{p}_2 b_2 - \mu \kappa.$$  

(6.4.64)

Similarly, from (6.4.42) and (6.4.51), we get

$$\frac{dV_2}{dt} + \frac{dV_5}{dt} \leq -\frac{\mu}{2}V_2 - \mu V_5 - Z^T A_0 Z_t + O \left( (|z|_{H^2} + |x_s - x_0|)^3 \right).$$

(6.4.65)

while from (6.4.46) and (6.4.53), we have

$$\frac{dV_3}{dt} + \frac{dV_6}{dt} \leq -\frac{\mu}{2}V_3 - \mu V_6 - Z^T A_1 Z_t + O \left( (|z|_{H^2} + |x_s - x_0|)^3 \right)$$

(6.4.66)

with

$$A_1 = A_0 + \begin{pmatrix} O(|z|_{H^2}) & 0 & 0 \\ 0 & O(|z|_{H^2}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

(6.4.67)

If $A$ is positive definite, from (6.4.58) and (6.4.67) and by continuity, $A_0$ and $A_1$ are also positive definite provided that $|z|_{H^2}$ and $|x_s - x_0|$ are sufficiently small. Hence, from (6.4.56), (6.4.65), (6.4.66) and Lemma 6.3.1 there exists $\delta_1(T) > 0$ such that, if $|z|_{H^2((0,x_0);\mathbb{R}^2)} \leq \delta_1(T)$ and $|x_s - x_0| \leq \delta_1(T)$, one has

$$\frac{dV}{dt} \leq -\frac{\mu}{2}V + O \left( (|z|_{H^2} + |x_s - x_0|)^3 \right),$$

(6.4.68)

which ends the proof of Lemma 6.4.3.

Let us now prove Theorem 6.4.1.
Proof of Theorem 6.4.1. From Lemma 6.4.1 and Lemma 6.4.2 all it remains to do is to show that for any $\gamma > 0$, under conditions (6.2.16) there exist $\mu$, $p_1$, $p_2$, $p_1$ and $p_2$ satisfying (6.4.10) and such that $V$ given by (6.4.3)-(6.4.9) decreases exponentially with rate $\gamma/2$ along any $C^3$ solution of the system (6.3.3)-(6.3.5). Using Lemma 6.4.3 we first show that for any $\gamma > 0$ there exists $\mu > \gamma$, and positive parameters $p_1$, $p_2$, $p_1$ and $p_2$ satisfying (6.4.16) and such that the matrix $A$ defined by (6.4.56)-(6.4.64) is positive definite, which implies that (6.4.39) holds. Then, we show that this implies the exponential decay of $V$ with decay rate $\gamma/2$ along any $C^3$ solution of (6.3.3)-(6.3.5).

Let us start by selecting $p_1$ and $p_2$ as

$$p_1 = \frac{\bar{p}_1}{2b_1}, \quad p_2 = \frac{\bar{p}_2}{2b_2}$$

(6.4.69)

Then the cross terms (6.4.61), (6.4.63) of the matrix $A$ become

$$a_{13} = a_{31} = \frac{\bar{p}_1}{2} e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}} - \frac{\kappa}{2}, \quad a_{23} = a_{32} = \frac{x_0}{L-x_0} \frac{\bar{p}_2}{2} e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}} - \frac{\kappa}{2}.$$

(6.4.70)

Let $\bar{p}_1$ and $\bar{p}_2$ be selected as

$$\bar{p}_1 = \kappa e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}}, \quad \bar{p}_2 = \kappa \frac{L-x_0}{x_0} e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}}.$$

(6.4.71)

Then we have

$$a_{13} = a_{31} = 0, \quad a_{23} = a_{32} = 0$$

(6.4.72)

such that $A$ can now be rewritten as

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.$$

(6.4.73)

Moreover from (6.4.69) and (6.4.71), we get

$$a_{33} = \frac{\bar{p}_1}{2b_1} b_1 + \frac{x_0}{L-x_0} \frac{\bar{p}_2}{2} b_2 - \mu \kappa = \frac{\kappa}{2} b_1 e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}} + \frac{\kappa}{2} b_2 e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}} - \mu \kappa.$$

(6.4.74)

As conditions (6.2.16) are strict inequalities, by continuity it follows that we can select $\mu > \gamma$ such that these conditions (6.2.16) are still satisfied with $\mu$ instead of $\gamma$ such that

$$\mu e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}} < b_1 < \frac{\mu e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}}}{1-e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}}}, \quad \mu e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}} < b_2 < \frac{\mu e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}}}{1-e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}}},$$

(6.4.75)

this together with (6.4.74) gives

$$a_{33} > 0.$$

(6.4.76)

From (6.4.17), (6.4.55), (6.4.69), (6.4.62), (6.4.69) and (6.4.71), we have

$$a_{11} = \frac{\kappa}{2b_1} (1 - k_1^2 e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}}) - \frac{\kappa}{2\mu^2} \left[ b_1 (e^{\frac{\mu e}{\eta} \frac{x_0}{x_0}} - 1) + b_2 (e^{\frac{\mu e}{\eta} \frac{x_0}{x_0}} - 1) \right].$$

(6.4.77)

$$a_{22} = \frac{\kappa}{2b_2} (1 - k_2^2 e^{-\frac{\mu e}{\eta} \frac{x_0}{x_0}}) - \frac{\kappa}{2\mu^2} \left[ b_1 (e^{\frac{\mu e}{\eta} \frac{x_0}{x_0}} - 1) + b_2 (e^{\frac{\mu e}{\eta} \frac{x_0}{x_0}} - 1) \right].$$

(6.4.78)

Then, under assumptions (6.2.16), it can be checked that

$$a_{11} > 0, \quad a_{22} > 0.$$  

(6.4.79)

This implies that $A$ is positive definite.

Thus from Lemma 6.4.3 for any $T > 0$, there exists $\delta_1(T) > 0$ such that for any $(z, x_s) \in C^3([0,T] \times [0, x_0]; \mathbb{R}^2)) \times C^3([0, T]; \mathbb{R})$ solution of (6.3.3)-(6.3.5) satisfying $|z|^0_{H^2((0,L);\mathbb{R}^2)} \leq \delta_1(T)$ and $|x_{s0} - x_0| \leq \delta_1(T)$, one has

$$\frac{dV}{dt} \leq -\frac{\mu}{2} V + O \left( |z|_{H^2} + |x_s - x_0|^3 \right).$$

(6.4.80)
Now let us remark that from condition (6.4.75) we have
\[ \max \left( \frac{b_1 \eta_1}{\mu} e^{-\frac{\eta_1}{\mu}}, \frac{L - x_0 b_2 \eta_2}{\mu} e^{-\frac{\eta_2}{\mu}}, \frac{L - x_0 b_2 \eta_2}{\mu} e^{-\frac{\eta_2}{\mu}} \right) < 2. \] (6.4.81)
Therefore, there exists \( \kappa > 1 \) such that
\[ \max \left( 2b_1 \eta_1 \mu e^{-\frac{\eta_1}{\mu}}, 2L - x_0 b_2 \eta_2 \mu e^{-\frac{\eta_2}{\mu}}, 2L - x_0 b_2 \eta_2 \mu e^{-\frac{\eta_2}{\mu}} \right) < 2, \] (6.4.82)
which means from (6.4.69) and (6.4.71) that (6.4.16) is satisfied. Hence from (6.4.80) and Lemma 6.4.1, since \( \mu > \gamma \), there exists \( \delta_0(T) \leq \delta_1(T) \) such that, if \( \| z \|_{H^2((0,x_0)_R^2)} \leq \delta_0(T) \) and \( |x_{a_0} - x_0| \leq \delta_0(T) \), then
\[ \frac{dV}{dt} \leq \frac{1}{2} \frac{\eta}{\mu} \] (6.4.83)
along the \( C^3 \) solutions of the system (6.3.3)–(6.3.5). Thus from Lemma 6.4.2 (6.4.83) holds along the \( C^0([0,T] ; H^2((0,x_0)_R^2)) \times C^1([0,T] ; \mathbb{R}) \) solutions of (6.3.3)–(6.3.5) in a distribution sense.
So far \( \delta_0(T) \) may depend on \( T \), while \( \delta^* \) in Theorem 6.4.1 does not depend on \( T \). The only thing left to check is that we can find \( \delta^* \) independent of \( T \) such that if \( \| z \|_{H^2((0,x_0)_R^2)} \leq \delta^* \) and \( |x_{a_0} - x_0| \leq \delta^* \), then (6.4.83) holds on \( (0,T) \) for any \( T > 0 \). As the constant \( \beta \) involved in Lemma 6.4.1 does not depend on \( T \), there exists \( T_1 > 0 \) such that
\[ \beta^{-2} e^{-\frac{\eta}{2}} \leq \frac{1}{2}. \] (6.4.84)
As \( T_1 \in (0, +\infty) \), from Lemma 6.3.1 we can choose \( \delta_0(T_1) > 0 \) satisfying \( C(T_1) \delta_0(T_1) < \beta/2 \), such that for every \( x_{a_0} \in (0,L) \) and \( z^0 \in H^2((0,x_0)_R^2) \) satisfying the compatibility conditions (6.3.7)–(6.3.8) and
\[ |z^0|_{H^2((0,x_0)_R^2)} \leq \delta_0(T_1), \quad |x_{a_0} - x_0| \leq \delta_0(T_1), \] there exists a unique solution \( (z, x_0) \in C^0([0,T_1] ; H^2((0,x_0)_R^2)) \times C^1([0,T_1] ; \mathbb{R}) \) to the system (6.3.3)–(6.3.5) satisfying
\[ \| z(t, \cdot) \|_{H^2((0,x_0)_R^2)} + |x_s(t) - x_0| < \beta \] (6.4.85)
and such that (6.4.83) holds on \( (0,T_1) \) in a distribution sense. From (6.4.85), Lemma 6.4.1 and (6.4.84),
\[ |z(T_1, \cdot)|_{H^2((0,x_0)_R^2)} \leq \delta_0(T_1), \quad |x_s(T_1) - x_0| \leq \delta_0(T_1). \] (6.4.86)
Moreover, the compatibility conditions hold now at time \( t = T_1 \) instead of \( t = 0 \). Thus, from Lemma 6.3.1 there exists a unique \( (z, x_s) \in C^0([T_1, 2T_1] ; H^2((0,x_0)_R^2)) \times C^1([T_1, 2T_1] ; \mathbb{R}) \) solution of (6.3.3)–(6.3.5) on \([T_1, 2T_1]\) and (6.4.83) holds on \((T_1, 2T_1)\) in a distribution sense. One can repeat this analysis on \([jT_1, (j+1)T_1]\) where \( j \in \mathbb{N} \setminus \{1\} \). Setting \( \delta^* = \delta_0(T_1) \), we get that (6.4.83) holds on \((0,T)\) for any \( T > 0 \) in a distribution sense along the \( C^0([0,T] ; H^2((0,x_0)_R^2)) \times C^1([0,T] ; \mathbb{R}) \) solutions of the system (6.3.3)–(6.3.5). In fact, it also implies the global existence and uniqueness of \( (z, x_s) \in C^0([0, +\infty) ; H^2((0,x_0)_R^2)) \times C^1([0, +\infty) ; \mathbb{R}) \) solution of (6.3.3)–(6.3.5) and the fact that (6.4.83) holds on \((0, +\infty)\). This concludes the proof of Theorem 6.4.1.

\[ \square \]

### 6.5 Extension to a general convex flux

We can in fact extend this method to a more general convex flux. Let \( f \in C^3(\mathbb{R}) \) be a convex function, and consider the equation
\[ \partial_t y + \partial_x (f(y)) = 0. \] (6.5.1)
For this conservation law, the Rankine-Hugoniot condition becomes
\[ \dot{x}_s = \frac{f(y(t, x_s(t)^+)) - f(y(t, x_s(t)^-))}{y(t, x_s(t)^+) - y(t, x_s(t)^-)}, \] (6.5.2)
and, let \((y^*, x_0)\) be an entropic shock steady state of \((6.5.1) - (6.5.2)\), without loss of generality we can assume that \(y^*(x_0^+) = -1\) and \(y^*(x_0^-) = 1\), thus \(f(1) = f(-1)\). Then, for any \(x_0 \in (0, L)\), we have the following result:

**Theorem 6.5.1.** Let \(f \in C^3(\mathbb{R})\) be a convex function such that \(f(1) = f(-1)\) and assume in addition that
\[
f'(1) \geq 1 \quad \text{and} \quad |f'(-1)| \geq 1.
\] (6.5.3)

Let \(\gamma > 0\). If the following conditions hold
\[
\begin{align*}
 b_1 &\in \left( \frac{2\gamma e^{-\gamma x_0}}{f'(1) + |f'(-1)|}, \frac{\gamma e^{-\gamma x_0}}{1 - e^{-\gamma x_0}} \right), \\
 b_2 &\in \left( \frac{2\gamma e^{-\gamma(L-x_0)}}{f'(1) + |f'(-1)|}, \frac{\gamma e^{-\gamma(L-x_0)}}{1 - e^{-\gamma(L-x_0)}} \right), \\
 k_1^2 &< e^{-\gamma x_0} \left(1 - f'(1) \frac{b_1}{\gamma} \left( b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}} \right) \right), \\
 k_2^2 &< e^{-\gamma(L-x_0)} \left(1 - f'(1) \frac{b_2}{\gamma} \left( b_1 \frac{1 - e^{-\gamma x_0}}{\gamma e^{-\gamma x_0}} + b_2 \frac{1 - e^{-\gamma(L-x_0)}}{\gamma e^{-\gamma(L-x_0)}} \right) \right),
\end{align*}
\] (6.5.4a,b,c)

then the steady state \((y^*, x_0)\) of the system \((6.5.1), (6.5.2), (6.2.3), (6.2.8)\) is exponentially stable for the \(H^2\)-norm with decay rate \(\gamma/4\).

One can use exactly the same method as previously. We give in Appendix 6.7.2 a way to adapt the proof of Theorem 6.4.1.

**Remark 6.5.1.** One could wonder why we require condition (6.5.3). This condition ensures that there always exist parameters \(b_i\) and \(k_i\) satisfying (6.5.4).
6.6 Conclusion and Open problems

The stabilization of shock-free regular solutions of quasilinear hyperbolic systems has been the subject of a large number of publications in the recent scientific literature. In contrast, there are no results concerning the Lyapunov stability of solutions with jump discontinuities, although they occur naturally in the form of shock waves or hydraulic jumps in many applications of fluid dynamics. For instance, the inviscid Burgers equation provides a simple scalar example of a hyperbolic system having natural solutions with jump discontinuities. The main contribution of this chapter is precisely to address the issue of the boundary exponential feedback stabilization of an unstable shock steady state for the Burgers equation over a bounded interval. Our strategy to solve the problem relies on introducing a change of variables which allows to transform the scalar Burgers equation with shock wave solutions into an equivalent 2 × 2 quasilinear hyperbolic system having shock-free solutions over a bounded interval. Then, by a Lyapunov approach, we show that, for appropriately chosen boundary conditions, the exponential stability in $H^2$-norm of the steady state can be achieved with an arbitrary decay rate and with an exact exponential stabilization of the desired shock location. Compared with previous results in the literature for classical solutions of quasilinear hyperbolic systems, the selection of an appropriate Lyapunov function is challenging because the equivalent system is parameterized by the time-varying position of the jump discontinuity. In particular, the standard quadratic Lyapunov function used in the book [13] has to be augmented with suitable extra terms for the analysis of the stabilization of the jump position. Based on the result, some open questions could be addressed. Could these results be generalized to any convex flux, especially when (6.5.3) is not satisfied? As we show the rapid stabilization result, is it possible to obtain finite time stabilization? Could we replace the left/right state at the shock by measurements nearby or by averages close to the shock? If not, could the error on both the state and shock location be bounded?

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6.7 Appendix

6.7.1 Proof of Lemma 6.3.1

Proof. We adapt the fixed point method used in [13] Appendix B (see also [118] [140]). We first deal with the case where

$$T \in (0, \min(x_0, L-x_0)).$$

(6.7.1)

For any $\nu > 0$, $x_{s0} \in \mathbb{R}$ and $z^0 \in H^2((0,x_0);\mathbb{R}^2)$, let $C_{\nu}(z^0, x_{s0})$ be the set of

$$z \in L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2)) \cap W^{1,\infty}((0,T); H^2((0,x_0);\mathbb{R}^2)) \cap W^{2,\infty}((0,T); L^2((0,x_0);\mathbb{R}^2))$$

such that

$$|z|_{L^\infty((0,T); H^2((0,x_0);\mathbb{R}^2))} \leq \nu,$$

(6.7.2)

$$|z|_{W^{1,\infty}((0,T); H^2((0,x_0);\mathbb{R}^2))} \leq \nu,$$

(6.7.3)

$$|z|_{W^{2,\infty}((0,T); L^2((0,x_0);\mathbb{R}^2))} \leq \nu,$$

(6.7.4)

$$z(\cdot,x_0) \in H^2((0,T);\mathbb{R}^2), \quad |z(\cdot,x_0)|_{H^2((0,T);\mathbb{R}^2)} \leq \nu^2,$$

(6.7.5)

$$z_0(0,\cdot) = z^0,$$

(6.7.6)

$$z_t(0,\cdot) = -A(z^0,\cdot, x_s(z(\cdot,x_0))(0))z^0,$$

(6.7.7)

where we write $x_s(z(\cdot,x_0))(t)$ in order to emphasize its dependence on $z(\cdot,x_0)$ in the following proof and

$$x_s(z(\cdot,x_0))(t) := x_{s0} + \int_0^t \frac{z_1(s,x_0) + z_2(s,x_0)}{2} ds.$$
In (6.7.7),

\[
A(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \begin{pmatrix}
a_1(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) & 0 \\
0 & a_2(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t))
\end{pmatrix}
\]  
(6.7.9)

with

\[
a_1(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \left(1 + z_1(t, x) - x \frac{z_1(t, x_0) + z_2(t, x_0)}{2x_0}\right) \frac{x_0}{x_s(\mathbf{z}(\cdot, x_0))(t)},
\]  
(6.7.10)

\[
a_2(\mathbf{z}, x, x_s(\mathbf{z}(\cdot, x_0))(t)) = \left(1 - z_2(t, x) + x \frac{z_1(t, x_0) + z_2(t, x_0)}{2x_0}\right) \frac{x_0}{L - x_s(\mathbf{z}(\cdot, x_0))(t)}.
\]  
(6.7.11)

The set \( C_\nu(\mathbf{z}^0, x_0) \) is not empty and is a closed subset of \( L^\infty((0, T); L^2((0, L); \mathbb{R}^2)) \) provided that \( |\mathbf{z}^0|_{H^2((0, x_0); \mathbb{R}^2)} \leq \delta \) and \( |x_{x_0} - x_0| \leq \delta \), with \( \delta \) sufficiently small (see for instance [13, Appendix B]).

Let us define a mapping:

\[
\mathcal{F} : C_\nu(\mathbf{z}^0, x_0) \rightarrow L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)) \cap W^{1, \infty}((0, T); H^1((0, x_0); \mathbb{R}^2)) \\
\cap W^{2, \infty}((0, T); L^2((0, x_0); \mathbb{R}^2))
\]

\[
\mathbf{v} = (v_1, v_2)^T \mapsto \mathcal{F}(\mathbf{v}) = \mathbf{z} = (z_1, z_2)^T
\]  
(6.7.12)

where \( \mathbf{z} \) is the solution of the linear hyperbolic equation

\[
\dot{\mathbf{z}}_t + A(\mathbf{v}, x, x_s(\mathbf{v}(\cdot, x_0))(t))\mathbf{z}_x = 0,
\]  
(6.7.13)

\[
\mathbf{z}(0, x) = \mathbf{z}^0(x),
\]  
(6.7.14)

with boundary conditions

\[
z_1(t, 0) = k_1z_1(t, x_0) + b_1\psi(t),
\]  
(6.7.15)

\[
z_2(t, 0) = k_2z_2(t, x_0) + b_2\psi(t),
\]  
(6.7.16)

where

\[
\psi(t) = x_0 - x_s(\mathbf{v}(\cdot, x_0))(t).
\]  
(6.7.17)

In the following, we will treat \( z_1 \) in details. For the sake of simplicity, we denote

\[
f_1(t, x) := a_1(\mathbf{v}(t, x), x, x_s(\mathbf{v}(\cdot, x_0))(t))
\]  
(6.7.18)

It is easy to check from (6.7.10) that if \( \nu \) is sufficiently small, then \( f_1(t, x) \) is strictly positive for any \((t, x) \in [0, T] \times [0, x_0]\). Let us now define the characteristic curve \( \xi_1(s; t, x) \) passing through \((t, x)\) as

\[
\frac{d\xi_1(s; t, x)}{ds} = f_1(s, \xi_1(s; t, x)),
\]  
(6.7.19)

\[
\xi_1(t; t, x) = x.
\]  

One can see that for every \((t, x) \in [0, T] \times [0, x_0]\), \( \xi_1(\cdot; t, x) \) is uniquely defined on some closed interval in \([0, T]\). From (6.7.1), only two cases can occur (see Figure 6.2): If \( \xi_1(t; 0, 0) < x \leq x_0 \), there exists \( \beta_1 \in [0, x_0] \) depending on \((t, x)\) such that

\[
\beta_1 = \xi_1(0; t, x).
\]  
(6.7.20)

If \( 0 < x < \xi_1(t; 0, 0) \), there exists \( \alpha_1 \in [0, t] \) depending on \((t, x)\) such that

\[
\xi_1(\alpha_1; t, x) = 0,
\]  
(6.7.21)

and in this case, there exists \( \gamma_1 \in [0, x_0] \) depending on \( \alpha_1 \) such that

\[
\gamma_1 = \xi_1(0; \alpha_1, x_0).
\]  
(6.7.22)

Moreover, we have the following lemma which will be used in the estimations hereafter (the proof can be found at the end of this appendix).
Lemma 6.7.1. There exist \( \nu_0 > 0 \) and \( C > 0 \) such that, for any \( T \) satisfying (6.7.1), for any \( \nu \in (0, \nu_0] \) and for a.e. \( t \in (0, T) \), we have
\[
|f_1(t, \cdot)|_0 \leq C, \quad |f_{1x}(t, \cdot)|_0 \leq C \nu, \quad |f_{1t}(t, \cdot)|_0 \leq C \nu, \tag{6.7.23}
\]
\[
|\partial_x \xi_1(s, \cdot)|_0 \leq C, \quad |\partial_t \xi_1(s, \cdot)|_0 \leq C, \quad s \in [0, t], \tag{6.7.24}
\]
\[
|\partial_x \beta_1(t, \cdot)|_0 \leq C, \quad |(\partial_x \beta_1(t, \cdot))^{-1}|_0 \leq C, \tag{6.7.25}
\]
\[
|\partial_t \beta_1(t, \cdot)|_0 \leq C, \quad |(\partial_t \beta_1(t, \cdot))^{-1}|_0 \leq C, \tag{6.7.26}
\]
\[
|\partial_x \alpha_1(t, \cdot)|_0 \leq C, \quad |(\partial_x \alpha_1(t, \cdot))^{-1}|_0 \leq C, \tag{6.7.27}
\]
\[
|\partial_x \gamma_1(t, \cdot)|_0 \leq C, \quad |(\partial_x \gamma_1(t, \cdot))^{-1}|_0 \leq C, \tag{6.7.28}
\]
\[
\int_0^T |\partial_t \beta_1(t, x_0)|^2 \, dt \leq C \nu, \tag{6.7.29}
\]
\[
\int_0^T |\partial_{xx} \alpha_1(t, x)|^2 \, dx \leq C \nu, \tag{6.7.30}
\]
\[
\int_0^T |\partial_{xx} \beta_1(t, x)|^2 \, dx \leq C \nu, \tag{6.7.31}
\]
\[
\int_0^T |\partial_{xx} \gamma_1(t, x)|^2 \, dx \leq C \nu. \tag{6.7.32}
\]

In these inequalities, and hereafter in this section, \( |f|_0 \) denotes the \( C^0 \)-norm of a function \( f \) with respect to its variable and \( C \) may depend on \( x_0, x_{x_0}, \nu_0, k_1, k_2, b_1 \) and \( b_2 \), but is independent of \( \nu, T, v \) and \( z \).

Our goal is now to use a fixed point argument to show the existence and uniqueness of the solution to (6.3.3)–(6.3.5). Firstly, we show that for \( \nu \) and \( \delta \) sufficiently small, \( \mathcal{F} \) maps \( C_\nu(z^0, x_{x_0}) \) into itself, i.e.,
\[
\mathcal{F}(C_\nu(z^0, x_{x_0})) \subset C_\nu(z^0, x_{x_0}).
\]

Then, in a second step, we prove that \( \mathcal{F} \) is a contraction mapping.

1) \( \mathcal{F} \) maps \( C_\nu(z^0, x_{x_0}) \) into itself.

For any \( v \in C_\nu(z^0, x_{x_0}) \), let \( z = \mathcal{F}(v) \), we prove that \( z \in C_\nu(z^0, x_{x_0}) \). By the definition of \( \mathcal{F} \) in (6.7.12), using the method of characteristics, we can solve (6.7.13) to (6.7.16) for \( z_1 \) and obtain that
\[
z_1(t, x) = k_1 z_0^0(\gamma_1) + b_1 \psi_0(a_1), \quad 0 < x < \xi_1(t; 0, 0), \quad \frac{z_0^0(\beta_1)}{\gamma_1}(t; 0, 0) < x < x_0. \tag{6.7.33}
\]

Obviously \( z \) verifies the properties (6.7.6)–(6.7.7). Next, we prove that \( z \) verifies the property (6.7.5). Using the change of variables and from (6.7.26), we have
\[
\int_0^T z_1(t, x_0)^2 \, dt = \int_0^T z_1^0(\beta_1(t, x_0))^2 \, dt \leq C \int_0^{x_0} (z_1^0(x))^2 \, dx. \tag{6.7.34}
\]

In (6.7.34) and hereafter, \( C \) denotes various constants that may depend on \( x_0, x_{x_0}, \nu_0, k_1, k_2, b_1 \) and \( b_2 \), but are independent of \( \nu, T, v \) and \( z \). Similarly, by (6.7.26), we obtain
\[
\int_0^T z_{1t}(t, x_0)^2 \, dt = \int_0^T (\frac{z_0^0(\beta_1(t, x_0))}{\gamma_1(\beta_1(t, x_0))})^2 \, dt \leq C \int_0^T (z_{1x}^0(x))^2 \, dx. \tag{6.7.35}
\]

From (6.7.29) and using Sobolev inequality, one has
\[
\int_0^T z_{1tt}(t, x_0)^2 \, dt = \int_0^T (\frac{z_0^0(\beta_1(t, x_0))}{\gamma_1(\beta_1(t, x_0))})^2 + \frac{z_0^0(\beta_1(t, x_0))}{\gamma_1(\beta_1(t, x_0))} \partial_{tt} \beta_1(t, x_0)^2 \, dt
\leq C \int_0^{x_0} (z_{1x}^0(x))^2 \, dx + \frac{z_0^0(\beta_1(t, x_0))}{\gamma_1(\beta_1(t, x_0))} \partial_{tt} \beta_1(t, x_0)^2 \, dt
\leq C \int_0^{x_0} (z_{1x}^0(x))^2 \, dx + 2 \int_0^T (\partial_{tt} \beta_1(t, x_0))^2 \, dt \tag{6.7.36}
\]
Combining (6.7.34)–(6.7.36), we get
\[ |z_1(\cdot, x_0)|_{H^2((0, T), \mathbb{R})} \leq C |z_1^0|_{H^2((0, x_0), \mathbb{R})}, \]  
(6.7.37)

Applying similar estimate to \( z_2 \) gives
\[ |z_2(\cdot, x_0)|_{H^2((0, T), \mathbb{R})} \leq C |z_2^0|_{H^2((0, x_0), \mathbb{R})}. \]  
(6.7.38)

From (6.7.37) and (6.7.38), we can select \( \delta \) sufficiently small such that
\[ |z(\cdot, x_0)|_{H^2((0, T), \mathbb{R}^2)} \leq \nu^2, \]  
(6.7.39)

which shows both the regularity and the boundedness property (6.7.5). We can again use the method of characteristics to prove properties (6.7.2)–(6.7.4). For a.e. \( t \in (0, T) \),
\[ z_{1x}(t, x) = \begin{cases} 0 & \xi(t; 0, 0) < x < x_0, \\ k_1 z_{1x}^0(\gamma_1) \partial_x \gamma_1 + b_1 \psi(\alpha_1) \partial_x \alpha_1 & 0 < x < \xi(t; 0, 0), \end{cases} \]
(6.7.40)

and
\[ z_{1xx}(t, x) = \begin{cases} 0 & \xi(t; 0, 0) < x < x_0, \\ k_1 z_{1xx}^0(\gamma_1) \partial_{xx} \gamma_1 + k_1 z_{1xx}^0(\partial_x \gamma_1)^2 + b_1 \psi(\alpha_1) \partial_x \alpha_1 & 0 < x < \xi(t; 0, 0), \end{cases} \]
(6.7.41)

Note that the last equation is true in distribution sense but shows that \( z_1 \in L^\infty((0, T); H^2((0, x_0); \mathbb{R})) \). We first estimate \( |z|_{L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2))} \). From (6.7.33) and (6.7.17), using Sobolev inequality, we get
\[ |\psi|_0 \leq |x-x_0| + C |v(\cdot, x_0)|_{H^2((0, T), \mathbb{R}^2)}, \]
(6.7.42)

\[ |\tilde{\psi}|_0 \leq C |v(\cdot, x_0)|_{H^2((0, T), \mathbb{R}^2)}, \]
(6.7.43)

\[ |\tilde{\psi}|_0 \leq C |v(\cdot, x_0)|_{H^2((0, T), \mathbb{R}^2)}. \]
(6.7.44)

From (6.7.33), (6.7.40) and (6.7.41), we can compute directly using (6.7.25), (6.7.27)–(6.7.28) and (6.7.30)–(6.7.32) that
\[ \int_0^{x_0} z_{1x}^2 dx \leq (|\partial_x \beta_1(t, \cdot)|^{-1})_0^2 + 2k_1^2 |\partial_x \gamma_1(t, \cdot)|^{-1})_0^2 + z_{1x}^0(x)^2 dx + 2b_1^2 x_0 |\psi|_0^2, \]
(6.7.45)

\[ \leq C (|z_1^0|_{H^2((0, x_0); \mathbb{R})} + |x-x_0| + |v(\cdot, x_0)|_{H^2((0, T), \mathbb{R}^2)}), \]
(6.7.46)

\[ \int_0^{x_0} z_{1xx}^2 dx \leq (|\partial_{xx} \beta_1(t, \cdot)|_0^2 + 4k_1^2 |\partial_{xx} \gamma_1(t, \cdot)|_0^2) + (z_{1xx}^0(x))^2 dx + 2 |z_{1xx}^0|_{H^2((0, T), \mathbb{R}^2)}, \]
(6.7.47)

Combining (6.7.45)–(6.7.47), we obtain
\[ |z_1(t, \cdot)|_{H^2((0, x_0); \mathbb{R})} \leq C (|z_1^0|_{H^2((0, x_0); \mathbb{R})} + |x-x_0| + |v(\cdot, x_0)|_{H^2((0, T), \mathbb{R}^2)}), \]
(6.7.48)

Similarly, one can get
\[ |z_2(t, \cdot)|_{H^2((0, x_0); \mathbb{R})} \leq C (|z_2^0|_{H^2((0, x_0); \mathbb{R})} + |x-x_0| + |v(\cdot, x_0)|_{H^2((0, T), \mathbb{R}^2)}), \]
(6.7.49)
We can clearly perform similar estimates for $z_{\nu''}$ which proves (6.7.2). The same method as to prove (6.7.50) enables us to show that $z_1$ verifies also (6.7.3). One only has to realize that

\[
z_{1t}(t, x) = \begin{cases} k_1 z_{1x}(\gamma_1) \partial_t \gamma_1 + b_1 \dot{\psi}(\alpha_1) \partial_t \alpha_1, & 0 < x < \xi_1(t; 0, 0), \\ z_{1x}(\beta_1) \partial_t \beta_1, & \xi_1(t; 0, 0) < x < x_0. \end{cases}
\]

and to estimate \( \int_0^{\xi_1(t, t, 0)} |\partial_t \alpha_1|^2 dx \), \( \int_0^{\xi_1(t, t, 0)} |\partial_t \beta_1|^2 dx \), \( \int_0^{\xi_1(t, t, 0)} |\partial_t \gamma_1|^2 dx \), \( \int_0^{\xi_1(t, t, 0)} |\partial_t \beta_1|^2 dx \) and \( \int_0^{\xi_1(t, t, 0)} |\partial_t (\gamma_1)|^2 dx \) similarly as in (6.7.30), (6.7.32) using the fact that $v$ belongs to $L^\infty((0, T); H^2((0, x_0); \mathbb{R}^2)) \cap W^{1, \infty}((0, T); H^1((0, x_0); \mathbb{R}^2)) \cap W^{2, \infty}((0, T); L^2((0, x_0); \mathbb{R}^2))$ with bound $\nu$ in these norms. We can clearly perform similar estimates for $z_2$. Consequently there exist $\delta$ and $\nu_1 \in (0, \nu_0]$ sufficiently small depending only on $C$ such that, for any $\nu \in (0, \nu_1]$, $z = F(v)$ verifies properties (6.7.2)-(6.7.7) and therefore $F(C_{\nu}(z^0, x_0)) \subset C_{\nu}(z^0, x_0)$.

2) $F$ is a contraction mapping.

Next, we prove that $F$ is a contraction mapping satisfying the following inequality:

\[
\|F(v) - F(\bar{v})\|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}^2))} + M\|F(v) - F(\bar{v})\|_{L^2((0, T); \mathbb{R}^2)} \leq \frac{1}{2} \|v - \bar{v}\|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}^2))} + \frac{M}{2} \|v - \bar{v}\|_{L^2((0, T); \mathbb{R}^2)},
\]

where $M > 0$ is a constant. We start with $z_1$, and with the estimate of $|z_1 - \bar{z}_1|_{L^\infty((0, T); L^2((0, x_0); \mathbb{R}^2))}$. For any chosen $v$ and $\bar{v}$ from $C_{\nu}(z^0, x_0)$, without loss of generality, we may assume that $\xi_1(t; 0, 0) < \xi_1(t; 0, 0)$, where $\xi_1$ is the characteristic defined in (6.7.19) associated to $v$. From (6.7.33), we have

\[
\int_0^{x_0} |z_1(t, x) - \bar{z}_1(t, x)|^2 dx
\]

\[
= \int_0^{\xi_1(t, 0, 0)} \left|k_1 z_{1x}(\gamma_1) - k_1 z_{1x}(\bar{\gamma}_1) + b_1 \dot{\psi}(\alpha_1) - b_1 \dot{\bar{\psi}}(\bar{\alpha}_1)\right|^2 dx
\]

\[
+ \int_0^{\xi_1(t, 0, 0)} \left|z_{1x}(\beta_1) - (k_1 z_{1x}(\bar{\gamma}_1) + b_1 \dot{\bar{\psi}}(\bar{\alpha}_1))\right|^2 dx + \int_0^{x_0} \left|z_{1x}(\beta_1) - z_{1x}(\bar{\beta}_1)\right|^2 dx.
\]

From the definition of $\psi$ in (6.7.17) and (6.7.8), using Sobolev and Cauchy-Schwarz inequalities, we have

\[
\int_0^{\xi_1(t, 0, 0)} \left|b_1 \dot{\psi}(\alpha_1) - b_1 \dot{\bar{\psi}}(\bar{\alpha}_1)\right|^2 dx
\]

\[
= \int_0^{\xi_1(t, 0, 0)} b_1^2 \left|\frac{\partial_1(s, x_0) + \partial_2(s, x_0)}{2} - \frac{\partial_1(s, x_0) + \partial_2(s, x_0)}{2}\right|^2 ds \leq C|v(\cdot, x_0) - \bar{v}(\cdot, x_0)|^2_{L^2((0, T); \mathbb{R}^2)} + C|v(\cdot, x_0)|^2_{H^2((0, T); \mathbb{R}^2)} \int_0^{\xi_1(t, 0, 0)} |\alpha_1 - \bar{\alpha}_1|^2 dx.
\]
By the definition of $\gamma_1$ in (6.7.22) and the corresponding definition of $\tilde{\gamma}_1$ and using (6.7.24), we obtain
\[
\int_0^{\xi_1(t,0,0)} |k_1 z_1^0(\gamma_1) - k_1 z_1^0(\tilde{\gamma}_1)|^2 dx \leq C |z_1^0|_{H^2((0,x_0),\mathbb{R})}^2 \int_0^{\xi_1(t,0,0)} |\alpha_1 - \tilde{\alpha}_1|^2 dx. \tag{6.7.54}
\]
Combining (6.7.52) and (6.7.54), we get
\[
\int_0^{x_0} |z_1(t,x) - \xi_1(t,x)|^2 dx 
\leq \left( C |z_1^0|_{H^2((0,x_0),\mathbb{R})}^2 + |\tilde{\nabla}(\cdot,x_0)|_{L^2((0,T),\mathbb{R}^2)}^2 \right) \int_0^{\xi_1(t,0,0)} |\alpha_1 - \tilde{\alpha}_1|^2 dx 
+ |z_1^0|_{H^2((0,x_0),\mathbb{R})}^2 \int_0^{x_0} |\beta_1 - \tilde{\beta}_1|^2 dx 
+ \int_0^{\xi_1(t,0,0)} |z_1^0(\beta_1) - k_1 z_1^0(\tilde{\gamma}_1) + b_1 \tilde{\psi}(\tilde{\alpha}_1)|^2 dx 
+ C |\nabla(\cdot,x_0) - \tilde{\nabla}(\cdot,x_0)|_{L^2((0,T),\mathbb{R}^2)}^2. \tag{6.7.55}
\]
We estimate each term in (6.7.55) separately. By the definition of $\beta_1$ in (6.7.20) and the corresponding definition of $\tilde{\beta}_1$, we have
\[
\int_0^{x_0} |\beta_1 - \tilde{\beta}_1|^2 dx = \int_0^{x_0} |\xi_1(0; t,x) - \xi_1(0; 0, t,x)|^2 dx. \tag{6.7.56}
\]
Now, let us estimate $|\xi_1(0; t,x) - \xi_1(0; 0, t,x)|$. From the definition of $x_s$ in (6.7.8) and the definitions of $\xi_1$ and $\xi_1$, see (6.7.19), we get for any $s \in [0,t]$ that
\[
|\xi_1(s; t,x) - \xi_1(s; t,x)| = \left| \int_s^t f_1(\theta, \xi_1(\theta; t,x)) d\theta \right| 
\leq \left| \int_s^t \left( \left( 1 + v_1(\theta, \xi_1) - \xi_1 \right) \frac{v_1(\theta, x_0) + v_2(\theta, x_0)}{2x_0} \right) \frac{x_0}{x(\theta,x_0)(\theta) x_0(\theta,x_0)(\theta)} d\theta \right| 
+ \left| \int_s^t \left| \frac{x_0}{x(\theta,x_0)(\theta)} \right| \left| v_1(\theta, \xi_1) - \tilde{v}_1(\theta, \tilde{\xi}_1) + \tilde{\xi}_1 \frac{\tilde{v}_1(\theta, x_0) + \tilde{v}_2(\theta, x_0)}{2x_0} \right| d\theta \right| 
+ C \left| \nabla(\cdot,x_0) - \tilde{\nabla}(\cdot,x_0) \right|_{L^2((0,T),\mathbb{R}^2)} + C \int_s^t |\xi_1(\theta; t,x) - \xi_1(\theta; 0, t,x)| d\theta.
\tag{6.7.57}
\]
From (6.7.57), we get for $v \in (0,\nu_0)$ sufficiently small and for $\tilde{\xi}_1(t; 0, x) \leq x \leq x_0$ that
\[
|\xi_1(t; x) - \tilde{\xi}_1(t; x)|_{C^0([0,t],\mathbb{R})} \leq C |\nabla(\cdot,x_0) - \tilde{\nabla}(\cdot,x_0)|_{L^2((0,T),\mathbb{R}^2)} + C \int_0^t |v_1(\theta, \xi_1(\theta; t,x)) - \tilde{v}_1(\theta, \tilde{\xi}_1(\theta; t,x))| d\theta. \tag{6.7.58}
\]
Thus, from (6.7.56) and (6.7.58) we have
\[
\int_0^{x_0} |\beta_1 - \tilde{\beta}_1|^2 dx \leq C |\nabla(\cdot,x_0) - \tilde{\nabla}(\cdot,x_0)|_{L^2((0,T),\mathbb{R}^2)}^2 
+ C \int_0^{x_0} \left( \int_0^t |v_1(\theta, \xi_1(\theta; t,x)) - \tilde{v}_1(\theta, \tilde{\xi}_1(\theta; t,x))| d\theta \right)^2 dx 
\leq C |\nabla(\cdot,x_0) - \tilde{\nabla}(\cdot,x_0)|_{L^2((0,T),\mathbb{R}^2)}^2.
\]
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\[ + C \int_0^t \int_{\xi(t,0,0)} v_1(\theta_1(\xi_1; \theta, t, x)) - \bar{v}_1(\theta, \xi_1(\theta; t, x)) \, dx \, d\theta \]
\[ \leq C|v(\cdot, x_0) - \bar{v}(\cdot, x_0)|^2_{L^2((0,T), \mathbb{R}^2)} + C|v_1 - \bar{v}_1|^2_{L^\infty((0,T), L^2((0,x_0), \mathbb{R}))}. \]  
(6.7.59)

The last inequality is obtained using the change of variable \( y = \xi_1(\theta; t, x) \), well-defined for \( 0 \leq \theta \leq t \leq T \) and \( \xi_1(t,0,0) < x < x_0 \). Let us now estimate \( |\alpha_1 - \bar{\alpha}_1|_{L^2((\xi_1(t,0,0),0), \mathbb{R})} \). Without loss of generality, we may assume that \( \alpha_1 \leq \bar{\alpha}_1 \). By definition of \( \alpha_1 \) in (6.7.21) and the corresponding definition of \( \bar{\alpha}_1 \), we have
\[ \int_{\alpha_1}^t f_1(s, \xi_1(s; t, x)) ds = \int_{\alpha_1}^t f_1(s, \bar{\xi}_1(s; t, x)) ds. \]  
(6.7.60)

Hence, similarly to (6.7.57), we get
\[ |\alpha_1 - \bar{\alpha}_1| \leq \frac{1}{\inf_{(t,x) \in [0,T] \times [0,x_0]} f_1(t, x)} \int_{\alpha_1}^t \left| f_1(s, \xi_1(s; t, x)) - f_1(s, \bar{\xi}_1(s; t, x)) \right| ds \]
\[ \leq C|v(\cdot, x_0) - \bar{v}(\cdot, x_0)|^2_{L^2((0,T), \mathbb{R}^2)} + C \int_{\alpha_1}^t |\xi_1(\theta; t, x) - \bar{\xi}_1(\theta; t, x)| d\theta \]
\[ + C \int_{\alpha_1}^t |v_1(\theta_1(\xi_1; \theta, t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta. \]  
(6.7.61)

Similarly to the proof of (6.7.58), for \( \nu \in (0, v_0) \) sufficiently small, we can obtain that (note that \( \xi_1(s; t, x) \) and \( \bar{\xi}_1(s; t, x) \) for any \( s \in [\alpha_1, t] \) are well defined as we assume that \( \alpha_1 \leq \bar{\alpha}_1 \))
\[ |\xi_1(\cdot; t, x) - \bar{\xi}_1(\cdot; t, x)|_{C^0([\alpha_1,t], \mathbb{R})} \leq C|v(\cdot, x_0) - \bar{v}(\cdot, x_0)|^2_{L^2((0,T), \mathbb{R}^2)} \]
\[ + C \int_{\alpha_1}^t |v_1(\theta_1(\xi_1; \theta, t, x)) - \bar{v}_1(\theta, \bar{\xi}_1(\theta; t, x))| d\theta. \]  
(6.7.62)

Using this inequality in (6.7.61) and performing similarly as in (6.7.59), we can obtain
\[ \int_0^{\xi_1(t,0,0)} |\alpha_1 - \bar{\alpha}_1|^2 \, dx \leq C|v(\cdot, x_0) - \bar{v}(\cdot, x_0)|^2_{L^2((0,T), \mathbb{R}^2)} + C|v_1 - \bar{v}_1|^2_{L_{\infty}((0,T), L^2((0,x_0), \mathbb{R}))}. \]  
(6.7.63)

Let us now focus on the estimation of the term \( \int_{\xi_1(t,0,0)}^{\bar{\xi}_1(t,0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\gamma_1) + b_1 \bar{\psi}(\alpha_1))|^2 \, dx \) in (6.7.55). Using the compatibility condition (6.3.7), we have
\[ \int_{\xi_1(t,0,0)}^{\bar{\xi}_1(t,0,0)} |z_1^0(\beta_1) - (k_1 z_1^0(\gamma_1) + b_1 \bar{\psi}(\alpha_1))|^2 \, dx \]
\[ = \int_{\xi_1(t,0,0)}^{\bar{\xi}_1(t,0,0)} |z_1^0(\beta_1) - z_1^0(0) + z_1^0(0) - (k_1 z_1^0(\gamma_1) + b_1 \bar{\psi}(\alpha_1))|^2 \, dx \]
\[ = \int_{\xi_1(t,0,0)}^{\bar{\xi}_1(t,0,0)} |z_1^0(\beta_1) - z_1^0(0) + k_1 z_1^0(x_0) + b_1 (x_0 - x_0) - (k_1 z_1^0(\gamma_1) + b_1 \bar{\psi}(\alpha_1))|^2 \, dx \]
\[ \leq C|z_1^0|^2_{H^2((0,x_0), \mathbb{R})} \int_{\xi_1(t,0,0)}^{\bar{\xi}_1(t,0,0)} |\beta_1|^2 \, dx \]
\[ + C|z_1^0|^2_{H^2((0,x_0), \mathbb{R})} \int_{\xi_1(t,0,0)}^{\bar{\xi}_1(t,0,0)} |x_0 - \bar{\gamma}_1|^2 \, dx \]
\[ + C \int_{\xi_1(t,0,0)}^{\bar{\xi}_1(t,0,0)} \left| \frac{\bar{v}_1(s,x_0) + \bar{\psi}(x_0)}{2} \right|^2 \, dx. \]  
(6.7.64)

We first estimate \( \int_{\xi_1(t,0,0)}^{\bar{\xi}_1(t,0,0)} |\beta_1|^2 \, dx \). As \( \xi_1(s; t, x) \) is increasing with respect to \( s \in [0,t] \), we have
\[ |\beta_1| \leq |\xi_1(\alpha_1; t, x) - \bar{\xi}_1(\alpha_1; t, x)| \leq |\xi_1(\cdot; t, x) - \bar{\xi}_1(\cdot; t, x)|_{C^0([\alpha_1,t], \mathbb{R})}, \]  
(6.7.65)
Finally, using estimations (6.7.66) and performing the same proof as in (6.7.69), we get
\[ \int_{\xi_1(t,0,0)} \left| \beta_1 \right|^2 \, dx \leq C \left| \mathbf{v}(\cdot, x_0) - \mathbf{v}(\cdot, x_0) \right|^2_{L^2((0,T);\mathbb{R}^2)} + C \left| v_1 - \bar{v}_1 \right|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}. \]  
(6.7.66)

Let us now look at the second term in (6.7.64), from (6.7.24) and the definition of $\gamma_1$, we have
\[ \int_{\xi_1(t,0,0)} |x_0 - \gamma_1|^2 \, dx = \int_{\xi_1(t,0,0)} |\xi_1(0; 0, x_0) - \xi_1(0; \alpha_1, x_0)|^2 \, dx \leq |\partial_\xi \xi_1|^2 \int_{\xi_1(t,0,0)} |\alpha_1|^2 \, dx \leq C \int_{\xi_1(t,0,0)} |\alpha_1|^2 \, dx. \]  
(6.7.67)

It is easy to deal with the last term in (6.7.64), one has
\[ \int_{\xi_1(t,0,0)} \left| \int_0^{\xi_1(t,0,0)} \frac{\bar{v}_1(s, x_0) + \bar{v}_2(s, x_0)}{2} \, ds \right|^2 \, dx \leq C \left| \mathbf{v}(\cdot, x_0) \right|^2_{H^2((0,T);\mathbb{R}^2)} \int_{\xi_1(t,0,0)} |\alpha_1|^2 \, dx. \]  
(6.7.68)

Thus, we only have to estimate $\int_{\xi_1(t,0,0)} |\alpha_1|^2 \, dx$. Noticing that for any fixed $(t, x)$, the characteristic $\xi_1(s; t, x)$ is increasing with respect to $s \in [\alpha_1, t]$ and that $\xi_1^{-1}(\cdot; t, x)(\beta_1) = 0$, we obtain
\[ \bar{\alpha}_1 < \xi_1^{-1}(\cdot; t, x)(\beta_1) - \xi_1^{-1}(\cdot; t, x)(\beta_1). \]

Moreover,
\[ \beta_1 = x + \int_t^{\xi_1^{-1}(\cdot; t, x)(\beta_1)} f_1(s; \xi_1(s; t, x)) \, d\theta, \]
\[ \beta_1 = x + \int_t^{\xi_1^{-1}(\cdot; t, x)(\beta_1)} \bar{f}_1(s; \bar{\xi}_1(s; t, x)) \, d\theta. \]

Then similarly as for (6.7.61), we can prove that
\[ \left| \xi_1^{-1}(\cdot; t, x)(\beta_1) - \xi_1^{-1}(\cdot; t, x)(\beta_1) \right| \leq C \left| \mathbf{v}(\cdot, x_0) - \mathbf{v}(\cdot, x_0) \right|_{L^2(0,T)} + C \nu \left| \int_0^t \left| \xi_1(\theta; t, x) - \bar{\xi}_1(\theta; t, x) \right| \, d\theta \right| + C \int_0^t \left| v_1(\theta, \bar{\xi}_1(\theta; t, x)) - v_1(\theta, \xi_1(\theta; t, x)) \right| \, d\theta. \]

Thus, similarly as in the proof for (6.7.63), we get
\[ \int_{\xi_1(t,0,0)} |\alpha_1|^2 \, dx \leq C \left| \mathbf{v}(\cdot, x_0) - \mathbf{v}(\cdot, x_0) \right|^2_{L^2((0,T);\mathbb{R}^2)} + C \left| v_1 - \bar{v}_1 \right|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}. \]  
(6.7.69)

Finally, using estimations (6.7.66) and (6.7.67)–(6.7.69), (6.7.64) becomes
\[ \int_{\xi_1(t,0,0)} \left| \beta_1 \right|^2 \, dx \leq C \left| \left( \bar{\xi}_1(t,0,0) \right)_{H^2((0,x_0);\mathbb{R})} + \left| \mathbf{v}(\cdot, x_0) \right|^2_{H^2((0,T);\mathbb{R}^2)} \right\} \left| \mathbf{v}(\cdot, x_0) - \mathbf{v}(\cdot, x_0) \right|^2_{L^2((0,T);\mathbb{R}^2)} + \left| v_1 - \bar{v}_1 \right|^2_{L^\infty((0,T);L^2((0,x_0);\mathbb{R}))}. \]  
(6.7.70)
The last inequality is obtained by changing the variable \( y = \xi(t; t, x_0) \). Similar estimates can be done for \( z_2 \). Hence, from (6.7.71) and (6.7.72), there exists \( M > 0 \) such that for \( \delta \) sufficiently small and \( \nu \in (0, \nu_2] \), where \( \nu_2 \in (0, \nu_1] \) is sufficiently small and depends only on \( C \), we have

\[
|z - \tilde{z}|_{L^\infty((0,T); L^2((0,x_0); \mathbb{R}))} + M|z(\cdot, x_0) - \tilde{z}(\cdot, x_0)|_{L^2((0,T); \mathbb{R}^2)} \leq \frac{1}{2} |v - \tilde{v}|_{L^\infty((0,T); L^2((0,x_0); \mathbb{R}^2))} + M|v - \tilde{v}|_{L^2((0,T); \mathbb{R}^2)}.
\]

(6.7.73)

Hence \( \mathcal{F} \) is a contraction mapping and has a fixed point \( z \in C_0(\mathbb{R}^n, x_0) \), i.e., there exists a unique solution \( z \in C_0(\mathbb{R}^n, x_0) \) to the system (6.3.3)–(6.3.5). Noticing (6.7.46), we get that \( x_0 \in C[0, T]; \mathbb{R} \). To get the extra regularity \( z \in C^0([0, T]; H^2((0,x_0); \mathbb{R}^2)) \), we adapt the proof given by Majda [150, p.44-46]. There, the author used energy estimates method for an initial value problem. Using this method for our boundary value problem, we have to be careful with the boundary terms when integrating by parts. Substituting \( \psi \) by \( z \) in (6.7.37)–(6.7.38) again, the estimate for the \( v(\cdot, x_0) \) part follows.

Next, we show the uniqueness of the solution in \( C^0([0, T]; H^2((0,x_0); \mathbb{R}^2)) \). Suppose that there is another solution \( \tilde{z} \in C^0([0, T]; H^2((0,x_0); \mathbb{R}^2)) \), we prove that \( \tilde{z} \in C^0(\mathbb{R}^n, x_0) \), for \( \delta \) sufficiently small. To that end, assume that \( z(t, \cdot) = \tilde{z}(t, \cdot) \) for any \( t \in [0, \tau] \) with \( \tau \in [0, T] \). If \( \tau = T \), by (6.3.10), for \( \delta \) sufficiently small and as \( \tilde{z} \in C^0([0, T]; H^2((0,x_0); \mathbb{R}^2)) \), one can choose \( \tau' \in (\tau, T) \) small enough such that \( \tilde{z} \in C_0(\mathbb{R}^n, x_0(\tau, \cdot)) \) with \( T \) is replaced by \( \tau' - \tau \) and by considering \( \tau \) as the new initial time. Thus, \( z(t, \cdot) = \tilde{z}(t, \cdot) \) for any \( t \in [0, \tau'] \).
As $\tilde{z}(t, \cdot)$ is uniformly continuous on $[0, T]$, and as, moreover $C$ and $\nu$ do not depend on $T$, we can repeat this process and finally get $z(t, \cdot) = \tilde{z}(t, \cdot)$ on $[0, T]$.

For general $T > 0$, one just needs to take $T_1$ satisfying (6.7.1) and, noticing that $C$ and $\nu$ do not depend on $T_1$, one can apply the above procedure at most $\lceil T/T_1 \rceil + 1$ times. This concludes the proof of Lemma 6.3.1.

Proof of Lemma 6.7.1. From (6.7.19), we have

$$
\begin{aligned}
\frac{\partial^2 \xi_1(s; t, x)}{\partial s \partial x} &= f_{1x} \frac{\partial \xi_1(s; t, x)}{\partial x}, \\
\frac{\partial \xi_1(t; t, x)}{\partial x} &= 1,
\end{aligned}
$$

(6.7.74)

and

$$
\begin{aligned}
\frac{\partial^2 \xi_1(s; t, x)}{\partial s \partial t} &= f_{1x} \frac{\partial \xi_1(s; t, x)}{\partial t}, \\
\frac{\partial \xi_1(s; s, x)}{\partial s} + \frac{\partial \xi_1(s; s, x)}{\partial t} &= 0.
\end{aligned}
$$

(6.7.75)

Thus,

$$
\begin{aligned}
\frac{\partial \xi_1(s; t, x)}{\partial s} &= e^{- \int_s^t f_{1x}(\xi_1(\theta; t, x))d\theta}, \\
\frac{\partial \xi_1(s; t, x)}{\partial t} &= -f_{1}(t, x)e^{- \int_s^t f_{1x}(\xi_1(\theta; t, x))d\theta}.
\end{aligned}
$$

(6.7.76)

(6.7.77)

From (6.7.76)–(6.7.77) and noticing $\beta_1 = \xi_1(0; t, x)$, we have

$$
\begin{aligned}
\frac{\partial \beta_1}{\partial t} &= -f_{1}(t, x)e^{- \int_0^t f_{1x}(\xi_1(\theta; t, x))d\theta}, \\
\frac{\partial \beta_1}{\partial x} &= e^{- \int_0^t f_{1x}(\xi_1(\theta; t, x))d\theta}.
\end{aligned}
$$

(6.7.78)
Observe that for a.e. \( s \) we have to be careful when we estimate (6.7.29). By (6.7.18) and using estimates (6.7.23), (6.7.24), we get
\[
\int_{0}^{x} f_{1}(s, \xi_{1}(s, t, x)) ds, \quad (6.7.79)
\]
and as \( \gamma_{1} = \xi_{1}(0; \alpha_{1}, x_{0}) \), we obtain from (6.7.77) that
\[
\int_{0}^{x} f_{1}(s, \xi_{1}(s, t, x)) ds - \int_{0}^{x} f_{1}(s, \xi_{1}(s, t, x)) ds. \quad (6.7.80)
\]
Observe that for a.e. \( s \in (0, T) \) and \( x \in [0, x_{0}] \),
\[
|v_{1}(s, x)| \leq \left| \int_{0}^{x} v_{1}(s, l) dl \right| + |v_{1}(s, \theta)|, \quad \forall \theta \in [0, x_{0}] \quad (6.7.81)
\]
and as \( v_{1} \) is \( H^{1} \) in \( x \) and its \( L^{2} \)-norm is bounded by \( \nu \), there exists \( \theta \) such that \( |v_{1}(s, \theta)| \leq \nu / \sqrt{x_{0}} \), therefore
\[
|v_{1}(s, x)| \leq C_{\nu}, \quad (6.7.82)
\]
and similarly as \( v_{1} \) is \( H^{2} \) in \( x \) with the same bound and \( v_{1x} \) is in \( L^{\infty}((0, T); H^{1}((0, x_{0}); \mathbb{R})) \) with bound \( \nu \) from (6.7.3)
\[
x \in [0, x_{0}], \quad |v_{1x}(s, x)| \leq C_{\nu}, \text{ for a.e. } s \in (0, T),
\]
\[
x \in [0, x_{0}], \quad |v_{1}(s, x)| \leq C_{\nu}, \text{ for a.e. } s \in (0, T). \quad (6.7.83)
\]
From the expression of \( f_{1} \) defined in (6.7.18) and using (6.7.76)–(6.7.80), after some direct computations, estimates (6.7.23)–(6.7.28) can be obtained. We now demonstrate the estimate (6.7.29) in details, while (6.7.30)–(6.7.32) can be treated in a similar way, thus we omit them. From (6.7.78), we have
\[
\partial_{tt} \beta_{1} = \left( -f_{1x}(l, x) + f_{1}(l, x) \left( f_{1x}(l, x) + \int_{0}^{x} f_{1xx}(\theta, \xi_{1}(\theta; t, x)) d\theta \right) \right) e^{-\int_{0}^{y} f_{1x}(\theta, \xi_{1}(\theta; t, x)) d\theta}. \quad (6.7.84)
\]
Looking at (6.7.18), as \( v \) is only in \( L^{\infty}((0, T); H^{2}((0, x_{0}); \mathbb{R}^{2})) \cap W^{2, \infty}((0, T); H^{1}((0, x_{0}); \mathbb{R}^{2})) \cap W^{2, \infty}((0, T); L^{2}((0, x_{0}); \mathbb{R}^{2})) \), this equation is expressed a priori formally in the distribution sense. Thus, we have to be careful when we estimate (6.7.29). By (6.7.18) and using estimates (6.7.23), (6.7.24), we get by Cauchy-Schwarz inequality together with the change of variable \( y = \xi_{1}(\theta; t, x) \) that
\[
\int_{0}^{T} |\partial_{tt} \beta_{1}(t, x_{0})|^{2} dt \leq C_{\nu} + C \int_{0}^{T} \left| \int_{0}^{x} f_{1xx}(\theta, \xi_{1}(\theta; t, x)) d\theta \right|^{2} dt
\]
\[
\leq C_{\nu} + C \int_{0}^{T} \int_{0}^{x} v_{1xx}^{2}(\theta, \xi_{1}(\theta; t, x)) d\theta dt
\]
\[
= C_{\nu} + C \int_{0}^{T} \int_{0}^{y_{0}} v_{1xx}^{2}(\theta, \xi_{1}(\theta; t, x)) d\theta dt
\]
\[
\leq C_{\nu} + C \int_{0}^{T} \int_{0}^{y_{0}} v_{1xx}^{2}(\theta, y) dy d\theta
\]
\[
\leq C_{\nu}. \quad (6.7.84)
\]
\[\square\]
6.7.2 Proof of Theorem 6.5.1

First observe that, after the change of variables (6.3.1), (6.3.2), the new equations are

\[
\begin{aligned}
 z_{1t} + \left(f'(1) + (f'(z_1 + 1) - f'(1)) - \frac{\dot{x}_s}{x_s}\right) \frac{z_{1x}}{x_s} &= 0, \\
 z_{2t} + \left(-f'(-1) + (f'(-1) - f'(z_2 - 1)) + \frac{\dot{x}_s}{x_s}\right) \frac{z_{2x}}{L - x_s} &= 0, \\
 \dot{x}_s(t) &= \frac{f'(1)z_1(t, x_0) - f'(-1)z_2(t, x_0)}{2 + (z_1(t, x_0) - z_2(t, x_0))} \\
 &\quad + \frac{(f(z_1(t, x_0) + 1) - f'(1)z_1(t, x_0) - f(1)) - (f(z_2(t, x_0) - 1) - f'(-1)z_2(t, x_0) - f(-1))}{2 + (z_1(t, x_0) - z_2(t, x_0))}
\end{aligned}
\]  

(6.7.85)

and the boundary conditions remain given by (6.3.4). Note that in (6.7.85) the expression of \( \dot{x}_s \) can actually be written as

\[
\dot{x}_s(t) = \frac{f'(1)z_1(t, x_0) - f'(-1)z_2(t, x_0)}{2} + O \left( |z(t, x_0)|^2 \right).
\]  

(6.7.86)

Thus, to prove Theorem 6.5.1 suffices to show Theorem 6.4.1 with (6.7.85) instead of (6.3.3). We still define the Lyapunov function candidate as previously by (6.4.3)–(6.4.9). Then Lemma 6.4.1 and Lemma 6.4.2 remain unchanged. To adapt Lemma 6.4.3 one can check that, when differentiating \( V_1, V_2 \) and \( V_3 \) along the \( C^3 \) solutions of (6.7.85), (6.3.4) with associated initial conditions and noticing that under assumption \( f(-1) = f(1), \) one has \( f'(-1) \leq 0, f'(1) \geq 0 \) from the property of convex function, we obtain as previously (6.4.40), (6.4.42) and (6.4.46) but with \( F'(1)p_1 \) instead of \( p_1 \) and \( F'(-1)p_2 \) instead of \( p_2 \) in the boundary terms and \( \mu \) replaced by \( \mu \min \{f'(1), \mu \} \). Then, from (6.5.3) and dealing with \( V_4 \), we finally get:

\[
\frac{dV_1}{dt} + \frac{dV_4}{dt} \leq - \mu (V_1 + V_4) + \max \left\{ \frac{\theta_1}{\epsilon_1}, \frac{\theta_2}{\epsilon_2} \right\} V_1 \\
+ \left[ \frac{x_0}{x_s} p_1 (k_1^2 - e^{-\frac{\mu a}{\theta_1}}) f'(1) + \frac{\epsilon_1 + \epsilon_2}{8} f'(1)^2 \right] z_1^2(x_0) \\
+ \left[ \frac{x_0}{L - x_s} p_2 (k_2^2 - e^{-\frac{\mu a}{\theta_2}}) |f'(-1)| + \frac{\epsilon_1 + \epsilon_2}{8} |f'(-1)|^2 \right] z_2^2(x_0) \\
+ f'(1) \left[ -2 \frac{x_0}{x_s} p_1 b_1 k_1 - \frac{x_0}{x_s} \tilde{p}_1 (e^{-\frac{\mu a}{\theta_1}} - k_1) + \kappa \right] z_1(x_0)(x_s - x_0) \\
+ |f'(-1)| \left[ -2 \frac{x_0}{L - x_s} p_2 b_2 k_2 - \frac{x_0}{L - x_s} \tilde{p}_2 (e^{-\frac{\mu a}{\theta_2}} - k_2) + \kappa \right] z_2(x_0)(x_s - x_0) \\
+ \left[ \frac{x_0}{x_s} p_1 b_1 f'(1) + \frac{x_0}{L - x_s} p_2 b_2 |f'(-1)| - \frac{x_0}{x_s} \tilde{p}_1 b_1 f'(1) - \frac{x_0}{L - x_s} \tilde{p}_2 b_2 |f'(-1)| + \mu \kappa \right] (x_s - x_0)^2 \\
+ O \left( \|z\|_{H^2} + |x_s - x_0|^3 \right),
\]  

(6.7.87)

and a similar expression for \( V_2 + V_5 \) and \( V_3 + V_6 \) as previously. Thus Lemma 6.4.3 still holds but with \( A \) now defined by

\[
\begin{aligned}
 a_{11} &= p_1 (e^{-\frac{\mu a}{\theta_1}} - k_1^2) f'(1) - \frac{\epsilon_1 + \epsilon_2}{8} f'(1)^2, \\
 a_{13} &= a_{31} = f'(1) p_1 b_1 k_1 + f'(1) \tilde{p}_1 (e^{-\frac{\mu a}{\theta_1}} - k_1) - f'(1) \frac{\kappa}{2}, \\
 a_{22} &= \frac{x_0}{L - x_s} p_2 (e^{-\frac{\mu a}{\theta_2}} - k_2^2) |f'(-1)| - \frac{\epsilon_1 + \epsilon_2}{8} |f'(-1)|^2,
\end{aligned}
\]  

(6.7.88–6.7.90)

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Instead of (6.4.60)–(6.4.64). We can then choose $p_1$, $p_2$, $\bar{p}_1$, $\bar{p}_2$ as previously by (6.4.69)–(6.4.71) and $A$ becomes again diagonal with the expression of its elements given by

\begin{align*}
a_{33} &= \frac{\kappa}{2} f'(1) b_1 e^{-\frac{\mu x_0}{\eta_1}} + \frac{\kappa}{2} f'(-1) b_2 e^{-\frac{\mu x_0}{\eta_2}} - \mu \kappa \tag{6.7.93}
\end{align*}

Instead of (6.4.74), (6.4.77) and (6.4.78) respectively. Then to prove Theorem 6.4.1 with (6.7.85) instead of (6.3.3), we only need to show now that under assumption (6.5.4) there exists $\mu > \gamma$ and $\kappa > 1$ such that $a_{ii} > 0$, $i = 1, 2, 3$ and such that (6.4.16) holds where $\Theta_i$, $i = 1, 2$ are still defined by (6.4.17). But this can be checked exactly as in the proof of Theorem 6.4.1. With condition (6.5.3), one can now check as in Remark 6.2.4 that there always exist parameters $b_i$ and $k_i$ such that conditions (6.5.4) are satisfied.
Chapter 7

Boundary feedback stabilization of hydraulic jumps

This chapter is taken from the following article (also referred to as [19]):


Abstract. In an open channel, a hydraulic jump is an abrupt transition between a torrential (supercritical) flow and a fluvial (subcritical) flow. In this chapter, hydraulic jumps are represented by discontinuous shock solutions of hyperbolic Saint-Venant equations. Using a Lyapunov approach, we prove that we can stabilize the state of the system in $H^2$-norm as well as the hydraulic jump location, with simple feedback boundary controls and an arbitrary decay rate, by appropriately choosing the gains of the feedback boundary controls.

7.1 Introduction and main result

Nonlinear hyperbolic equations are well-known to give rise to discontinuities in finite time that are physically meaningful. Hydraulic jump is one of the most known example. A hydraulic jump is a phenomenon that frequently occurs in open channel flow, such as rivers and spillways. It describes a transition between a torrential (or supercritical) regime and a fluvial (or subcritical) regime, i.e., an abrupt transition between a fast flow and a slow flow with a higher height. As a consequence, a part of the initial kinetic energy of the flow is converted into an increase in potential energy, while some energy is irreversibly lost through turbulence and heat. This phenomenon can be seen not only in rivers and spillways but also in air flows of the atmosphere. This is for instance believed to explain the phenomenon of “Morning Glory cloud” [44] and may be at the origin of some gliders’ crashes [129]. Hydraulic jumps are important not only because they occur naturally but also because they are sometimes engineered on purpose and are very useful in hydraulic applications to dissipate energy in water and prevent in this way the erosion of the streambed or damages on hydraulic installations [104]. However, when studying the flow equations, the stabilization of hydraulic jumps is seldom considered and almost all the studies focus on the stabilization of the dynamics of the fluvial regime [12, 13, 16, 18, 52, 71, 98, 135]. In this chapter, we explicitly address the issue of the stabilization of a hydraulic jump represented by a discontinuous shock solution of the flow equations, switching from the torrential regime to the fluvial regime. In other words, the two eigenvalues of the hyperbolic system modeling the shallow water are both positive in the torrential regime and one of them changes sign and switches to a negative value in the fluvial regime. Our goal is to achieve the stability of the channel with a general class of local feedback controls at the boundary. Fundamentally, the stabilization of shock steady states for hyperbolic systems, while being very interesting, has rarely been studied. One can refer to [20] and [103] for the scalar case and to our knowledge, no such result exists for systems. By a Lyapunov approach we prove the exponential $H^2$-stability of the steady state, with an arbitrary decay rate and with an exact exponential
stabilization of the desired location of the hydraulic jump.

We consider a channel with a rectangular cross section with constant width, which is taken to be 1 without loss of generality. We denote by \( Q(t, x) \) the flux and \( H(t, x) \) the water depth, where \( t \) and \( x \) are, respectively, the time and space independent variables as usual. As the channel has a finite length \( L > 0 \), the spatial domain is bounded and noted \([0, L]\). The Saint-Venant model which, neglecting friction, consists in a continuity equation and an equilibrium of forces, is written as

\[
\begin{align*}
\partial_t H + \partial_x Q &= 0, \\
\partial_t Q + \partial_x \left( \frac{g H^2}{2} + \frac{Q^2}{H} \right) &= 0.
\end{align*}
\] (7.1.1)

We are interested with solution trajectories \((H(t, x), Q(t, x))^T\) that may have a jump discontinuity at some point \( x_s(t) \in (0, L) \) and are classical otherwise. Thus, in order to close the system, we need a relationship between \( Q \) and \( H \) before and after this jump. From the Rankine-Hugoniot condition applied to (7.1.1), two quantities are conserved through the jump in the jump’s referential: the flux \( Q \) and the momentum \( g H^2/2 + Q^2/H \). This gives the following relationships at the jump \( x_s(t) \):

\[
\begin{align*}
[Q]_+^e &= \dot{x}_s[H]_+^e, \\
[Q]_+^e \dot{x}_s &= \left[ \frac{Q^2}{H} + \frac{1}{2} g H^2 \right]_+^e.
\end{align*}
\] (7.1.2)

where, as usual, \( \dot{x}_s \) denotes the time derivative of \( x_s \), i.e., the speed of the jump. These relationships can be reformulated as:

\[
\dot{x}_s = \left[ \frac{Q_+^e}{[H]_+^e} \right],
\] (7.1.3)

and 

\[
([Q]_+^e)^2 = \left[ \frac{Q^2}{H} + \frac{1}{2} g H^2 \right]_+^e
\] (7.1.4)

where we define for any bounded function \( f \) in a neighbourhood of \( x_s \):

\[
[f]_+^e = f(x_+^e(t)) - f(x_-^e(t)).
\]

This relation (7.1.4) can be regarded as the generalisation for non-stationary states of the well-known Bélanger equation (7.1.8) below.

Our goal is to stabilize the steady states of the system (7.1.1), (7.1.3) and (7.1.4) where a (single) hydraulic jump occurs, meaning that the flow switches from the torrential regime to the fluvial regime with a discontinuity in height. Therefore, such steady states \((H^*, Q^*)^T, x_s^*)\) satisfy the following conditions:

1. \( Q^* \) is constant and positive, \( x_1^* \in (0, L) \) and

\[
H^* = \begin{cases} 
H_1^*, & x \in [0, x_1^*), \\
H_2^*, & x \in (x_1^*, L],
\end{cases}
\] (7.1.5)

where \( H_1^*, H_2^* \) are positive constants.

2. The steady state flow is in the torrential regime before the jump and in the fluvial regime after the jump. This means that in the torrential regime the two system eigenvalues are positive,

\[
\lambda_1 = \frac{Q^*}{H_1^*} - \sqrt{g H_1^*} > 0, \quad \lambda_2 = \frac{Q^*}{H_1^*} + \sqrt{g H_1^*} > 0, \quad \text{for } x \in [0, x_1^*),
\] (7.1.6)

while there is one positive and one negative eigenvalue in the fluvial regime \([13] \),

\[
-\lambda_3 = \frac{Q^*}{H_2^*} - \sqrt{g H_2^*} < 0, \quad \lambda_4 = \frac{Q^*}{H_2^*} + \sqrt{g H_2^*} > 0, \quad \text{for } x \in (x_1^*, L].
\] (7.1.7)

In particular this implies that \( H_1^* < H_2^* \).
3. Furthermore, the Rankine-Hugoniot conditions applied to (7.1.1) in the stationary case are equivalent to the following well-known Bélainger equation [42]

$$\frac{H_2}{H_1} = -1 + \sqrt{1 + \frac{8(Q^*)^2}{g(H_1^*)^2}}.$$  

(7.1.8)

**Physical remarks:**

- The switch from the torrential regime to the fluvial regime corresponds to a transition (shock) between a state where the system (7.1.1) has two positive eigenvalues and a state where the system has one positive and one negative eigenvalue. As we will see later (from Theorem 7.1.1 together with (7.1.6) and (7.1.7)), this transition (shock) induces a discontinuity not only for the eigenvalue that changes sign but also for the eigenvalue that keeps the same sign. More precisely, if we denote by $\lambda_s$ the eigenvalue that changes sign, then $\lambda_s(x_s^+(t)) > 0 > \lambda_s(x_s^-(t))$ for all $t > 0$. And if we denote by $\lambda_c$ the eigenvalue that does not change sign, then $\lambda_c(x_s^+(t)) \neq \lambda_c(x_s^-(t))$ for all $t > 0$. We point out that smooth transitions could happen around critical equilibria or when source terms are considered (see [58], [102]). Such smooth transitions are also related to coupling conditions for networks for the transition from supersonic to subsonic fluid states, such as natural gas pipeline transportation systems that have been analyzed in [90].

- Note that when the solutions are classical, the formulation (7.1.1) of the Saint-Venant equations with the level $H$ and the flux $Q$ as state variables is equivalent to the alternative formulation with the level $H$ and the velocity $\mathbf{v}$ where $\mathbf{v} = \left(\frac{Q}{H}\right)$ is obtained by replacing the equilibrium of forces by an energy equation and is used for instance in [13, 16, 110]. When the solutions are not classical however, the two formulations are not equivalent anymore and this can be seen by looking at the stationary states: the formulation (7.1.1) in level and flux is compatible with shock and discontinuity of $H^*(x)$ while the version with the energy equation is not. This is logical as there is a pointwise loss of energy in the hydraulic jump, which implies that the energy conservation does not hold anymore.

- From (7.1.3), the location of the shock $x_s$ may be moving around its initial location and potentially all along the channel. This can be seen in practical phenomena such as tidal bores. The main challenge of this work is to also stabilize this location when stabilizing the state of the system. This is not obvious as one can see that for given heights and flux $(H_1^*, H_2^*, Q^*)$ satisfying (7.1.6) and (7.1.8), any shock location $x_s^* \in [0, L]$ induces an admissible steady state $((H^*, Q^*)^T, x_s^*)$, where $H^*$ is given by (7.1.5). Thus the steady states are not isolated and therefore not asymptotically stable in open loop. Indeed, any small perturbation on $x_s^*$ corresponds to another steady state with the same heights and flux at the two ends.

As illustrated in Figure [7.1], let us consider a channel which is equipped with devices allowing a feedback control on $H(t, 0) = H_0(t), Q(t, 0) = Q_L(t)$ and $Q(t, 0) \approx Q_0(t)$ (quasi-steady state approximation). Let the set point for the control be a steady state $((H^*, Q^*)^T, x_s^*)$ defined as previously by (7.1.5)-(7.1.8). We assume that static boundary control laws are selected so that the boundary conditions can be written in the following general form:

$$
\begin{pmatrix}
H(t, 0) - H_1^* \\
Q(t, 0) - Q^* \\
Q(t, L) - Q^*
\end{pmatrix}
= G
\begin{pmatrix}
Q(t, x_s^-) - Q^* \\
Q(t, x_s^+) - Q^* \\
H(t, x_s^-) - H_1^* + (H_2^* - H_1^*)\frac{x_s - x_s^*}{Q(t, x_s^+)}
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
G_4(H(t, L) - H_2^*)
\end{pmatrix}
$$

(7.1.9)

where $G = (G_1, G_2, G_3)^T : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ and $G_4 : \mathbb{R} \rightarrow \mathbb{R}$ are of class $C^2$ and satisfy

$$
G(0) = 0, \quad G_4(0) = 0, \quad G_4'(0) = -\lambda_4.
$$

(7.1.10)

Obviously, by (7.1.5), the steady state $((H^*, Q^*)^T, x_s^*)$ satisfies the boundary conditions (7.1.9), as $H^*(0) = H_1^*$ and $H^*(L) = H_2^*$. Note that this boundary feedback is quite simple to implement as it only requires a pointwise measure of $H(t, L), x_s(t), H(t, x_s^-), Q(t, x_s^+)$ and $Q(t, x_s^-)$.
In order to state the main stability result of this article, we first introduce the following notations:

\[
D(x, \gamma) = \text{diag} \left( \frac{s_i(1 - s_i \frac{\lambda_i}{\lambda_3})}{b_i} e^{\frac{\gamma x}{\lambda_i}}, i \in \{1, 2, 3\} \right),
\]

\[
\bar{D}(\gamma) = \text{diag} \left( \sum_{j=1}^{3} e^{\frac{\gamma x}{\lambda_j}}, (1 - s_i \frac{\lambda_i}{\lambda_3})^2, i \in \{1, 2, 3\} \right),
\]

\[
K = \begin{pmatrix}
\lambda_2 \lambda_1 & -\lambda_1 & 0 \\
\lambda_2 & -\lambda_1 & 0 \\
0 & 0 & \lambda_3 + \lambda_4
\end{pmatrix} G'(0) \begin{pmatrix}
1 & 1 & 0 \\
\lambda_2 & \lambda_4 & 1 + \frac{\lambda_4}{\lambda_3} \\
\frac{1}{\lambda_3} & \frac{1}{\lambda_2} & 0 \\
0 & 0 & 0
\end{pmatrix}, \tag{7.1.11}
\]

\[
d = \frac{1}{H_1 - H_2},
\]

\[
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix} = \begin{pmatrix}
\lambda_2 \lambda_1 & -\lambda_1 & 0 \\
\lambda_2 & -\lambda_1 & 0 \\
0 & 0 & \lambda_3 + \lambda_4
\end{pmatrix} G'(0) \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

with \( s_1 = s_2 = 1, s_3 = -1, x_1 = x_2 = 1, x_3 = x_s^*/(L - x_s^*) \) and \( x_4 = x_s^*/(x_s^* - L) \).

We consider the following initial condition

\[
H(0, x) = H_0(x), \quad Q(0, x) = Q_0(x), \quad x_s(0) = x_{s,0} \tag{7.1.12}
\]

where \( x_{s,0} \in (0, L) \) and \((H_0(x), Q_0(x))^T \in H^2((0, x_{s,0}); \mathbb{R}^2) \cap H^2((x_{s,0}, L); \mathbb{R}^2) \). We assume that the initial condition satisfies the first order compatibility conditions derived from (7.1.9), (see [13] for a proper definition of the first order compatibility condition which is omitted here for the sake of simplicity).

Now, we give the following definition:

**Definition 7.1.1.** The steady state \(((H^*, Q^*)^T, x_s^*)\) is locally exponentially stable for the \(H^2\)-norm with decay rate \(\gamma\), if there exists \(\delta^* > 0\) and \(C^* > 0\) such that for any initial data \((H_0(x), Q_0(x))^T \in H^2((0, x_{s,0}); \mathbb{R}^2) \cap H^2((x_{s,0}, L); \mathbb{R}^2)\) and \(x_{s,0} \in (0, L)\) satisfying

\[
|H_0 - H_1^*, Q_0 - Q^*)^T|_{H^2((0, x_{s,0}); \mathbb{R}^2)} + |(H_0 - H_2^*, Q_0 - Q^*)^T|_{H^2((x_{s,0}, L); \mathbb{R}^2)} \leq \delta^*, \tag{7.1.13}
\]

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\[ |x_{s,0} - x_s^*| \leq \delta^*, \quad (7.1.14) \]

and the corresponding first order compatibility conditions derived from (7.1.9), and for any \( T > 0 \), the system (7.1.1), (7.1.3), (7.1.4), (7.1.9), and (7.1.12) has a unique solution \( (H, Q) \in C^0([0, T]; H^2((0, x_s(t)); \mathbb{R}^2) \cap H^4((x_s(t), L); \mathbb{R}^2)) \) and \( x_s \in C^1([0, T]) \) and

\[
|\langle (H(t, \cdot) - H^*_s, Q(t, \cdot) - Q^*)^T|_{H^2((0, x_s(t)); \mathbb{R}^2)} + |(H(t, \cdot) - H^*_s, Q(t, \cdot) - Q^*)^T|_{H^2((x_s(t), L), \mathbb{R}^2)} + |x_s(t) - x^*_s| \leq C e^{-\gamma t} \left( \left| (H_0 - H^*_s, Q_0 - Q^*)^T|_{H^2((0, x_{s,a}); \mathbb{R}^2)} + \right| x_{s,0} - x^*_s \right) \right), \quad \forall t \in [0, T).
\]

**Remark 7.1.1.** A function \( f \) in \( C^0([0, T]; H^2((0, x_s(t)); \mathbb{R}^2) \cap H^4((x_s(t), L); \mathbb{R}^2)) \) is a function in \( C^0([0, T]; L^2((0, L); \mathbb{R}^2)) \) such that, if one defines

\[
f_1(t, x) := f(t, x_s(t)x), \quad t \in (0, T), \quad x \in (0, 1), \quad (7.1.16)
\]

\[
f_2(t, x) := f(t, L + (x_s(t) - L)x), \quad t \in (0, T), \quad x \in (0, 1), \quad (7.1.17)
\]

then \( f_1 \) and \( f_2 \) are both in \( C^0([0, T]; H^2((0, 1); \mathbb{R}^2)) \). The transformation \( f \rightarrow (f_1, f_2) \) enables us to reduce the problem to a time-invariant domain and to define the stability of a function \( f \in C^0([0, T]; H^2((0, x_s(t)); \mathbb{R}^2) \cap H^4((x_s(t), L); \mathbb{R}^2)) \), a function that is piecewise \( H^2 \) with a discontinuity that is potentially moving. This transformation will also be used later on in the analysis of the problem (see (7.2.4) below).

Based on Definition 7.1.1, we have the following theorem.

**Theorem 7.1.1.** For any given steady state \( (H^*, Q^*)^T, x^*_s) \) of the system (7.1.1) satisfying (7.1.5) - (7.1.8) and the boundary conditions (7.1.9), for any \( \gamma > 0 \), if for \( i = 1, 2, 3 \)

\[
b_i \in \begin{cases} \frac{-\gamma e^{-\frac{\pi i \gamma}{\lambda_4} x^*_s}}{3ds_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right)}, & \text{if } s_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right) < 0, \\ \frac{-\gamma e^{-\frac{\pi i \gamma}{\lambda_4} x^*_s}}{3ds_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right)}, & \text{if } s_i \left(1 - s_i \frac{\lambda_i}{\lambda_4}\right) > 0, \end{cases} \quad (7.1.18)
\]

and if the matrix

\[
D(x^*_s, \gamma) - K^T D(0, \gamma) K - \sum_{k=1}^3 \frac{2d^2}{b_k} b_k (1 - s_k \frac{\lambda_k}{\lambda_4}) (e^{\frac{\pi i \gamma}{\lambda_4}} - 1) \tilde{D}(\gamma) \quad (7.1.19)
\]

is positive definite, with \( (b_1, b_2, b_3)^T \), \( D \), \( \tilde{D} \) and \( K \) defined in (7.1.11), then the steady state \( ((H^*, Q^*)^T, x^*_s) \) is locally exponentially stable for the \( H^2 \)-norm with decay rate \( \gamma/4 \).

**Remark 7.1.2.** Note that it is not obvious that there always exists \( G \) such that \( K \) and \((b_1, b_2, b_3)^T \) defined in (7.1.11) satisfy (7.1.18) - (7.1.19). We will prove in details that such \( G \) indeed exists in Appendix 7.5

### 7.2 Well-posedness of the system

In this section, we prove the well-posedness of the Saint-Venant equations (7.1.1) with the hydraulic jump conditions (7.1.3) and (7.1.4), the boundary feedback control conditions (7.1.9) and initial condition (7.1.12). We have the following well-posedness theorem.
Theorem 7.2.1. For any $T > 0$, there exists $\delta(T) > 0$ such that, for any given initial condition $(7.1.12)$ satisfying the first order compatibility conditions and

$$
|(H_0 - H_1^*, Q_0 - Q^*)^T|_{H^2((0,x_s,t),\mathbb{R}^2)} + |(H_0 - H_2^*, Q_0 - Q^*)^T|_{H^2((x_s,t),\mathbb{R}^2)} \leq \delta(T),
$$

(7.2.1)

$$
|x_{s,0} - x_s^*| \leq \delta(T),
$$

(7.2.2)

the system $(7.1.1)$, $(7.1.3)$, $(7.1.4)$, $(7.1.9)$ and $(7.1.12)$ has a unique solution $(H,Q)^T \in C^0([0,T]; H^2((0,x_s,t),\mathbb{R}^2) \cap H^2((x_s,t),\mathbb{R}^2))$ and $x_s \in C^1([0,T])$. Moreover, the following estimate holds for any $t \in [0,T]$

$$
|(H(t,\cdot) - H_1^*, Q(t,\cdot) - Q^*)^T|_{H^2((0,x_s(t)),\mathbb{R}^2)}
$$

$$+ |(H(t,\cdot) - H_2^*, Q(t,\cdot) - Q^*)^T|_{H^2((x_s(t),t),\mathbb{R}^2)} + |x_s(t) - x_s^*|
$$

$$
\leq C(T)\left(|(H_0 - H_1^*, Q_0 - Q^*)^T|_{H^2((0,x_s,0),\mathbb{R}^2)}
$$

$$+ |(H_0 - H_2^*, Q_0 - Q^*)^T|_{H^2((x_s,0,t),\mathbb{R}^2)} + |x_{s,0} - x_s^*|\right).
$$

(7.2.3)

Proof. One can see that the shock location $x_s$ depends on $t$ in general. In order to avoid the time-varying domains $[0,x_s(t)]$ and $[x_s(t),L]$, under the assumption that $x_s \in C^0([0,T])$, we perform, as in $(7.2.10)$, a transformation of the space coordinate $x$ which allows to define new state variables on the fixed domain $[0,x_s^*]$ as follows:

$$
H_1(t,x) = H(t,x,x_s/x_s^*),
$$

$$
Q_1(t,x) = Q(t,x,x_s/x_s^*),
$$

$$
H_2(t,x) = H(t,L + x - x_s/L-x_s^*),
$$

$$
Q_2(t,x) = Q(t,L + x - x_s/L-x_s^*).
$$

(7.2.4)

Let us denote by $h_i$ and $q_i$ the deviations

$$
h_i = H_i - H_1^*, \quad q_i = Q_i - Q^*, \quad i = 1,2.
$$

(7.2.5)

Then, the system $(7.1.1)$, $(7.1.3)$ and $(7.1.4)$ is equivalent to the following $4 \times 4$ system, which is diagonalisable by blocks and defined on $\mathbb{R}^7 \times [0,x_s^*]$:

$$
\partial_r h_1 - \left(\frac{x_s}{x_s^*}\right) \frac{x_s^*}{x_s} \partial_r h_1 + \frac{x_s^*}{x_s} \partial_r q_1 = 0,
$$

$$
\partial_r q_1 + \frac{2(q_1 + Q^*)}{h_1 + H_1^*} \frac{x_s}{x_s^*} \partial_r q_1 + \left(g(h_1 + H_1^*) - \frac{(q_1 + Q^*)^2}{(h_1 + H_1^*)^2}\right) \frac{x_s^*}{x_s} \partial_r h_1 = 0,
$$

$$
\partial_r h_2 + \left(\frac{x_s}{x_s^*}\right) \frac{x_s^*}{L-x_s} \partial_r h_2 - \frac{x_s^*}{L-x_s} \partial_r q_2 = 0,
$$

$$
\partial_r q_2 - \frac{2(q_2 + Q^*)}{h_2 + H_2^*} \frac{x_s}{x_s^*} \partial_r q_2 - \left(g(h_2 + H_2^*) - \frac{(q_2 + Q^*)^2}{(h_2 + H_2^*)^2}\right) \frac{x_s^*}{L-x_s} \partial_r h_2 = 0,
$$

(7.2.6)

where

$$
\dot{x}_s = \frac{q_2(t,x_s^*) - q_1(t,x_s^*)}{h_2(t,x_s^*) - h_1(t,x_s^*)} + H_2^* - H_1^*
$$

(7.2.7)

and with, from the jump condition $(7.1.4)$, the following boundary condition at $x = x_s^*$:

$$
(q_2 - q_1)^2 = (h_2 - h_1 + H_2^* - H_1^*) \left(\frac{(q_2 + Q^*)^2}{h_2 + H_2^*} + \frac{g}{2}(h_2 + H_2^*)^2 - \frac{(q_1 + Q^*)^2}{h_1 + H_1^*} + \frac{g}{2}(h_1 + H_1^*)^2\right).
$$

(7.2.8)
Now, we introduce the following Riemann coordinates

\[
u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} h_1 \\ q_1 \\ h_2 \\ q_2 \end{pmatrix}
\] (7.2.9)

with

\[
S_1 = \begin{pmatrix} \frac{1}{\lambda_1} & \frac{1}{\lambda_2} \\ 1 & 1 \end{pmatrix}^{-1}, \quad S_2 = \begin{pmatrix} -\frac{1}{\lambda_3} & \frac{1}{\lambda_4} \\ 1 & 1 \end{pmatrix}^{-1}
\] (7.2.10)

and \(\lambda_i\) defined in (7.1.6), (7.1.7). Then the system (7.2.6) can be rewritten as

\[
u_s + (A(x_s) + A(u, x_s) + \hat{x}_s B(x_s))\nu_s = 0,
\] (7.2.11)

where

\[
A = \begin{pmatrix} \frac{x_s}{x} & 0 & 0 & 0 \\ 0 & \frac{x_s}{x} & 0 & 0 \\ 0 & 0 & \frac{x_s}{x} & 0 \\ 0 & 0 & 0 & -\frac{x_s}{x} \lambda_4 \end{pmatrix}
\] (7.2.12)

and where \(A, B\) are two matrices of class \(C^2\) that can be obtained by direct computations (omitted here for simplicity) and such that \(A\) satisfies \(A(0, x_s) = 0\). Using the change of coordinates (7.2.9), equation (7.2.7) becomes:

\[
\dot{x}_s = \frac{u_1(t, x_s^*) + u_2(t, x_s^*) - u_3(t, x_s^*) - u_4(t, x_s^*)}{\sum_{i=1}^{3} \frac{u_i(t, x_s^*)}{\lambda_i}} - \frac{u_4(t, x_s^*)}{\lambda_4} + (H_1^s - H_2^s)
\] (7.2.13)

and the boundary condition (7.2.8) becomes:

\[
\frac{2Q^*}{H_2^s}(u_3 + u_4) - \frac{2Q^*}{H_1^s}(u_1 + u_2) + (gH_2^s - \frac{Q^*}{H_2^s})(u_4 - \frac{u_3}{\lambda_3}) - (gH_1^s - \frac{Q^*}{H_1^s})(\frac{u_1}{\lambda_1} + \frac{u_2}{\lambda_2}) = O(|u(t, x_s^*)|^2).
\] (7.2.14)

Here and hereafter, \(O(s)\) (with \(s \geq 0\)) means that for any \(\varepsilon > 0\), there exists \(C_1 > 0\) such that

\[(s \leq \varepsilon) \implies (|O(s)| \leq C_1 s).
\]

With the expression of the eigenvalues given by (7.1.6) and (7.1.7), (7.2.14) becomes

\[
\lambda_4 u_4(t, x_s^*) = \lambda_1 u_1(t, x_s^*) + \lambda_2 u_2(t, x_s^*) + \lambda_3 u_3(t, x_s^*) + O(|u(t, x_s^*)|^2).
\] (7.2.15)

Using (7.2.4), (7.2.5), (7.2.9) and (7.2.15), the boundary conditions (7.1.9) now become

\[
\begin{pmatrix} u_1(t, 0) \\ u_2(t, 0) \\ u_3(t, 0) \end{pmatrix} = B \begin{pmatrix} u_1(t, x_s^*) \\ u_2(t, x_s^*) \\ u_3(t, x_s^*) \end{pmatrix},
\] (7.2.16)

where \(B = (B_1, B_2, B_3)^T : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3\) is of class \(C^2\) and where \(B_1\) and \(B_2\) are defined by

\[
B_1 = (\lambda_2 G_1(u(t, x_s^*), x_s) - G_2(u(t, x_s^*), x_s)) \frac{\lambda_1}{\lambda_2 - \lambda_1},
\] (7.2.17)

\[
B_2 = (\lambda_1 G_1(u(t, x_s^*), x_s) - G_2(u(t, x_s^*), x_s)) \frac{\lambda_2}{\lambda_1 - \lambda_2}.
\] (7.2.18)
To define $B_3$, from the boundary conditions (7.1.9) and the change of variables (7.2.5), (7.2.9), we have
\[ u_3(t, 0) = \frac{-\lambda_4 \lambda_3}{\lambda_3 + \lambda_4} \left( \frac{u_4(t, 0)}{\lambda_4} - \frac{u_5(t, 0)}{\lambda_3} \right) + \frac{\lambda_3}{\lambda_3 + \lambda_4} G_3(u(t, x^*_s), x_s) - \frac{\lambda_3}{\lambda_3 + \lambda_4} G_4 \left( \frac{u_4(t, 0)}{\lambda_4} - \frac{u_5(t, 0)}{\lambda_3} \right). \] (7.2.19)

From condition (7.1.10), applying the implicit function theorem, one obtains
\[ B_3 = F(u_4(t, 0), G_3(u(t, x^*_s), x_s)) \] (7.2.20)
in a neighborhood of $u = 0$ with
\[ F(0, 0) = 0, \quad \partial_1 F(0, 0) = 0, \quad \partial_2 F(0, 0) = \frac{\lambda_3}{\lambda_3 + \lambda_4}, \] (7.2.21)
where $\partial_i F$, $i = 1, 2$, denote the partial derivative of $F$ with respect to its $i$-th variable.

**Remark 7.2.1.** For simplicity, in (7.2.17)-(7.2.20), we have used the following slight abuse of notation adapted from (7.1.9):
\[ G_i(u(t, x^*_s), x_s) = \begin{pmatrix} u_1(t, x^*_s) + u_2(t, x^*_s) \\ u_3(t, x^*_s) + u_4(t, x^*_s) \\ \frac{u_1(t, x^*_s) + u_2(t, x^*_s)}{\lambda_1} \end{pmatrix}, \quad i = 1, 2, 3. \] (7.2.22)

From (7.2.4), (7.2.5), (7.2.9) and (7.2.15), one can see that, as expressed in (7.2.16), $B$ only depends on $u_i(t, x^*_s)$, $i = 1, 2, 3$, $u_4(t, 0)$ and $x_s - x^*_s$ because from (7.2.15) $u_4(t, x^*_s)$ can be considered as a function of $u_i(t, x^*_s)$, $i = 1, 2, 3$.

The initial condition (7.1.12) becomes
\[ u(0, x) = u_0(x) = (u_{10}(x), u_{20}(x), u_{30}(x), u_{40}(x))^T, \] \[ x_s(0) = x_{s,0} \] (7.2.23)
that satisfies the first order compatibility conditions corresponding to (7.2.16). Thus, to study the well-posedness of (7.1.1), (7.1.3), (7.1.4), (7.1.9) and (7.1.12) is equivalent to study the well-posedness of (7.2.11), (7.2.13), (7.2.15)-(7.2.23).

**Lemma 7.2.1.** For any $T > 0$, there exists $\delta(T) > 0$ such that, for any $x, s, 0 \in (0, L)$ and $u_0 \in H^2((0, x^*_s); \mathbb{R}^4)$ satisfying the first order compatibility conditions and
\[ |u_0|_{H^2((0,x^*_s);\mathbb{R}^4)} \leq \delta(T) \] \[ |x_s,0 - x^*_s| \leq \delta(T), \] (7.2.24)
the system (7.2.11), (7.2.13), (7.2.15)-(7.2.23) has a unique solution $u \in C^0([0,T]; H^2((0,x^*_s);\mathbb{R}^4))$ and $x_s \in C^1([0,T])$. Moreover, the following estimate holds for any $t \in [0,T]$ \[ |u(t, \cdot)|_{H^2((0,x^*_s);\mathbb{R}^4)} + |x_s(t) - x^*_s| \leq C(T)(|u_0|_{H^2((0,x^*_s);\mathbb{R}^4)} + |x_{s,0} - x^*_s|). \] (7.2.25)

**Remark 7.2.2.** For the proof of Lemma 7.2.1 we refer to [20, Appendix], where the well-posedness of a $2 \times 2$ nonlinear hyperbolic system coupled with an ODE was studied. But the proof there can be easily adapted to the $4 \times 4$ nonlinear hyperbolic system coupled with an ODE. Noticing that $A(0, x_s) = 0$ and that, from (7.2.13), $x_s = 0$ when $u = 0$, one has
\[ \Lambda(x_s) + A(u, x_s) + x \hat{x}_s B(x_s) = \Lambda(x_s) \]
when $u = 0$. Thus, (7.2.11) is indeed strictly hyperbolic provided that $|u|_{C^0([0,T]; H^2((0,x^*_s);\mathbb{R}^4))}$ is small enough and can be diagonalized in a neighbourhood of $u = 0$. Then we can perform similar fixed point argument as in [20, Appendix] by carefully estimating the related norms of the solution along the characteristic curves. The $C^1$ regularity of $x_s$ is then obtained directly from (7.2.13). We omit the details.

This completes the proof of Theorem 7.2.1.  

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7.3 Exponential stability of the steady state for the $H^2$-norm

In this section we prove Theorem 7.1.1.

Proof of Theorem 7.1.1. It is worth noticing that due to the equivalence of the system (7.1.1), (7.1.3), (7.1.4), (7.1.9) and the system (7.2.11), (7.2.13), (7.2.15) and (7.2.16), one only needs to prove the exponential stability of the null-steady state of the system (7.2.11), (7.2.13), (7.2.15) and (7.2.16) for the $H^2$-norm.

Motivated by [50], see also [13, Section 4.4], and by [20], we introduce the following Lyapunov function:

$$V(u, x_s) = V_1(u) + V_2(u) + V_3(u) + V_4(u, x_s) + V_5(u, x_s) + V_6(u, x_s),$$

where:

$$V_1(u) = \int_0^T \sum_{i=1}^3 \rho_i e^{-\frac{\mu}{\lambda_i} x_i} u_i^2 dx,$$

$$V_2(u) = \int_0^T \sum_{i=1}^3 \rho_i e^{-\frac{\mu}{\lambda_i} x_i} u_i^2 dx,$$

$$V_3(u) = \int_0^T \sum_{i=1}^3 \rho_i e^{-\frac{\mu}{\lambda_i} x_i} u_i^2 dx,$$

$$V_4(u, x_s) = \int_0^T \sum_{i=1}^3 \rho_i e^{-\frac{\mu}{\lambda_i} x_i} u_i(t) (x_s - x_i^*) + C_0 (x_s - x_i^*)^2,$$

$$V_5(u, x_s) = \int_0^T \sum_{i=1}^3 \rho_i e^{-\frac{\mu}{\lambda_i} x_i} u_i(t) \dot{x}_s dx + C_0 (\dot{x}_s)^2,$$

$$V_6(u, x_s) = \int_0^T \sum_{i=1}^3 \rho_i e^{-\frac{\mu}{\lambda_i} x_i} u_i(t) \dot{x}_s dx + C_0 (\dot{x}_s)^2,$$

where $p_i$ and $C_0$ are positive constants that shall be determined later on, while $\rho'_i$ are constants, not necessarily positive, which will also be determined later on. Besides we impose $x_1 = x_2 = 1$, $x_3 = x_4^*/(L - x_3^*)$ and $x_4 = x_4^*/(x_3^* - L)$. In the following we may denote for simplicity $V_t := V_t(u, x_s)$ and $|u|_{H^2} := |u(t, \cdot)|_{H^2([0, x_s^*]; \mathbb{R}^3)}$ in the computations. Similarly to what is done in [20], from the Cauchy-Schwarz inequality and as $C_0 > 3/2$, it can be shown that the Lyapunov function $V$ considered here is equivalent to $(|u|_{H^2} + |x_s - x_s^*|)^2$ provided that $|u|_{H^2} + |x_s - x_s^*|$ is small enough and that

$$\max_i \left( \frac{\rho_i^2 x_i}{\mu^2 \lambda_i} (1 - e^{-\frac{\mu}{\lambda_i} x_i^*}) \right) < 2. $$

This means that, under condition (7.3.8), there exists $\bar{\rho} > 0$ and $\bar{C}$ such that, for every $T > 0$ and $u \in C^0([0, T]; H^2((0, x_s^*); \mathbb{R}^3))$ and for every $x_s \in C^1([0, T])$ solution of the system (7.2.11), (7.2.13), (7.2.15) and (7.2.16), if $|u|_{H^2} + |x_s - x_s^*| \leq \bar{\rho}$

$$\frac{1}{\bar{C}}(|u|_{H^2} + |x_s - x_s^*|)^2 \leq V(u, x_s) \leq \bar{C}(|u|_{H^2} + |x_s - x_s^*|)^2. $$

This can be proved by direct estimations (see [20] for more details). From the boundary condition (7.2.16), as $B$ is of class $C^2$, we have

$$v(t, 0) = \partial_1 B(0, 0, 0)v(t, x_s^*) + \partial_2 B(0, 0, 0)u_4(t, 0) + \partial_3 B(0, 0, 0)(x_s - x_s^*) + O(|u|_{H^2} + |x_s - x_s^*|)^2, $$

where $v = (u_1, u_2, u_3)^T$ is the vector of the components of $u$ on which the feedback (7.2.16) applies. This notation is practical as it isolates $u_1$, $u_2$ and $u_3$ from $u_4$ on which we have no control and whose boundary condition is imposed by the condition (7.2.15). In (7.3.10), the notation $\partial_1 B$ is the $3 \times 3$ Jacobian matrix of
We observe that now... By differentiating (7.2.11), similarly as (7.3.12), we can obtain noticing (7.2.21), it can be verified that the matrix $K$ in (7.2.16). From (7.2.17)-(7.2.21), one can check that the vector-valued function $\vec{V}$ derivative of $\dot{x}$ be given and let $\dot{x}, 0 \in (0, L)$ and $\dot{u}_0 \in H^2((0, x_s^*); \mathbb{R}^4)$ satisfying the first order compatibility conditions and (7.2.24). Let $u \in C^0([0, T]; H^2((0, x_s^*); \mathbb{R}^4))$ and $x_s \in C^1([0, T])$ be the solution of the system (7.2.11), (7.2.13), (7.2.15)-(7.2.23). Let us start with the case where $u$ is of class $C^3$. Taking the time derivative of $V_1$ along this solution, we obtain
\[ \frac{dV_1}{dt} = -\mu V_1 = -\mu \left[ \sum_{i=1}^{4} p_i x_i \lambda_i e^{-\frac{\mu}{\eta} x_i^2} u_{t_1}^2 \right]_0^{x_s^*} + \mathcal{O} \left( \|u\|_{H^2} + |x_s - x_s^*| \right)^3. \]

By differentiating (7.2.11), similarly as (7.3.12), we can obtain
\[ \frac{dV_2}{dt} = -\mu V_2 = -\mu \left[ \sum_{i=1}^{4} p_i x_i \lambda_i e^{-\frac{\mu}{\eta} x_i^2} u_{t_1}^2 \right]_0^{x_s^*} + \mathcal{O} \left( \|u\|_{H^2} + |x_s - x_s^*| \right)^3. \]

Now, let us deal with the $V_3$ term. To that end, we derive from (7.2.11) that
\[ u_{tt} + \Lambda(x_s) u_{tt} + 2 \dot{x} \Lambda'(x_s) u_x + (\Lambda''(x_s)(\dot{x})^2 + \Lambda'(x_s)) u_x + 2 \dot{x} \Lambda'(x_s) u_x + 2 \dot{x} \Lambda'(x_s) u_x = 0. \]

Thus,
\[ \frac{dV_3}{dt} = -\mu V_3 = -\mu \left[ \sum_{i=1}^{4} p_i x_i \lambda_i e^{-\frac{\mu}{\eta} x_i^2} u_{tt}^2 \right]_0^{x_s^*} + \mathcal{O} \left( \|u\|_{H^2} + |x_s - x_s^*| \right)^3. \]

We observe that now $\dot{x}$ appears. As $\dot{x}$ is proportional to $u_t(x_s^*)$, it can not be bounded by $\|u\|_{H^2}$. However, we can use Young’s inequality to compensate it with the boundary terms. Using (7.3.10), one has
\[ \frac{dV_3}{dt} \leq -\mu V_3 - \sum_{i=1}^{4} \left( \left( p_i x_i \lambda_i e^{-\frac{\mu}{\eta} x_i^2} + \mathcal{O}(\|u\|_{H^2}) \right) u_{tt}^2(x_s^*) - u_{tt}^2(0) \right) + \mathcal{O} \left( \|u\|_{H^2} + |x_s - x_s^*| \right)^3. \]

Differentiating (7.3.5), from (7.2.11), one has
\[ \frac{dV_4}{dt} = (x_s - x_s^*) \int_0^{x_s^*} \sum_{i=1}^{3} \frac{\tilde{p}_i}{\lambda_i} e^{-\frac{\mu}{\eta} x_i^2} u_t(t, x)dx \]
\[ + \dot{x} \int_0^{x_s^*} \sum_{i=1}^{3} \frac{\tilde{p}_i}{\lambda_i} e^{-\frac{\mu}{\eta} x_i^2} u_t(t, x)dx + 2 C_0 \dot{x} (x_s - x_s^*) \]
\[ = (x_s - x_s^*) \int_0^{x_s^*} \sum_{i=1}^{3} \tilde{p}_i e^{-\frac{\mu}{\eta} x_i^2} u_t(t, x)dx \]
\[ + d \left( u_1(x_s^*) + u_2(x_s^*) - u_3(x_s^*) - u_4(x_s^*) \right) \int_0^{x_s^*} \sum_{i=1}^{3} \frac{\tilde{p}_i}{\lambda_i} e^{-\frac{\mu}{\eta} x_i^2} u_t(t, x)dx \]
\[ + 2 d C_0 (x_s - x_s^*) (u_1(x_s^*) + u_2(x_s^*) - u_3(x_s^*) - u_4(x_s^*)) + \mathcal{O} \left( \|u\|_{H^2} + |x_s - x_s^*| \right)^3, \]

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where we recall that \( d = (H_1^* - H_2^*)^{-1} \) is defined in (7.1.11). Thus, integrating by parts and using (7.2.15),

\[
\frac{dV_4}{dt} = -(x_s - x_*) \left[ \sum_{i=1}^{3} x_i p_i e^{-\frac{u_i}{\lambda_i}} u_i(t, x) \right]_0^{x_* - x_s} - \mu (V_4 - C_0(x_s - x_*)^2) \\
+ d \left( u_1(x_*) \left( 1 - \frac{\lambda_1}{\lambda} \right) + u_2(x_*) \left( 1 - \frac{\lambda_2}{\lambda} \right) - u_3(x_*) \left( 1 + \frac{\lambda_3}{\lambda} \right) \right) \\
\left( 2C_0(x_s - x_*) + \int_0^{x_* - x_s} \sum_{i=1}^{3} \frac{p_i}{\lambda_i} e^{-\frac{u_i}{\lambda_i}} u_i(t, x) \, dx \right) + O \left( (|u|_{H^2} + |x_s - x_*|^3)^3 \right). \quad (7.3.18)
\]

Similarly for \( V_5 \), from (7.2.11), one has

\[
\frac{dV_5}{dt} = -\ddot{x}_s \left[ \sum_{i=1}^{3} x_i p_i e^{-\frac{u_i}{\lambda_i}} u_{it}(t, x) \right]_0^{x_* - x_s} - \mu (V_5 - C_0(\ddot{x}_*)^2) \\
+ d \left( \sum_{i=1}^{3} \left( 1 - s_i \frac{\lambda_i}{\lambda} \right) s_i u_{it}(x_*) \right) \left( 2C_0 \ddot{x}_s + \int_0^{x_* - x_s} \sum_{i=1}^{3} \frac{p_i}{\lambda_i} e^{-\frac{u_i}{\lambda_i}} u_{it}(t, x) \, dx \right) \quad (7.3.19)
\]

By (7.3.14), for \( V_6 \), one has

\[
\frac{dV_6}{dt} = -\ddot{x}_s \left[ \sum_{i=1}^{3} x_i p_i e^{-\frac{u_i}{\lambda_i}} u_{itt}(t, x) \right]_0^{x_* - x_s} - \mu (V_6 - C_0(\ddot{x}_*)^2) \\
+ d \left( \sum_{i=1}^{3} \left( 1 - s_i \frac{\lambda_i}{\lambda} \right) s_i u_{itt}(x_*) \right) \left( 2C_0 \ddot{x}_s + \int_0^{x_* - x_s} \sum_{i=1}^{3} \frac{p_i}{\lambda_i} e^{-\frac{u_i}{\lambda_i}} u_{itt}(t, x) \, dx \right) \quad (7.3.20)
\]

Dealing with the \( \ddot{x}_s \) term in (7.3.20) similarly as for \( V_3 \), we have

\[
\frac{dV_6}{dt} = -\ddot{x}_s \sum_{i=1}^{3} \left( (x_i p_i e^{-\frac{u_i}{\lambda_i}} + O (|u|_{H^2}) u_{it}(x_*) - x_i p_i u_{it}(0)) \right) - \mu (V_6 - C_0(\ddot{x}_*)^2) \\
+ d \left( \sum_{i=1}^{3} \left( 1 - s_i \frac{\lambda_i}{\lambda} \right) s_i u_{itt}(x_*) \right) \left( 2C_0 \ddot{x}_s + \int_0^{x_* - x_s} \sum_{i=1}^{3} \frac{p_i}{\lambda_i} e^{-\frac{u_i}{\lambda_i}} u_{itt}(t, x) \, dx \right) \quad (7.3.21)
\]

Note that \( V_2 + V_5 \) has the same structure as \( V_1 + V_4 \) with \( u_i \) and \( x_s - x_* \) being replaced by \( u_{it} \) and \( \ddot{x}_s \) respectively. The same applies for \( V_3 + V_6 \) by replacing \( u_i \) and \( x_s - x_* \) in \( V_1 + V_4 \) with \( u_{itt} \) and \( \ddot{x}_s \) respectively. Hence, we only need to analyze \( V_1 + V_4 \). From (7.3.12) and (7.3.18), recalling that \( s_i = 1 \) if \( i \in \{1, 2\} \) and \( s_3 = -1 \), one has

\[
\frac{d(V_1 + V_4)}{dt} = -\left[ \sum_{i=1}^{3} p_i \ddot{x}_i e^{-\frac{u_i}{\lambda_i}} u_i^2 \right]_0^{x_* - x_s} - \mu (V_1 + V_4) \\
- (x_s - x_*) \left[ \sum_{i=1}^{3} x_i p_i e^{-\frac{u_i}{\lambda_i}} u_i \right]_0^{x_* - x_s} + \mu C_0 (x_s - x_*)^2
\]
Therefore, combining (7.3.23)-(7.3.26), one has

\[ + O \left( \|u\|_{H^2} + |x_s - x_s^*|^3 \right). \]  

(7.3.22)

Using now the boundary conditions (7.3.10) and (7.2.15), (7.3.22) becomes

\[ d(V_1 + V_4) \frac{dt}{dt} = -\mu(V_1 + V_4) \]

\[ - v(x_s^*)^T (F(x_s^*, \mu) - K^T F(0, \mu) K) v(x_s^*) - \frac{\lambda_4 p_4}{\lambda_4} e^{-\frac{\pi}{\sqrt{\lambda_4}}} \lambda_1 u_1(x_s^*)^2 \]

\[ - \lambda_4 |x_d| p_2 u_2(0) + \sum_{i=1}^{3} x_i p_1 \lambda_i b_i^2 (x_s - x_s^*)^2 + 2 \sum_{i=1}^{3} x_i p_1 \lambda_i \left( \sum_{j=1}^{3} k_{ij} u_j(x_s^*) (x_s - x_s^*) \right) \]

\[ - \left( \sum_{i=1}^{3} u_i(x_s^*) s_i \right) \left( 1 - s_i \frac{\lambda_i}{\lambda_4} \right) \left( \int_{0}^{x_s^*} \sum_{j=1}^{3} \frac{p_j}{\lambda_j} e^{-\frac{\pi}{\sqrt{\lambda_j}}} u_j(t, x) dx \right) \]

\[ + O \left( \|u\|_{H^2} + |x_s - x_s^*|^3 \right), \]  

(7.3.23)

where

\[ F(x, \mu) = \text{diag} \left( \lambda_i p_i x_i e^{-\frac{\pi}{\sqrt{\lambda_i}}} x_i, i \in \{1, 2, 3\} \right). \]  

(7.3.24)

We observe that, except from the last product proportional to \( d \), a quadratic form in \((v(x_s^*)^T, u_4(0), x_s - x_s^*)\) appears. Using successively the Young and Cauchy-Schwarz inequalities to deal with the last product, and noticing that

\[ \int_{0}^{x_s^*} e^{-\frac{\pi}{\sqrt{\lambda_i}}} dx = \frac{\lambda_i x_i}{\mu} \left( 1 - e^{-\frac{\pi}{\sqrt{\lambda_i}}} x_s^* \right), \]  

(7.3.25)

we get that, for any \( j \in \{1, 2, 3\}, \)

\[ d \left( \sum_{i=1}^{3} u_i(x_s^*) s_i \right) \left( 1 - s_i \frac{\lambda_i}{\lambda_4} \right) \left( \int_{0}^{x_s^*} \frac{p_j}{\lambda_j} e^{-\frac{\pi}{\sqrt{\lambda_j}}} u_j(t, x) dx \right) \leq \left( \frac{\varepsilon_j}{\mu} \right) \left( \frac{p_j^2 x_j (1 - e^{-\frac{\pi}{\sqrt{\lambda_j}}} x_s^*)}{\lambda_j p_j} \right) \left( \int_{0}^{x_s^*} \frac{p_j}{\lambda_j} e^{-\frac{\pi}{\sqrt{\lambda_j}}} u_j(t, x) dx \right) + \frac{d^2}{4 \varepsilon_j} \left( \sum_{i=1}^{3} u_i(x_s^*) \left( 1 - s_i \frac{\lambda_i}{\lambda_4} \right) s_i \right)^2. \]

Using again the Cauchy-Schwarz inequality, we get that

\[ \frac{d^2}{4 \varepsilon_j} \left( \sum_{i=1}^{3} u_i(x_s^*) \left( 1 - s_i \frac{\lambda_i}{\lambda_4} \right) s_i \right)^2 \leq \frac{d^2}{4 \varepsilon_j} \left( \sum_{i=1}^{3} u_i^2(x_s^*) \left( 1 - s_i \frac{\lambda_i}{\lambda_4} \right) \right)^2 \left( \sum_{j=1}^{3} e^{\frac{\pi}{\sqrt{\lambda_j}}} \frac{\mu_j}{\lambda_j} \right) \]  

(7.3.26)

Therefore, combining (7.3.23)-(7.3.26), one has

\[ \frac{d(V_1 + V_4)}{dt} \leq -\mu(V_1 + V_4) - v(x_s^*)^T (F(x_s^*, \mu) - K^T F(0, \mu) K) \]

\[ - \frac{d^2}{4} \sum_{k=1}^{3} \frac{1}{\varepsilon_k} \text{diag} \left( \sum_{j=1}^{3} e^{\frac{\pi}{\sqrt{\lambda_j}}} \frac{\mu_j}{\lambda_j} \right) \left( 1 - s_i \frac{\lambda_i}{\lambda_4} \right)^2 \]  

\[ v(x_s^*) \]

(7.3.27)
\[- \frac{x_4 p_4}{\lambda_4} e^{-\frac{\pi x}{\varepsilon_i x^2}} (\lambda_1 u_1(x_s^*) + \lambda_2 u_2(x_s^*) + \lambda_3 u_3(x_s^*))^2 - \lambda_4 |x_4| p_4 u_4^2(0) \]
\[+ \left( \mu C_0 + \sum_{i=1}^3 (x_i p_i \lambda_i b_i^2 + x_i p_i b_i) \right) (x_s - x_s^*)^2 \]
\[+ \sum_{i=1}^3 \left( \frac{\varepsilon_i}{\mu} \left( \frac{p_i^2 x_i (1 - e^{-\frac{\pi x}{\varepsilon_i x^2}})}{\lambda_i p_i} \right) \left( \int_0^{x_s^*} p_i e^{-\frac{\pi x}{\varepsilon_i x^2}} u_i^2(t, x) dx \right) \right) \]
\[+ \sum_{j=1}^3 \left( 2d C_0 s_j \left( 1 - s_j \frac{\lambda_j}{\lambda_4} \right) - x_j p_j' e^{-\frac{\pi x}{\varepsilon_i x^2}} \right) u_j(x_s^*)(x_s - x_s^*) \]
\[+ O \left( (|u|_{L^2} + |x_s - x_s^*|)^3 \right). \tag{7.3.27} \]

In order to obtain an exponential decay, we first choose \( \varepsilon_i \) such that
\[\frac{1}{\varepsilon_i} = \frac{2 p_i^2 x_i (1 - e^{-\frac{\pi x}{\varepsilon_i x^2}})}{\mu^2 \lambda_i p_i}, \quad i = 1, 2, 3. \tag{7.3.28} \]

Therefore, (7.3.27) becomes
\[\frac{d(V_1 + V_4)}{dt} \leq - \frac{\mu}{2} V_1 - \mu V_4 - \mathbf{v}(x_s^*)^T \left( F(x_s^*, \mu) - K^T F(0, \mu) K - \frac{d^2}{4} \left( \sum_{k=1}^3 \frac{1}{\varepsilon_k} \right) \tilde{D}(\mu) \right) \mathbf{v}(x_s^*) \]
\[\quad - \frac{x_4 p_4}{\lambda_4} e^{-\frac{\pi x}{\varepsilon_i x^2}} (\lambda_1 u_1(x_s^*) + \lambda_2 u_2(x_s^*) + \lambda_3 u_3(x_s^*))^2 - \lambda_4 |x_4| p_4 u_4^2(0) \]
\[\quad + \left( \mu C_0 + \sum_{i=1}^3 (x_i p_i \lambda_i b_i^2 + x_i p_i b_i) \right) (x_s - x_s^*)^2 \]
\[\quad + \sum_{j=1}^3 \left( 2d C_0 s_j \left( 1 - s_j \frac{\lambda_j}{\lambda_4} \right) - x_j p_j' e^{-\frac{\pi x}{\varepsilon_i x^2}} + \sum_{i=1}^3 (2x_i p_i \lambda_i b_i k_{ij} + x_i p_i' k_{ij}) \right) u_j(x_s^*)(x_s - x_s^*) \]
\[\quad + O \left( (|u|_{L^2} + |x_s - x_s^*|)^3 \right). \tag{7.3.29} \]

We clearly see now two terms proportional to \( V_1 \) and \( V_4 \) respectively that will bring the exponential decay, and a quadratic form in \( \mathbf{v}(x_s^*)^T, u_4(0), x_s - x_s^* \) appears. In order to simplify the quadratic form by cancelling the cross terms, we choose
\[p'_i = \frac{d C_0 s_j}{x_i} \left( 1 - s_j \frac{\lambda_j}{\lambda_4} \right) e^{\frac{\pi x}{\varepsilon_i x^2}} x_i^2, \quad i = 1, 2, 3. \tag{7.3.30} \]

Observe that from (7.1.18), one always has \( b_i x_i p'_i < 0 \) for \( i = 1, 2, 3 \), thus we can choose
\[p_i = \frac{p'_i}{2b_i \lambda_i} > 0, \quad i = 1, 2, 3. \tag{7.3.31} \]

Therefore we have, using (7.3.30), (7.3.31) and Young’s inequality
\[\frac{d(V_1 + V_4)}{dt} \leq - \frac{\mu}{2} V_1 - \mu V_4 - \mathbf{v}(x_s^*)^T \left( F(x_s^*, \mu) - K^T F(0, \mu) K - \frac{d^2}{4} \left( \sum_{k=1}^3 \frac{1}{\varepsilon_k} \right) \tilde{D}(\mu) \right) \mathbf{v}(x_s^*) \]
\[\quad - \text{diag} \left( 3|x_4| p_4 u_4^2(0) e^{-\frac{\pi x}{\varepsilon_i x^2}} \right)_{i \in \{1, 2, 3\}} \mathbf{v}(x_s^*) \]
\[\quad - \lambda_4 |x_4| p_4 u_4^2(0) + \left( \mu C_0 - \frac{3}{2} \sum_{i=1}^3 |x_i p'_i| \right) (x_s - x_s^*)^2 \]

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Observe that the conditions (7.1.18) and (7.1.19) are satisfied for $\gamma > 0$, but as the inequalities are strict, there exists $\mu > \gamma$ such that (7.1.18) and (7.1.19) are also verified with $\mu$ instead of $\gamma$. We choose such $\mu$ and using (7.3.30), one can see that
\[
\left( \mu C_0 - \frac{1}{2} \sum_{i=1}^{3} |x_i p'_i b_i| \right) < 0,
\]
and one can also check from (7.2.17)–(7.2.20), (7.3.28), (7.3.30), (7.3.31), conditions (7.1.18) and (7.1.19) that the matrix defined by
\[
F(x^*_s, \mu) - K^T F(0, \mu) K - \frac{d^2}{4} \left( \sum_{k=1}^{3} \frac{1}{x^*_k} \right) \tilde{D}(\mu)
\]
is positive definite. This implies that there exists $p_4 > 0$ such that the quadratic form in $v(x^*_s)$ in (7.3.32) is non-positive. Therefore
\[
\frac{d(V_1 + V_4)}{dt} \leq -\frac{\mu}{2} V_1 - \mu V_4 + O \left( |u|_{H^2} + |x_s - x^*_s| \right)^3.
\]
As $\mu > \gamma$, at least if $|u|_{H^2} + |x_s - x^*_s|$ is small enough which can be guaranteed from Lemma (7.2.1) by requiring $\delta(T)$ small enough
\[
\frac{d(V_1 + V_4)}{dt} \leq -\frac{\gamma}{2} (V_1 + V_4),
\]
thus,
\[
\frac{dV}{dt} \leq -\frac{\gamma}{2} V.
\]
We have derived (7.3.37) under the assumption that the trajectories of (7.2.11), (7.2.13), (7.2.15) and (7.2.16) are of class $C^3$, but one can use a density argument to generalize the result for trajectories in $C^0([0, \bar{T}]; H^2((0, x^*_s); \mathbb{R}^4))$ by noticing that $\gamma$ does not depend on any $C^2$ or $C^3$-norm of $u$. The inequality (7.3.37) is then understood in the distribution sense. One can refer to [20] or [13, Comment 4.6] for more details.

By the equivalence between the Lyapunov function $V$ and $\left( |u|_{H^2} + |x_s - x^*_s| \right)^2$ if this last quantity is small, we get immediately the exponential stability of the null steady state of the system (7.2.11), (7.2.13), (7.2.15) and (7.2.16) for the $H^2$-norm with decay rate $\gamma/4$. It remains to check that under assumption (7.1.19), (7.3.8) holds with $p'_i$ and $p_i$ defined as (7.3.30) and (7.3.31). Indeed,
\[
\max_i \left( \frac{p'_i x_i}{\mu_i p_i} (1 - e^{-\mu_i x_i}) \right) < \frac{4C_0}{3},
\]
therefore there exists $C_0 > 3/2$ such that the condition (7.3.8) is satisfied. So far $\delta(T)$ depends on $T$, we next prove that for any given $\bar{T} > 0$, we can choose $\delta$ independent of $T$ such that (7.3.37) holds on $(0, T)$ as required in Definition (7.1.1).

Let us now assume that $x_{s,0} \in (0, L)$ and $u_0 \in H^2((0, x^*_s); \mathbb{R}^4)$ satisfying the first order compatibility conditions and
\[
|u_0|_{H^2((0, x^*_s); \mathbb{R}^4)} + |x_{s,0} - x^*_s| < \bar{p} \text{ and } V(u_0, x_{s,0}) \leq \nu,
\]
where $\nu > 0$ is going to be chosen small enough. Then, for any $t \in [0, \bar{T}]$, at least if $\nu > 0$ is small enough, from (7.2.25), (7.3.9) and (7.3.37),
\[
|u(t)|_{H^2((0, x^*_s); \mathbb{R}^4)} + |x_{s}(t) - x^*_s| < \tilde{\rho} \text{ and } V(u(t), x_{s}(t)) \leq \nu.
\]
Using (7.3.41) for \( t = \bar{T} \) one can keep going on \([\bar{T}, 2\bar{T}]\) and then on \([2\bar{T}, 3\bar{T}]\), etc. So we get that, for every \( j = 1, 2, 3, \ldots \),

\[
V(u(t), x_s(t)) \leq \nu, \quad t \in [(j - 1)\bar{T}, j\bar{T}],
\]

(7.3.41)

\[
(|u(t)|_{H^2((0, T); \mathbb{R}^3)} + |x_s(t) - x^*_s|) < \bar{\rho}, \quad t \in [(j - 1)\bar{T}, j\bar{T}],
\]

(7.3.42)

\[
\frac{dV}{dt} \leq -\frac{\gamma}{2} V \text{ in the distribution sense on } (0, j\bar{T}).
\]

(7.3.43)

Noticing (7.2.9), there exists a \( \delta^* \) such that if (7.1.13)-(7.1.14) hold, one has (7.3.39). Thus, noticing also that for any \( T > 0 \) there exists \( j \in \mathbb{N} \) such that \((0, T) \subset (0, j\bar{T})\), one gets that the steady state \((H^*, Q^*)^T, x^*_s\) is locally exponentially stable for the \( H^2\)-norm with decay rate \( \gamma/4 \). The proof of Theorem 1.1 is thus complete.

**Remark 7.3.1.** Given the assumptions of Theorem 7.1.1, it is obvious that this stability result is robust with respect to small variations of \( G \) in the feedback control. However, it is actually also robust with respect to small variations of \( G_4 \). Indeed, if \(|G'_4(0) + \lambda_4|\) is sufficiently small but with a bound independent of the state \((H, Q)^T\) and \(x_s\), we can still define \( B \) as in (7.2.17)–(7.2.20) using the implicit function theorem. Then looking at (7.3.10), \( \partial_x B(0, 0, 0) \neq 0 \), but for any \( \delta > 0 \), \( \partial_x B(0, 0, 0) < \delta \) provided \(|G'_4(0) + \lambda_4|\) is sufficiently small. Then all the additional terms about \( u^2_i(0) \) and \( u^2_i(x^*_s) \), \( i = 1, 2, 3 \) will be compensated by the fact that \( p_1 > 0 \) in (7.3.29) and that \(|G'_4(0) + \lambda_4|\) is sufficiently small. The rest of the proof is the same as in the case where \( G'_4(0) = -\lambda_4 \).

### 7.4 Conclusion

In this article, we have considered the problem of the boundary feedback stabilization of an open channel with a hydraulic jump. We focused on the case where the channel has a rectangular cross section without friction or slope. The channel dynamics are modelled by a version of the homogeneous Saint-Venant equations with the water level \( H \) and the flow rate \( Q \) as state variables. The hydraulic jump is represented by a discontinuous shock solution of the system. The main contribution of this chapter is to analyze the boundary feedback stabilization of the system with a general class of static feedback controls that require pointwise measurements of the level and the flux at the boundary and in the immediate vicinity of the hydraulic jump. In order to prove the well-posedness of the system, we first introduce a change of variables which allows to transform the Saint-Venant equations with shock wave solutions into an equivalent \( 4 \times 4 \) quasilinear hyperbolic system which is parametrized by the jump position but has shock-free solutions. Then, by a Lyapunov approach, we show that, for the considered class of boundary feedback controls, the exponential stability in \( H^2\)-norm of the steady state can be achieved with an arbitrary decay rate and with an exponential stabilization of the desired location of the hydraulic jump. Compared with previous results in the literature for classical solutions of quasilinear hyperbolic systems, the \( H^2\)-Lyapunov function introduced in [50] (see also [13, Section 4.4]) has to be augmented with suitable extra terms for the analysis of the stabilization of the jump position. In the case where the cross section is irregular and with friction or slope, the jump stabilization issue is much more challenging and remains an open problem.

### 7.5 Appendix

In this appendix we prove that there always exists \( G \) such that \( K \) and \((b_1, b_2, b_3)^T\) defined in (7.1.11) satisfy (7.1.18)–(7.1.19). Let us first point out that, for every \( K \in \mathbb{R}^{3 \times 3} \), there exists a linear map \( G : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) such that the third equation of (7.1.11) holds. Hence it remains only to show that there always exist \( K \) and \((b_1, b_2, b_3)^T\) satisfying (7.1.18) and (7.1.19). In the special case where \( K = \text{diag}(k_i, i \in \{1, 2, 3\}) \), the condition that the matrix defined in (7.1.19) is positive definite becomes

\[
k_i^2 < e^{-\frac{\gamma}{2\pi}} T^2 D_{t_i}, \quad \forall i \in \{1, 2, 3\},
\]

(7.5.1)
with
\[ D_i := 1 - \frac{2d^2 b_i}{\gamma^2 s_i(1 - s_i \frac{\lambda_i}{\lambda_d})} \left( \sum_{k=1}^{3} b_k s_k (1 - s_k \frac{\lambda_k}{\lambda_d}) (e^{\frac{\gamma^2 s_k}{x_k \lambda_k}} - 1) \right) \left( \sum_{j=1}^{3} e^{\frac{\gamma^2 s_j}{x_j \lambda_j}} \right) (1 - s_i \frac{\lambda_i}{\lambda_d})^2. \] (7.5.2)

Let us look at a limiting case in (7.1.18) and take 
\[ b_i = -\gamma e^{-\gamma s_i/(x_i \lambda_i)} / 3ds_i \left( 1 - s_i \frac{\lambda_i}{\lambda_d} \right). \] Then we have
\[ D_i = 1 - \frac{2}{9} \left( \sum_{k=1}^{3} (1 - e^{\frac{\gamma^2 s_k}{x_k \lambda_k}}) \right) \left( \sum_{j=1}^{3} e^{\frac{-\gamma^2 s_j}{x_j \lambda_j}} \right). \] (7.5.3)

We denote 
\[ y = \left( \sum_{k=1}^{3} e^{\frac{-\gamma^2 s_k}{x_k \lambda_k}} \right). \] Thus we get
\[ D_i := 1 - \frac{2}{3} y + \frac{2}{9} y^2. \] (7.5.4)

This is a second order polynomial with negative discriminant, thus \( D_i \) is always strictly positive. As \( D_i \) depends continuously on \( b_i \), this implies that there exist \( K = \text{diag}(k_i, i \in \{1, 2, 3\}) \) and \( (b_1, b_2, b_3)^T \), satisfying (7.1.18) and (7.1.19).

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Part III

PI Controllers
Chapter 8

PI controllers for 1-D nonlinear transport equation

This chapter is taken from the following article (also referred to as [59]):


Abstract. In this chapter, we introduce a method to obtain necessary and sufficient stability conditions for systems governed by 1-D nonlinear hyperbolic partial-differential equations with closed-loop integral controllers, when the linear frequency analysis cannot be used anymore. We study the stability of a general nonlinear transport equation where the control input and the measured output are both located on the boundaries. The principle of the method is to extract the limiting part of the stability from the solution using a projector on a finite-dimensional space and then use a Lyapunov approach. This chapter improves a result of Trinh, Andrieu and Xu, and gives an optimal condition for the design of the controller. The results are illustrated with numerical simulations where the predicted stable and unstable regions can be clearly identified.

8.1 Introduction

Stabilisation of systems with Proportional-Integral (PI) controllers has been well-studied in more recent decades as it is the most famous boundary control in engineering applications. The use of PI controllers in practical applications goes back to the end of the 18th century with the Perier brothers’ pump regulator [72, Pages 50-51 and figure 231, Plate 26], [25, Chapter 2] and later on with Fleeming Jenkin’s regulator studied by Maxwell in [154]. Of course these regulators were not yet referred as PI control but in practice they worked similarly. Mathematically the PI control was studied first by Minorsky at the beginning of the 20th century for finite-dimensional systems [157]. In the last decades, the stability of 1-D linear systems with PI control has been well-investigated both for finite-dimensional systems [4, 5] and infinite-dimensional systems (see for instance [21, 54, 130, 167, 192, 197, 198] for hyperbolic systems) and is now very well-known. For infinite-dimensional nonlinear systems, however, only few results are known comparatively, most of them conservative [13, Theorem 2.10], [192]. From a mathematical point of view, dealing nonlinear systems is a very challenging and interesting question. From a practical point of view, it can be seen as a necessity as numerous physical systems are based on infinite dimensional nonlinear models that are sometimes linearized afterward. The intuitive belief that the stability condition for a nonlinear system should be the same as the stability condition for its linearized counterpart when close to the equilibrium is wrong in general, as shown for example in [61].

The reason for this gap in knowledge between linear and nonlinear systems in infinite dimension is that the main method to obtain the stability of 1-D linear systems with PI control is the frequency (or spectrum) analysis (e.g. [198]), a powerful tool based on the Spectral Mapping Property which gives, among other
things, the limit of stability from the differential operator’s eigenvalues (e.g. [141, 159, 174]). This powerful tool is not anymore available when dealing with nonlinear systems. Thus, most studies use a Lyapunov approach instead that has the advantage of enabling robust results [47, 119] but as a counterpart is often conservative, meaning that the stability conditions raised are only sufficient and not necessary. Among the necessary and sufficient conditions one can refer for instance to [13, Theorem 2.9]. Another point to mention is that, for nonlinear systems, the exponential stability in the different topologies are not equivalent [61].

In this article we introduce a method to obtain a necessary and sufficient condition on the stability. We study the general scalar transport equation with a PI boundary controller which was studied in [192], and in which the authors obtained a sufficient, although conservative, stability condition.

Not only is this equation interesting in itself [27], as it covers for instance the inviscid Burgers equation around a non-zero constant steady-state, which can be used as a basic model for fluid flows or road traffic, but it is also interesting as, even if it is the most simple nonlinear evolution equation, it already has some of the key features of nonlinear hyperbolic models whose stabilization has been quite studied in the recent years using various methods [13, 106, 114]. This problem has an associated linearized problem where the first eigenvalues making the system unstable are discrete and in finite number. We first extract from the solution of the nonlinear problem the part that would be associated to these eigenvalues in the linear case, using a projector on a finite-dimensional space. In the linearized problem this projected part of the solution is the limiting factor on the stability and it is therefore natural to think that it can also be the limiting factor in the non-linear case. Besides, we know precisely the dynamic of this projection and we can control precisely its decay. Then, a key point is to find a good Lyapunov function for the remaining part of the solution. As the remaining part of the solution is not the limiting factor, the Lyapunov function can be conservative with no harm provided that it gives a sufficient condition that goes beyond the limiting condition corresponding to the projected part.

8.2 Stability of non-linear transport equation with PI boundary condition

We are interested with the following problem

\[
\begin{align*}
\partial_t z + \lambda(z)\partial_x z &= 0, \\
z(0, t) &= -k_I I(t), \\
I(t) &= z(L, t),
\end{align*}
\]

where \( \lambda \) is a \( C^2 \) function with \( \lambda(0) = \lambda_0 > 0 \) and \( k_I \) is a constant. Let \( T > 0 \), one can show that the system is well-posed in \( C^0([0, T], H^2(0, L)) \times C^2([0, T]) \) for initial conditions small enough and sufficiently regular. More precisely one has [192]

**Theorem 8.2.1.** Let \( T > 0 \). There exists \( \delta(T) > 0 \) such that for any \( \phi_0 \in H^2(0, L) \) satisfying \( \|\phi_0\|_{H^2} \leq \delta \), the system (8.2.1)-(8.2.3) with initial condition \((\phi_0, I_0)\) such that

\[
I_0 = -k_I^{-1}\phi_0(0), \quad \phi_0(L) = k_I^{-1}\lambda(\phi_0(0))\phi_0'(0),
\]

has a unique solution \((\phi, I) \in C^0([0, T], H^2(0, L)) \times C^2([0, T])\). Moreover there exists \( C(T) > 0 \) such that

\[
\|\phi(t, \cdot)\|_{H^2} \leq C(T) \|\phi_0(\cdot)\|_{H^2}.
\]

The interest of this system comes from the fact that it is the most simple nonlinear system with a proportional integral control. However it already constitutes a challenge and, to our knowledge, the most advanced result so far is the following result developed in the recent years [192]:

**Theorem 8.2.2.** If \( 0 < k_I < \lambda(0)\Pi(2 - \sqrt{2})/2L \), then the nonlinear system (8.2.1)-(8.2.3) is exponentially stable for the \( H^2 \) norm, where

\[
\Pi(x) = \sqrt{x(2-x)}e^{-x/2}.
\]

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Note that $\Pi(2 - \sqrt{2})/2 \approx 0.34$. In \[192\] it is also shown that this result is conservative. In order to study this system, it is interesting to compare it with the corresponding linear case namely the case where $\lambda$ does not depend on $z$ and (8.2.1) is replaced by

$$\partial_t z + \lambda_0 \partial_x z = 0.$$  (8.2.7)

In this case, a necessary and sufficient condition for the stability can be simply obtained from the frequency analysis, by looking at the eigenvalues of the system (8.2.7), (8.2.2), (8.2.3). It is easy to see that these eigenvalues satisfy the following equation \[192\].

$$k_I + \rho e^{\frac{\pi}{\rho}} = 0.$$  (8.2.8)

This implies from \[24\] that the linear system (8.2.7), (8.2.2), (8.2.3) is exponentially stable if and only if

$$k_I \in \left(0, \frac{\pi \lambda_0}{2L}\right).$$  (8.2.9)

In the nonlinear case, it is not possible anymore to use a frequency analysis method. One has to use other methods, as for instance the Lyapunov method, which is one of the most famous as it guarantees some robustness of the result. This method was for instance used in \[192\] to prove Theorem 8.2.2. However, this method is often conservative as, except in simple cases, it is often difficult to find the right Lyapunov function leading to an optimal condition. As stated in the introduction, we tackle this problem by extracting from the solution the part that limits the stability with a projector and apply our Lyapunov function to the remaining part. Our main result is the following

**Theorem 8.2.3.** The nonlinear system (8.2.1)–(8.2.3) is exponentially stable for the $H^2$ norm if

$$k_I \in \left(0, \frac{\pi \lambda(0)}{2L}\right).$$  (8.2.10)

The sharpness of this nonlinear result is suggested from the linear condition (8.2.9). This sharpness can also be illustrated by the following proposition

**Proposition 8.2.4.** There exists $k_1 > \pi \lambda(0)/2L$, such that for any $k_I \in (\pi \lambda(0)/2L, k_1)$ the nonlinear system (8.2.1)–(8.2.3) is unstable for the $H^2$ norm.

In Section 8.3 we introduce a new Lyapunov function that can be seen as a good Lyapunov function for this system but we show why it still leads to a conservative result. In Section 8.4 we introduce a projector to extract from the solution the limiting part for the stability. In Section 8.5 we prove Theorem 8.2.3 and Proposition 8.2.4 using the Lyapunov function and the projector respectively introduced in Section 8.3 and Section 8.4. In Section 8.6 we illustrate these results with a numerical simulation.

### 8.3 A quadratic Lyapunov function

In this section we first introduce a new Lyapunov function for the system (8.2.1)–(8.2.3). This Lyapunov function can be seen as a good candidate to study the stability for the $H^2$ norm, but, although it already gives a sufficient condition relatively close to the linear condition (8.2.9), we will show that it is not enough to achieve the optimal condition (8.2.10), which will be the motivation for the next section. As this part is only here to motivate the method of this chapter, we will give a sketch of proof for a Lyapunov function equivalent to the $L^2$ norm, but the same would apply for a similar Lyapunov function equivalent to the $H^2$ norm (see Section 8.5).

Let us define $V_0 : L^2(0, L) \times \mathbb{R} \to \mathbb{R}$ by

$$V_0(Z, I) := \int_0^L f(x) e^{-\frac{x}{\alpha}} Z^2(x) dx + \left(\int_0^L \alpha Z dx + \beta I\right)^2,$$  (8.3.1)
where \( f \) is a positive \( C^1 \) function to be determined later on and \( \alpha \) and \( \beta \) are non-zero constants to be determined later on as well. For any \((Z, I) \in L^2(0, L) \times \mathbb{R}\) one has from Cauchy-Schwarz inequality:

\[
\min \left\{ f(x)e^{-\frac{\lambda_0}{2}z^2}: x \in [0, L] \right\} \left( \int_0^L Z dx \right)^2 + \alpha^2 \left( \int_0^L Z dx \right)^2 + 2\beta \alpha I \left( \int_0^L Z dx \right) + (\beta I)^2 \geq V_0(Z, I) \leq C_1 \left( |Z|_{L^2(0, L)}^2 + \beta I^2 \right).
\]  

(8.3.2)

Using that for any \( p > 0 \), there exists \( n_1 \in \mathbb{N}^* \) such that

\[
(p + 1)a^2 + b^2 - 2ab \geq \frac{p}{n_1} (a^2 + b^2), \quad \forall (a, b) \in \mathbb{R}^2,
\]

(8.3.3)

there exists \( C_2 > 0 \) such that

\[
\frac{1}{C_2} \left( |Z|_{L^2(0, L)}^2 + \beta I^2 \right) \leq V_0(Z, I) \leq C_2 \left( |Z|_{L^2(0, L)}^2 + \beta I^2 \right).
\]

(8.3.4)

Thus, our function \( V_0 \) is equivalent to the norm on \( L^2(0, L) \times \mathbb{R} \) defined by \( |(Z, I)| = \left( |Z|_{L^2(0, L)}^2 + \beta I^2 \right) \). It is therefore enough to find \( f \) in \( C^1([0, L], (0, +\infty)) \), \( \alpha \) and \( \beta \) such that \( V_0 \) is exponentially decreasing along all \( C^0([0, T], H^2 \times \mathbb{R}) \) solutions of system (8.2.1)–(8.2.3) to prove that the null steady-state of the system (8.2.1)–(8.2.3) is exponentially stable for the \( L^2 \) norm. Let \( T > 0 \), and let \((z, I)\) be a \( C^1([0, T] \times [0, L]) \times C^0([0, T]) \) solution of the system (8.2.1)–(8.2.3) (we could get the result for \( C^0([0, T], H^2 \times \mathbb{R}) \) later on by density as in [20] Section 4), this will not be done in this section as it is only a sketch proof. Let us denote \( V_0(z(x, \cdot), I(t)) \) by \( V_0(t) \). Differentiating \( V_0 \) with respect to \( t \), using (8.2.1), (8.2.3) and integrating by parts one has

\[
\frac{dV_0}{dt} = - \left[ \lambda(z(t, x))f(x)e^{-\frac{\lambda}{2}z^2(t, x)} \right]_0^L + \int_0^L \lambda(0)f'(x)e^{-\frac{\lambda}{2}z^2}(t, x)dx - \mu \int_0^L f(x)e^{-\frac{\lambda}{2}z^2}(t, x)dx + \mu \int_0^L \frac{\lambda_0 - \lambda(z(t, x))}{\lambda_0} f'(x)e^{-\frac{\lambda}{2}z^2}(t, x)dx
\]

\[
+ \int_0^L f(x)e^{-\frac{\lambda}{2}z^2}(t, x)dx + (\lambda(z(t, x)) - \lambda(0))f'(x)e^{-\frac{\lambda}{2}z^2}(t, x)dx + 2 \left( \int_0^L \alpha zdx + \beta I(t) \right) \left( -\alpha \lambda z_0 + \int_0^L \alpha \frac{\partial \lambda}{\partial z} zdx + \beta z(t, L) \right).
\]

(8.3.5)

Thus using (8.2.2), one has

\[
\frac{dV_0}{dt} = -\lambda(z(t, L))f(L)e^{-\frac{\lambda}{2}z^2} - \mu \int_0^L f(x)e^{-\frac{\lambda}{2}z^2}(t, x)dx + \lambda(z(t, 0))f(0)f'(0)I^2(t)\mu_1^2
\]

\[
- \int_0^L (-\lambda(0)f'(x))e^{-\frac{\lambda}{2}z^2}(t, x)dx + \mu \int_0^L \frac{\lambda_0 - \lambda(z(t, x))}{\lambda_0} f(x)e^{-\frac{\lambda}{2}z^2}(t, x)dx
\]

\[
+ 2 \left( \int_0^L \alpha zdx + \beta I(t) \right) \left( -\alpha \lambda_0 z(t, t) + \beta z(t, L) - \alpha k_1 I(t) \lambda(z(t, 0)) - \alpha \lambda(z(t, L)) - \beta \lambda(z(t, L)) \right)
\]

(8.3.6)

\[
+ \int_0^L f(x) \frac{\partial \lambda}{\partial z} zdx + (\lambda(z(t, x)) - \lambda(0))f'(x)e^{-\frac{\lambda}{2}z^2}(t, x)dx + 2 \left( \int_0^L \alpha zdx + \beta I(t) \right) \int_0^L \alpha \frac{\partial \lambda}{\partial z} zdx.
\]
We can now choose $\beta = \lambda_0 \alpha$. Equation (8.3.6) becomes

$$
\frac{dV_0}{dt} = -\lambda(z(t, L))e^{-\frac{\lambda}{2\mu} t}f(L)z^2(t, L) - \mu \left( \int_0^L f(x)e^{-\frac{\lambda}{2\mu} t}z^2(t, x)dx + I(t)^2 \right) \\
+ (\lambda(0)f(0)k_f^2 + \mu)I^2(t) - \int_0^L (-\lambda(0)f'(x))e^{-\frac{\lambda}{2\mu} t}z^2(t, x)dx \\
- 2 \int_0^L \alpha^2 k_f \lambda(0)zI(t)dx - 2\alpha^2 \lambda(0)k_f I^2(t) \\
- 2 \int_0^L \alpha^2 k_f (\lambda(z(t, 0)) - \lambda(0))zI(t)dx \\
- 2\alpha^2 \lambda(0)(\lambda(z(t, 0))) + \lambda(0))k_f I^2(t) \\
+ (\lambda(z(t, 0)) - \lambda(0))f(0)k_f^2 I^2(t) \\
+ 2 \left( \int_0^L \alpha z dx + \beta I(t) \right) (-\alpha(\lambda(z(t, L)) - \lambda_0)z(t, L)) \\
+ \int_0^L f(x) \frac{\partial \lambda}{\partial z} z e^{-\frac{\lambda}{2\mu} t}z^2 + (\lambda(z) - \lambda(0))f'(x)e^{-\frac{\lambda}{2\mu} t}z^2(t, x)dx \\
+ 2 \left( \int_0^L \alpha z dx + \beta I(t) \right) \int_0^L \alpha \frac{\partial \lambda}{\partial z} z dx.
$$

Using the equivalence between $V_0$ and $\int |z(t, \cdot)|_{L^2}^2 + |I|^2$, there exists a constant $C_3 > 0$, maybe depending continuously on $\mu$ but positive for $\mu \in [0, \infty)$ such that

$$
\mu \left( \int_0^L f(x)e^{-\frac{\lambda}{2\mu} t}z^2(t, x)dx + I(t)^2 \right) \geq \mu C_3 V_0,
$$

and as $\lambda$ is $C^1$, (8.3.7) can be simplified in

$$
\frac{dV_0}{dt} \leq -\mu C_3 V_0 - \lambda(z(t, L))f(L)e^{-\frac{\lambda}{2\mu} t}z^2(t, L) \\
- Q + O \left( |z(t, \cdot)|_{H^2} + |I(t)|^2 \right),
$$

where $O(r)$ means that there exist $\eta > 0$ and $C > 0$, both independent of $\phi$, $I$, $T$ and $t \in [0, T]$, such that

$$
(|r| \leq \eta) \implies (|O(r)| \leq C_1 |r|),
$$

and where $Q$ is the quadratic form defined by

$$
Q := I^2(t) \left( 2\alpha^2 \lambda(0)k_f^2 - \lambda(0)f(0)k_f^2 - \mu \right) \\
+ \int_0^L (-\lambda(0)f'(x))e^{-\frac{\lambda}{2\mu} t}z^2(t, x) + 2\alpha^2 \lambda(0)k_f I(t)dx.
$$

To ensure the decay of $V_0$, we would like to make this quadratic form in $z$ and $I$ positive definite with $f > 0$. This implies that $f$ is decreasing and $k_I > 0$. Then, bringing all the terms inside the integral we would need the discriminant to be positive, i.e.

$$
\frac{1}{L} \left( 2\alpha^2 \lambda(0)k_I - \lambda(0)f(0)k_f^2 - \mu \right) (-\lambda(0)f'(x))e^{-\frac{\lambda}{2\mu} t}z^2(t, x) - \alpha^4 k_f^2 \lambda(0)^2 > 0
$$

If we place ourselves in the limiting favourable case where $Q$ is only semi-definite positive, and $f(L) = \mu = 0$, one has

$$
f'(x) \left( 2\alpha^2 \lambda(0)k_I - f(0)k_f^2 \right) = -L\alpha^4 k_f^2.
$$
Thus $f'$ is constant and, as $f(L) = 0$,

\[-2\lambda(0)\alpha^2 f(0)k_1 + f^2(0)k_1^2 + L^2\alpha^4 k_1^2 = 0.\]  \hspace{1cm} (8.3.13)

With $\lambda(0) = \lambda_0$, this equation has a positive solution if and only if

\[4\alpha^4 k_1^2 (\lambda_0^2 - k_1^2 L^2) \geq 0.\]  \hspace{1cm} (8.3.14)

This is equivalent to $|k_1| \leq \lambda_0/L$. This is the limiting case, to get $Q$ definite positive and $V_0$ exponentially decreasing we would need to add $V_{0,1}(t) = V_0(z_t, \hat{I})$ and $V_{0,2}(t) = V(z_t, \hat{I})$ to make the Lyapunov function equivalent to the $H^2$ norm to deal with $O(|z(t)|_{H^2} + |I(t)|)$ as in Section 8.5 and we would get the following sufficient condition: $k_1 \in (0, \lambda_0/L)$ which is better than the condition given by Theorem 8.2.2, but conservative compared to the necessary condition (8.2.9). This motivates the next section.

8.4 Extracting the limiting part of the solution

In this section we introduce the projector that will enable us to extract from the solution the limiting part for the stability. We start by introducing the operator $A$,

\[A \left( \begin{array}{c} \phi \\ I \end{array} \right) := \left( \begin{array}{c} -\lambda_0 \phi_x \\ \phi(L) \end{array} \right)\]  \hspace{1cm} (8.4.1)

defined on the domain $\mathcal{D}(A) = \{ (\phi, I)^T | \phi \in H^2(0, L), I \in \mathbb{R}, \phi(0) = -k_1 I \}$. And we note that looking for solutions to the linearized problem (8.2.7), (8.2.2), (8.2.3) can be seen as looking for solutions $(\phi, I)^T \in C^0([0, T], \mathcal{D}(A))$ to the differential problem

\[\left( \begin{array}{c} \dot{\phi} \\ \dot{I} \end{array} \right) = A \left( \begin{array}{c} \phi \\ I \end{array} \right).\]  \hspace{1cm} (8.4.2)

As mentioned in Section 8.2 we know that any eigenvalue $\rho$ of this operator satisfies (8.2.8), which, denoting $\rho \lambda_0^{-1} = \sigma + i\omega$ with $(\sigma, \omega) \in \mathbb{R}^2$, is equivalent to

\[
\begin{align*}
\lambda_0 e^{\sigma L} (\omega \sin(\omega L) - \sigma \cos(\omega L)) &= k_1, \\
\omega \cos(\omega L) + \sigma \sin(\omega L) &= 0.
\end{align*}
\]  \hspace{1cm} (8.4.3)

Assuming (8.2.9), there is a unique solution to (8.4.3) that also satisfies $\omega \in (-\pi/2L, \pi/2L)$ [190, Page 22]. We denote by $\rho_1$ the corresponding eigenvalue. In [190] it was shown that this eigenvalue and its conjugate are the eigenvalues with the largest real part and are the limiting factor to the stability in the linear case. Although we do not need this claim in what follows, it explains why we consider this eigenvalue. We suppose that $\omega := \omega_{\rho_1} \neq 0$. The special case $\omega_{\rho_1} = 0$ is simpler can be treated similarly (see Remark 8.4.1).

We introduce the following operator:

\[p := \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right) \in \mathcal{L}(\mathcal{D}(A), \text{Span}\{e^{\frac{\pi x}{2L}}, e^{\frac{-\pi x}{2L}}\})\]  \hspace{1cm} (8.4.4)

defined by

\[
\begin{align*}
p_1 \left( \begin{array}{c} \phi \\ I \end{array} \right) &:= \alpha_1 \left( \int_0^L \phi(x) e^{\frac{\pi x}{2L}} dx + \lambda_0 e^{\frac{\pi L}{2L}} I \right) e^{\frac{-\pi x}{2L}} \\
&\quad + \bar{\alpha}_1 \left( \int_0^L \phi(x) e^{\frac{-\pi x}{2L}} dx + \lambda_0 e^{\frac{-\pi L}{2L}} I \right) e^{\frac{\pi x}{2L}},
\end{align*}
\]  \hspace{1cm} (8.4.5)

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\[ p_2 \left( \phi \right) = \alpha_1 \left( \int_0^L \phi(x) e^{\frac{\imath}{\lambda_0} x} dx + \lambda_0 e^{\frac{\imath}{\lambda_0} L} I \right) e^{-\frac{\imath}{\lambda_0} L} \]

\[ + \frac{\alpha_1}{\bar{\alpha}_1} \left( \int_0^L \phi(x) e^{\frac{\imath}{\lambda_0} x} dx + \lambda_0 e^{\frac{\imath}{\lambda_0} L} I \right) e^{-\frac{\imath}{\lambda_0} L} , \]

(8.4.6)

where \( \bar{z} \) stands for the conjugate of \( z \) and \( \alpha_1 := \varrho_1/(\varrho_1 L + \lambda_0) \). Here we used a slight abuse of notation and the notation \( e^{-\frac{\imath}{\lambda_0} x} \) outside the brackets refers actually to the function \( x \to e^{-\frac{\imath}{\lambda_0} x} \) defined on \([0, L]\). One can see that \( p \) is real even though \( \varrho_1 \) is complex, as \( p \) is the sum of a function and its conjugate. Denoting \( \varrho_1 \lambda_0^{-1} = \sigma + \imath \omega \), the formulation \((8.4.5) - (8.4.6)\) is equivalent to

\[ p_1 \left( \phi \right) = \left( \int_0^L \phi(x) e^{\sigma x} \cos(\omega x) dx + \lambda_0 I(t) \cos(\omega L) e^{\sigma L} \right) \left( \text{Re}(\alpha_1) \sin(\omega x) e^{-\sigma x} + \text{Im}(\alpha_1) \cos(\omega x) e^{-\sigma x} \right) \]

\[ + \left( \int_0^L \phi(x) e^{\sigma x} \sin(\omega x) dx + \lambda_0 I(t) \sin(\omega L) e^{\sigma L} \right) \left( \text{Re}(\alpha_1) \cos(\omega x) e^{-\sigma x} - \text{Im}(\alpha_1) \sin(\omega x) e^{-\sigma x} \right) . \]

\[ p_2 \left( \phi \right) = \left( \int_0^L \phi(x) e^{\sigma x} \cos(\omega x) dx + \lambda_0 I(t) \cos(\omega L) e^{\sigma L} \right) \left( \text{Re}(\frac{\alpha_1}{\varrho_1}) \sin(\omega L) e^{-\sigma L} + \text{Im}(\frac{\alpha_1}{\varrho_1}) \cos(\omega L) e^{-\sigma L} \right) \]

\[ + \left( \int_0^L \phi(x) e^{\sigma x} \sin(\omega x) dx + \lambda_0 I(t) \sin(\omega L) e^{\sigma L} \right) \left( \text{Re}(\frac{\alpha_1}{\varrho_1}) \cos(\omega L) e^{-\sigma L} - \text{Im}(\frac{\alpha_1}{\varrho_1}) \sin(\omega L) e^{-\sigma L} \right) . \]

(8.4.7)

However in the following, for simplicity, we will keep the complex formulation. We first show that \( p \) commutes with the operator \( A \) given by \((8.4.1)\). Indeed one can check that, with

\[ p_{1, \varrho_1} := \alpha_1 \left( \int_0^L \phi(x) e^{\frac{\imath}{\lambda_0} x} dx + \lambda_0 e^{\frac{\imath}{\lambda_0} L} I \right) e^{-\frac{\imath}{\lambda_0} x} , \]

(8.4.8)

one has

\[ p_{1, \varrho_1} \left( A \left( \phi \right) \right) = p_{1, \varrho_1} \left( \left( -\lambda_0 \phi \right) \right) = \alpha_1 \left( -\lambda_0 \int_0^L \phi_x(x) e^{\frac{\imath}{\lambda_0} x} dx + \lambda_0 e^{\frac{\imath}{\lambda_0} L} \phi(L) \right) e^{-\frac{\imath}{\lambda_0} x} \]

\[ = \alpha_1 \left( -\lambda_0 \phi(L) e^{\frac{\imath}{\lambda_0} L} + \lambda_0 \phi(0) + \varrho_1 \int_0^L \phi(x) e^{\frac{\imath}{\lambda_0} x} dx + \lambda_0 e^{\frac{\imath}{\lambda_0} L} \phi(L) \right) e^{-\frac{\imath}{\lambda_0} x} . \]

(8.4.9)

Using that \((\phi, I)^T \) belongs to the space \(\{(\phi, I) \in L^2(0, L) \times \mathbb{R} | \phi(0) = -k_I I\}\), together with \((8.2.8)\), one gets that

\[ p_{1, \varrho_1} \left( A \left( \phi \right) \right) = \alpha_1 \varrho_1 \left( \int_0^L \phi(x) e^{\frac{\imath}{\lambda_0} x} dx + \lambda_0 e^{\frac{\imath}{\lambda_0} L} I \right) e^{-\frac{\imath}{\lambda_0} x} \]

\[ = -\lambda_0 \left( p_{1, \varrho_1} \left( \phi \right) \right) . \]

(8.4.10)

As \( \varrho_1 \) also verifies \((8.2.8)\), we get the same for \( p_{1, \varrho_1} \), which is defined as \( p_{1, \varrho_1} \) in \((8.4.8)\) with \( \varrho_1 \) instead of \( \varrho_1 \). Thus from \((8.4.5) \) and \((8.4.8)\)

\[ p_1 \left( A \left( \phi \right) \right) = \left( A \left( p \left( \phi \right) \right) \right) . \]

(8.4.11)

Then from \((8.2.8) \) and \((8.4.6)\), one easily gets that, for any \((\phi, I) \in L^2(0, L) \times \mathbb{R}, \)

\[ p_2((\phi, I)^T) = -k_I^{-1} p_1((\phi, I)^T)(0), \]

thus \( p \left( \phi \right) \in \mathcal{D}(A) \) and

\[ p \left( A \left( \phi \right) \right) = A \left( p \left( \phi \right) \right) . \]

(8.4.12)
Now, we show that $p$ is a projector, meaning that $p \circ p = p$. To avoid overloading the computations, we denote

$$d_1 = \alpha_1 \left( \int_0^L \phi(x)e^{\frac{\varphi}{\vartheta}x}dx + \lambda_0 e^{\frac{\varphi}{\vartheta}L}I \right),$$

(8.4.13)

and $\bar{d}_1$ is defined similarly with $\bar{g}_1$ instead of $g_1$. Therefore one has

$$p_1 \left( p \left( \phi \right) I \right) = \alpha_1 \left( d_1 + d_1 e^{\frac{\varphi}{\vartheta} + \bar{d}_1} - \lambda_0 \int \frac{d_1 e^{\varphi L} - \lambda_0 e^{\varphi L}}{g_1 - \bar{g}_1} \phi \right) e^{\varphi x}$$

(8.2.8)

Integrating and using $(8.4.6)$, one has

$$p_1 \left( p \left( \phi \right) I \right) = \alpha_1 \left( d_1 L + \lambda_0 \int \frac{d_1 e^{\varphi L}}{g_1 - \bar{g}_1} - \lambda_0 \int \frac{d_1 e^{\varphi L}}{g_1 - \bar{g}_1} \phi \right) e^{\varphi x}$$

(8.4.14)

Integrating and using $(8.2.8)$, one has

$$p_1 \left( p \left( \phi \right) I \right) = \alpha_1 \left( d_1 L + \lambda_0 \int \frac{d_1 e^{\varphi L}}{g_1 - \bar{g}_1} - \lambda_0 \int \frac{d_1 e^{\varphi L}}{g_1 - \bar{g}_1} \phi \right) e^{\varphi x}$$

(8.4.15)

But, still from $(8.2.8)$, observe that

$$\frac{e^{\frac{\varphi}{\vartheta}L} - \lambda_0}{g_1 - \bar{g}_1} = \left( \frac{k_1}{k_2} \right) \left( \frac{\varphi}{\vartheta} \right) - \frac{1}{g_1 - \bar{g}_1} = -1 \frac{1}{g_1 - \bar{g}_1}$$

(8.4.16)

and recall that $\alpha_1 = g_1/(g_1 L + \lambda_0)$, thus

$$p_1 \left( p \left( \phi \right) I \right) = d_1 e^{\frac{\varphi}{\vartheta}x} + \bar{d}_1 e^{\frac{\varphi}{\vartheta}x} = p_1 \left( \phi \right) I.$$
Thus
\[ \int_0^L \phi_2(x) \left( e^{\frac{\omega_1}{L} (x-L)} - e^{\frac{\omega_1}{L} (x-L)} \right) dx = 0. \] (8.4.21)

Or equivalently, denoting as previously \( \varrho \)
\[ \int_0^L \varrho_2(x) e^{\sigma(x-L)} \sin(\omega(x-L)) dx = 0. \] (8.4.22)

**Remark 8.4.1.** • In the special case \( \omega = 0 \), we can define \( p \) similarly as previously but with \( \alpha_1 = \bar{\alpha}_1 = 1/2 \) instead. Then still holds, but, as \( \varrho_1 = \bar{\varrho}_1 \), \( p \) is now a projector on the one-dimensional space \( \text{Span}\{e^{-\frac{\omega_1}{L} x}\} \) and is defined by
\[ p_1((\phi, I)^T) = \left( \int_0^L \phi(x) e^{\frac{\omega_1}{L} x} dx + \lambda_0 e^{\frac{\omega_1}{L} L} I \right) e^{-\frac{\omega_1}{L} x}, \] (8.4.23)

and \( p_2((\phi, I)^T) = \varrho_1^{-1} p_1((\phi, I)^T)(L) \). Nevertheless still holds and is straightforward. Indeed, we can still define \((\phi_1, I_1)^T = p((\phi, I)^T)\) and \((\phi_2, I_2) = (\phi - \phi_1, I - I_1)\), and, as \( \omega = 0 \), holds directly.

• Note that, when \( \omega \neq 0 \), contains two equations, as \( p \) is a projector on a space of dimension 2. Therefore another relation can be inferred from in addition to namely
\[ \int_0^L \varrho_2(x) e^{\sigma(x-L)} \cos(\omega(x-L)) dx = -\lambda_0 I_2. \] (8.4.24)

However this relation will not be used in the following.

### 8.5 Exponential stability analysis

In this section we use the results of the above sections to prove Theorem 8.2.3. We first separate the solution of the system in a projected part and a remaining part using the projector defined in Section 8.4. Then we use the Lyapunov function defined in Section 8.3 to deal with the remaining part.

**Linear case** We first prove Theorem 8.2.3 using our method in the linear case. This could seem senseless as the result in the linear case is already known. However, the purpose is to make clear how this method works while keeping the computations simple. Then we will show that this method still works when adding the nonlinearities. Let \( T > 0 \) and let \( \phi \) be a solution of the linear system (8.2.7), (8.2.2), (8.2.3) in \( C^1([0, L] \times [0, T]) \). Using the last section, we define the following functions
\[ \begin{pmatrix} \phi_1(t, x) \\ I_1(t, x) \end{pmatrix} = p \begin{pmatrix} \phi(t, x) \\ I(t) \end{pmatrix}, \] (8.5.1)
\[ \begin{pmatrix} \phi_2(t, x) \\ I_2(t, x) \end{pmatrix} = \begin{pmatrix} \phi(t, x) \\ I(t) \end{pmatrix} - \begin{pmatrix} \phi_1(t, x) \\ I_1(t) \end{pmatrix}. \] (8.5.2)

We expect to have extracted from \((\phi, I)\) the limiting factor for the stability that is now contained in \((\phi_1, I_1)\). The function \((\phi_1, I_1)\) is a simple projection on a space of finite dimension, it has therefore a simple dynamic and is easy to control, while we will use our Lyapunov function introduced earlier in Section 8.3 to deal with \((\phi_2, I_2)\). In other words we will consider the following total Lyapunov function
\[ V(t) = V_1(t) + V_2(t), \] (8.5.3)
where \( V_1 \) is a Lyapunov function for \((\phi_1, I_1)\) to be defined and \( V_2(t) = V_0(\phi_2(t, \cdot), I(t)) \). Recall that the definition of \( V_0 \) is given in (8.3.1).
Let us look at \( \phi_1 \). From the definition of \( p_1 \) given by (8.4.5), \( p_1 = p_{1,\bar{\varphi}_1} + p_{1,\varphi_1} \), where \( p_{1,\varphi_1} \) is given by (8.4.8) and \( p_{1,\bar{\varphi}_1} \) is given by the same definition with \( \bar{\varphi}_1 \) instead of \( \varphi_1 \). Similarly \( p_2 = p_{2,\bar{\varphi}_1} + p_{2,\varphi_1} \) with

\[
p_{2,\varphi_1}((\phi, I)^T) = \frac{p_{1,\varphi_1}((\phi, I)^T)(L)}{\varphi_1},
\]

(8.5.4)

and \( p_{2,\bar{\varphi}_1} \) defined similarly but with \( \bar{\varphi}_1 \) instead of \( \varphi_1 \). Therefore we can define

\[
\left( \phi_{\varphi_1}(t, x) \right) := p_{\varphi_1} \left( \phi(t, x) \right) := \left( p_{1,\varphi_1}(\phi, I)^T(t, x) \right),
\]

(8.5.5)

and we can define its conjugate \((\phi_{\bar{\varphi}_1}, I_{\bar{\varphi}_1})^T\) similarly. Thus we can decompose \((\phi_1, I_1)^T\) in

\[
\left( \phi_1(t, x) \right) = \left( \phi_{\varphi_1}(t, x) \right) + \left( \phi_{\bar{\varphi}_1}(t, x) \right),
\]

(8.5.6)

Let us now define \( V_1(t) \) by

\[
V_1(t) := \int_0^L |\phi_{\varphi_1}(t, x)|^2 dx + |I_{\varphi_1}(t)|^2,
\]

(8.5.7)

then

\[
|\phi_1|^2_{L^2} + |I_1|^2 \leq 4|\phi_{\varphi_1}|^2_{L^2} + 4|I_{\varphi_1}|^2 \leq 4V_1.
\]

(8.5.8)

Differentiating \( V_1 \) one has

\[
\frac{dV_1}{dt} = \int_0^L 2\text{Re}(\partial x \phi_{\varphi_1}) dx + 2\text{Re}(\dot{I}_{\varphi_1} I_{\varphi_1}).
\]

(8.5.9)

From (8.4.5), (8.4.6), and (8.5.1)

\[
\left( \partial_t \phi_{\varphi_1}(t, x) \right) := p_{\varphi_1} \left( \partial_t \phi(t, x) \right) = p_{\varphi_1} \left( A \left( \phi(t, x) \right) \right)
\]

(8.5.10)

Observe that the commutation property (8.4.11) still holds with \( p_{\varphi_1} \) instead of \( p \), and that \( p_{\bar{\varphi}_1} \) is still a linear operator. Therefore,

\[
\frac{dV_1}{dt} = 2\text{Re}(\partial_t \dot{\varphi}_1) \lambda_0 \int_0^L |\phi_{\varphi_1}|^2 dx + 2\text{Re}(\dot{\varphi}_1) |I_{\varphi_1}(t)|^2.
\]

(8.5.11)

As \( \text{Re}(\varphi_1) < 0 \) from (8.2.10) and (8.2.8),

\[
\frac{dV_1}{dt} \leq -2|\text{Re}(\varphi_1)| \text{min}(\lambda_0, 1) V_1
\]

(8.5.12)

which will now imply the exponential decay.

Let us now look at \( V_2 \). From (8.2.7), (8.2.2), (8.2.3), (8.4.6), (8.5.2), \( (\phi_2, I_2) \) is also a solution to the linear system (8.2.7), (8.2.2), (8.2.3). Thus acting similarly as in Section 8.3 (8.3.5)-(8.3.9), we have

\[
\frac{dV_{2,1}}{dt} \leq -\mu C_3 V_{2,1} - \lambda(\phi(t, L)) f(L) e^{-\frac{\nu}{\lambda}} \phi_2^2(t, L) - I_2^2(t) (2\alpha^2 \lambda(0)^2 k_I - \lambda(0) f(0) k_f^2 - \mu)
\]

(8.5.13)

\[
- \int_0^L (-\lambda(0) f'(x)) e^{-\frac{\nu}{\lambda}} \phi_2^2(t, x) dx + 2\alpha^2 k_I \lambda(0) \phi_2 I(t) dx.
\]

If we look now at the quadratic form in \( I_2 \) and \( \phi_2 \) that appears, we can see that it is exactly the same as previously in (8.3.10). However, since \( \phi_2 \) is the complement of \( \phi_1 \) in \( \phi \), we now have an additional
information on $\phi_2$ given by \(8.4.22\). Thus, denoting again this quadratic form by $Q$, and using \(8.4.22\) we have

\[
Q = \int_0^L (-\lambda_0 f'(x)) e^{-\frac{\kappa}{2} L} \phi_2^2(t, x) dx + 2\alpha^2 k_I \lambda_0 I(t) \int_0^L \phi_2(1 - \kappa \theta(x)) dx \\
+ I^2(t) (2\alpha^2 \lambda_0^2 k_I - \lambda_0 f(0) k_I^2 - \mu)
\]

\[
\geq \inf_{x \in [0, L]} (-\lambda_0 f'(x)) e^{-\frac{\kappa}{2} L} \left( \int_0^L \phi_2^2(t, x) dx \right) \\
- 2\alpha^2 k_I \lambda_0 |I(t)| \left( \int_0^L \phi_2^2 dx \right)^{1/2} \left( \int_0^L (1 - \kappa \theta(x))^2 dx \right)^{1/2} \\
+ I^2(t) (2\alpha^2 \lambda_0^2 k_I - \lambda_0 f(0) k_I^2 - \mu)
\]

(8.5.14)

where

\[
\theta(x) := e^{\sigma(x - L)} \sin(\omega(x - L))
\]

(8.5.15)

and $\kappa$ is a constant that can be chosen arbitrarily. As the right-hand side is now a quadratic form in $|\phi_2|^2_L$ and $I$, a sufficient condition for $Q$ to be positive is

\[
\inf_{x \in [0, L]} (-\lambda_0 f'(x)) e^{-\frac{\kappa}{2} L} (2\alpha^2 \lambda_0^2 k_I - \lambda_0 f(0) k_I^2 - \mu) \\
> (\alpha^2 k_I \lambda_0)^2 \left( \int_0^L (1 - \kappa \theta(x))^2 dx \right).
\]

(8.5.16)

Of course we have all interest in choosing $\kappa$ such that it minimizes the integral of $(1 - \kappa \theta(x))^2$. We have

\[
\int_0^L (1 - \kappa \theta(x))^2 dx = \kappa^2 \left( \int_0^L \theta^2(x) dx \right) - 2\kappa \left( \int_0^L \theta(x) dx \right) + L.
\]

(8.5.17)

This is a second order polynomial in $\kappa$ thus, assuming $\omega \neq 0$, its minimum is

\[
L + \frac{\left( \int_0^L \theta(x) dx \right)^2}{\left( \int_0^L \theta^2(x) dx \right)^2} \left( \int_0^L \theta^2(x) dx \right) - 2 \frac{\left( \int_0^L \theta(x) dx \right)}{\left( \int_0^L \theta^2(x) dx \right)} \left( \int_0^L \theta(x) dx \right)
\]

(8.5.18)

Choosing such $\kappa$, and $f'$ constant, condition \(8.5.16\) becomes

\[
e^{-\frac{\kappa}{2} L} \frac{f(0)}{L} - \frac{f(L)}{L} (2\alpha^2 \lambda_0^2 k_I - \mu - \lambda_0 f(0) k_I^2) \\
- (\alpha^2 k_I \lambda_0)^2 L \left( 1 - \frac{\left( \int_0^L \theta(x) dx \right)^2}{L \left( \int_0^L \theta^2(x) dx \right)} \right) > 0.
\]

(8.5.19)

which is equivalent to

\[
- \lambda_0^2 k_I^2 f^2(0) + (2\alpha^2 \lambda_0^2 k_I - \mu + f(L) \lambda_0 k_I^2) \lambda_0 f(0) - (2\alpha^2 \lambda_0^2 k_I - \mu) \lambda_0 f(L) \\
- e^{\frac{\kappa}{2} L} \lambda_0^2 (\alpha^2 k_I \lambda_0)^2 L^2 \left( 1 - \frac{\left( \int_0^L \theta(x) dx \right)^2}{L \left( \int_0^L \theta^2(x) dx \right)} \right) > 0.
\]

(8.5.20)
We place ourselves in the limiting case, when \( \mu = 0 \) and \( f(L) = 0 \). As the left-hand side is a second order polynomial in \( f(0) \), there exists a positive solution \( f(0) \) to the inequality if and only if

\[
\left( 1 - \frac{\int_0^L \theta(x)dx}{L \int_0^L \theta^2(x)dx} \right) k_f^2 L^2 < \lambda_0^2. \tag{8.5.21}
\]

Under assumption (8.2.10), we can show that this is always verified, this is done in the Appendix. When \( \omega = 0 \), taking again \( f \) constant and the limiting case where \( f(L) = 0 \) and \( \mu = 0 \), \( Q \) is definite positive provided that

\[
- \lambda^2(0)k_f^2f(0) + (2\alpha^2 \lambda_0^2 k_f - \mu + f(L) \lambda_0 k_f) \lambda_0 f(0)
- (2\alpha^2 \lambda_0^2 k_f - \mu) \lambda_0 f(L) - e^{\frac{\alpha^2 L}{2}}(\alpha^2 k_f \lambda_0)^2 L^2 > 0. \tag{8.5.22}
\]

There exists a positive solution \( f(0) \) to this inequality if and only if

\[
k_f^2 < \left( \frac{\lambda_0}{L} \right)^2, \tag{8.5.23}
\]

but, as \( g_1 \) is real and \( k_f \) is positive, \( k_f = -\left( \lambda_0/L \right) (g_1 L/\lambda_0) e^{-\gamma t} < \lambda_0/L \), thus (8.5.23) is satisfied. Thus, by continuity, there always exists \( \mu_1 > 0 \) and \( f \) positive such that \( Q > 0 \) and therefore

\[
\frac{dV_2}{dt} \leq -\mu_1 C_3 V_2 - (\lambda_0 f(L) e^{-\gamma t}) \phi_2^2(t, L). \tag{8.5.24}
\]

This implies from (8.5.3) and (8.5.12) that

\[
\frac{dV}{dt} \leq -\min\{2|\text{Re}(g_1)|, 2|\text{Re}(g_1)|\lambda_0, \mu_1 C_3\} V \tag{8.5.25}
\]

This shows the exponential decay for \( V \). It remains now only to show that it also implies the exponential decay for \( (\phi, I) \) in the \( L^2 \) norm. But from (8.3.4) and (8.5.7), \( V \) is equivalent to the norm \( (|\phi_2|_{L^2} + |I_2|_{L^2} + |I_2(t)|)^2 \). Besides, we have from (8.4.8), (8.5.7), and using Cauchy-Schwarz inequality

\[
V_1(t) \leq \left( \int_0^t \left| \alpha_1 e^{-\frac{\alpha_1}{\lambda_0} t} \right|^2 dx + \left| \alpha_1 e^{-\frac{\alpha_1}{\lambda_0} t} \right|^2 \right) \left( \int_0^t e^{2\alpha_1 e^{-\frac{\alpha_1}{\lambda_0} t} dx} \right)^{1/2} \left( \int_0^t \phi^2 dx \right)^{1/2} + I e^{\frac{\alpha_1}{\lambda_0} L} \right)^2 \tag{8.5.26}
\]

where \( C_5 \) is a constant that does not depend on \( I \) or \( \phi \). Also, from (8.3.4), (8.5.26) and noting that \( \phi_2 = \phi - \phi_1 \) and \( I_2 = I - I_1 \),

\[
V_2(t) \leq C_2 (|\phi_2(t, \cdot)|_{L^2} + |I_2(t)|^2) \leq C_6 (|\phi(t, \cdot)|_{L^2} + |I(t)|^2). \tag{8.5.27}
\]

Thus, from (8.5.26) and (8.5.27), there exists \( C_7 > 0 \) independent of \( \phi \) and \( I \) such that

\[
V(t) \leq C_7 (|\phi(t, \cdot)|_{L^2} + |I(t)|). \forall t \in [0, T]. \tag{8.5.28}
\]

And from (8.3.4), (8.5.8), and (8.5.25),

\[
|\phi(t, \cdot)|_{L^2}^2 + |I(t)|^2 \leq 4V_1(t) + C_2 V_2(t) \leq \max(4C_2) e^{-\gamma t} V(0). \tag{8.5.29}
\]

Thus, there exists \( C_8 > 0 \) independent of \( \phi \) and \( I \) such that

\[
(|\phi(t, \cdot)|_{L^2} + |I(t)|) \leq C_8 e^{-\gamma t} (|\phi(0, \cdot)|_{L^2} + |I(t)|). \tag{8.5.30}
\]

This concludes the proof of Theorem 8.2.3 in the linear case.
Nonlinear case In this paragraph we show how to adapt the previous method when the system is nonlinear instead. Let \( T > 0 \) and let \( \phi \) be a solution to the nonlinear system \((8.2.1)–(8.2.3)\). We suppose in the following that
\[
|\phi(t, \cdot)|_{H^2} \leq \varepsilon, \quad \forall \ t \in [0, T],
\]
with \( \varepsilon \in (0, 1) \) to be chosen later on. This assumption can be done as we are looking for a local result with respect to the perturbations (i.e., the initial conditions), and, from \((8.2.5)\), for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( |\phi_0|_{H^2} \leq \delta \) then \((8.5.31)\) holds. Let us assume in addition that \( \phi \in C^2([0, L] \times [0, T]) \) (we will relax this assumption later on using a density argument). We define again \((\phi_1, I_1)^T\) and \((\phi_2, I_2)^T\) as in \((8.5.1)–(8.5.2)\), and \((\phi, I)^T\) can still be decomposed using \((8.5.6)\). We consider the Lyapunov function
\[
V(t) = V_1(t) + V_2(t),
\]
where \( V_2(t) = V_{2,1}(t) + V_{2,2}(t) + V_{2,3}(t) \), with \( V_{2,k}(t) = \phi_0(\partial_t^{k-1}\phi_2(t, \cdot), \partial_t^{k-1}I(t)) \) (the definition of \( \phi_0 \) is given in \((8.3.1)\) and \( V_1 \) is defined by
\[
\begin{align*}
V_1(t) := & \int_0^L |\phi_{\varepsilon_1}(t, x)|^2 + |\partial_x\phi_{\varepsilon_1}(t, x)|^2 + |\partial_t^2\phi_{\varepsilon_1}(t, x)|^2 \, dx \\
& + |I_{\varepsilon_1}(t)|^2 + |\dot{I}_{\varepsilon_1}(t)|^2 + |I_{\varepsilon_1}(t)|^2.
\end{align*}
\]

Remark 8.5.1. Note that, strictly speaking, both \( V_1 \) and \( V_2 \) can be expressed as a functional on time-independent functions belonging to \( H^2(0, L) \times \mathbb{R} \), using for instance the following notations for \((\phi, I) \in H^2(0, L) \times \mathbb{R}\):
\[
\begin{align*}
\dot{I} := & \phi(t, L), \quad \ddot{I} := -\lambda(\phi(L))\partial_x\phi(L), \\
\partial_I \phi := & -\lambda(\phi)\partial_x\phi, \quad \partial_t^2 \phi := -\lambda'(\phi)(\partial_x\phi)^2 - \lambda(\phi)\partial_x^2\phi.
\end{align*}
\]

Of course these notations correspond to the time-derivatives of the functions when \((I, \phi)\) is time-dependent and a solution of \((8.2.1)–(8.2.3)\).

Using \((8.5.33)\), there exists \( \varepsilon_1 \in (0, 1) \) such that for \( \varepsilon < \varepsilon_1 \), one has
\[
\frac{1}{2} \min(1, \lambda_0^4) \left( |\phi_{\varepsilon_1}|_{H^2}^2 + |I_{\varepsilon_1}|^2 + |\dot{I}_{\varepsilon_1}|^2 + |\ddot{I}_{\varepsilon_1}|^2 \right) \\
\leq V_1 \leq 2 \max(1, \lambda_0^4) \left( |\phi_{\varepsilon_1}|_{H^2}^2 + |I_{\varepsilon_1}|^2 + |\dot{I}_{\varepsilon_1}|^2 + |\ddot{I}_{\varepsilon_1}|^2 \right),
\]
and therefore
\[
\begin{align*}
|\phi_1|_{H^2}^2 + |I_1|^2 + |\dot{I}_1|^2 + |\ddot{I}_1|^2 \\
\leq 4|\phi_{\varepsilon_1}|_{H^2}^2 + 4|I_{\varepsilon_1}|^2 + 4|\dot{I}_{\varepsilon_1}|^2 + 4|\ddot{I}_{\varepsilon_1}|^2 \\
\leq 8 \max(1, \lambda_0^4)V_1.
\end{align*}
\]

which is similar to \((8.5.8)\).

Differentiating \( V_1 \) one has, similarly as in \((8.5.9)\),
\[
\frac{dV_1}{dt} = \int_0^L 2\text{Re} \left( \partial_t \phi_{\varepsilon_1}(t, x) \right) + 2\text{Re} \left( \partial_t^2 \phi_{\varepsilon_1}(t, x) \right) + 2\text{Re} \left( \partial_t^3 \phi_{\varepsilon_1}(t, x) \right) \, dx \\
+ 2\text{Re} \left( \dot{I}_{\varepsilon_1} \dot{I}_1 \right) + 2\text{Re} \left( \ddot{I}_{\varepsilon_1} \ddot{I}_1 \right) + 2\text{Re} \left( \dddot{I}_{\varepsilon_1} \dddot{I}_1 \right).
\]

From \((8.4.5)\), \((8.4.6)\), and \((8.5.1)\)
\[
\begin{align*}
\left( \partial_t \phi_{\varepsilon_1}(t, x), \dot{I}_{\varepsilon_1}(t) \right) = & \ p_{\varepsilon_1} \left( \partial_t \phi(t, x), \dot{I}(t) \right) = \ p_{\varepsilon_1} \left( A_1 \left( \phi(t, x), I(t) \right) \right),
\end{align*}
\]
where $A_1$ is now defined for any $(\phi, I) \in D(A)$ by

$$A_1 \left( \frac{\phi}{I} \right) := \left( -\lambda(\phi) \frac{\partial \phi}{\partial x} - \lambda(\phi) \right) = A \left( \frac{\phi}{I} \right) + \left( (\lambda_0 - \lambda(\phi)) \frac{\partial \phi}{\partial x} \right). \tag{8.5.39}$$

Thus using the commutation property with $p_{\psi_1}$,

$$\left( \frac{\partial \phi_{\psi_1}(t,x)}{\partial t} \right)_{I_{\psi_1}(t)} = A \left( p_{\psi_1} \left( \frac{\phi(t,x)}{I(t)} \right) \right) + p_{\psi_1} \left( (\lambda_0 - \lambda(\phi)) \frac{\partial \phi(t,x)}{\partial x} \right)$$

$$= \left( -\lambda_0(\phi_{\psi_1})(t,x) \right)_{I_{\psi_1}(t)} + \left( \frac{\alpha_1 e^{-\frac{\lambda_0}{2} x} \int_{0}^{L} (\lambda_0 - \lambda(\phi)) \frac{\partial \phi(t,x)}{\partial x} e^{\frac{\lambda_0}{2} x} dx \right)$$

$$= \left( \phi_{\psi_1}(t,x) \right)_{I_{\psi_1}(t)} + \left( \frac{\alpha_1 e^{-\frac{\lambda_0}{2} x} \int_{0}^{L} (\lambda_0 - \lambda(\phi)) \frac{\partial \phi(t,x)}{\partial x} e^{\frac{\lambda_0}{2} x} dx \right). \tag{8.5.40}$$

Besides, let $k \in \{0, 1, 2\}$, as $\lambda$ is $C^1$, integrating by parts and using $\phi_{(t,x)}, \phi$ and noticing that all the quadratic terms in $\lambda(\phi)$ can be bounded by $C_0$ is a constant independent of $\phi$ that depends only on $\lambda$, $\varphi_1$, $L$ and $k_1$. Thus, using $\phi_{(t,x)}$, with $k = 0$, and noting that $|\varphi_0| + |\varphi_2|0$ can be bounded by $|\varphi|_{L^2}$ from Sobolev inequality, the last term of $\phi_{(t,x)}$ is a quadratic perturbation that can be bounded by $|\varphi|_{L^2}^2 + |I|^2 + \phi(t,L)^2$. One can do similarly with the second and third time-derivative noticing that

$$\left( \frac{\partial^2 \phi}{\partial x^2} \right)_{I} = \left( -\lambda_0 \frac{\partial \phi}{\partial x} \right)_{I}$$

$$+ \left( -\lambda'(\phi) \frac{\partial \phi}{\partial x} + (\lambda_0 - \lambda(\phi)) \frac{\partial \phi}{\partial x} \right)$$

$$= \left( -\lambda_0 \frac{\partial \phi}{\partial x} \right)_{I} + \left( -\lambda'(\phi) \frac{\partial \phi}{\partial x} + (\lambda_0 - \lambda(\phi)) \frac{\partial \phi}{\partial x} \right) \tag{8.5.42}$$

and noticing that all the quadratic terms in $\phi$ involve at most a second derivative in $\phi$. Thus as $\lambda$ is $C^2$, all the quadratic terms belong to $L^1$ and their $L^1$ norm can be bounded by $|\varphi|^2_{L^2}$. The $L^1$ norm of the third
order derivative can be bounded by \(|\phi|_{L^2} + |\tilde{I}|^2 + (\partial_t \phi(t, L))^2\) using (8.5.41) and \(k = 2\). Therefore, noting from (8.4.5) that \(|\partial_t^k \phi_1(t, L)| \leq 2|\tilde{g}_1||\partial_t^k I_{g_1}|\), and as \(\text{Re}(g_1) < 0\), as previously,

\[
\frac{dV_1}{dt} \leq -2|\text{Re}(g_1)| \min(\lambda_0, 1)V_1 + O \left(|\phi_{g_1}|_{L^2} + |\phi_2|_{L^2} + |I_{g_1}| + |\tilde{I}_{g_1}| + |I_2| + |\tilde{I}_2|\right)^2
\]

(8.5.43)

\[
+ C(|\phi_{g_1}|_{L^2} + |I_{g_1}|) (|\phi_2|^2(t, L) + \partial_t^2 \phi_2^2(t, L) + \partial_t^2 \phi_2^2(t, L)),
\]

where \(C\) is a positive constant that only depends on \(\lambda, g_1, k_1, L\). The first term will imply the exponential decay as previously, while there is now two others terms that will be compensated using \(V_2\).

Let us now look at \(V_2\). From (8.2.1)–(8.2.3), (8.2.8), (8.4.6), (8.5.2), and (8.5.40), \((\phi_2, I_2)\) is not anymore a solution to the original system but a solution to the following system

\[
\partial_t \phi_2 + \lambda(\phi) \partial_x \phi_2 = (\lambda_0 - \lambda(\phi)) \partial_x \phi_1 + p_1 \left( \left(\frac{\lambda_0 - \lambda(\phi)}{\lambda_0 - \lambda(\phi)} \partial_x \phi_1 \right) \right)
\]

\[
\phi_2(t, 0) = -k_1 I_2(t)
\]

\[
I_2 = \phi_2(t, L).
\]

Thus acting again similarly as in Section 8.3, (8.3.5)–(8.3.9), and using (8.5.41), we have

\[
\frac{dV_2}{dt} \leq -\mu C_2 V_2 - \lambda(\phi(t, L)) f(L) e^{-\frac{\mu}{2}L} \partial_t^2 \phi_2^2(t, L) - I_2^2(t) \left(2\alpha^2 \lambda(0) k_1 - \lambda(0) f(0) k_1^2 - \mu\right)
\]

\[
- \int_0^L (-\lambda(0) f'(x)) e^{-\frac{\mu}{2}L} \partial_t^2 \phi_2^2(t, x) + 2\alpha^2 k_1 \lambda(0) \phi_2 I(t) dx
\]

\[
+ O \left(|\phi_{g_1}|_{L^2} + |\phi_2|_{L^2} + |I_1| + |I_2| + |\tilde{I}_2|\right)^3 + C_{2,1} |\phi_0| \phi_2^2(t, L),
\]

(8.5.45)

where \(C_{2,1}\) is a positive constant independent of \(\phi\) and \(I\). The quadratic form in \(I_2\) and \(\phi_2\) that appears is the same as in the linear case, thus, as in (8.5.13)–(8.5.23), and by continuity there exists \(\mu_1 > 0\) and \(f\) positive such that the quadratic form is positive definite and therefore

\[
\frac{dV_2}{dt} \leq -\mu_1 C_2 V_2 - (\lambda(\phi(t, L)) f(L)) e^{-\frac{\mu}{2}L} - C_{2,1} |\phi_0| \phi_2^2(t, L)
\]

\[
+ O \left(|\phi_{g_1}|_{L^2} + |\phi_2|_{L^2} + |I_1| + |I_2(t)| + |\tilde{I}_2(t)|\right)^3.
\]

(8.5.46)

Let us now deal with \(V_{2,2}\) and \(V_{2,3}\). Observe that from (8.5.44), one has for \(\phi_2 \in C^3\),

\[
\partial_t^2 \phi_2 + \lambda(\phi) \partial_x (\partial_t \phi_2) = (\lambda_0 - \lambda(\phi)) \partial_t^2 \phi_1 + p_1 \left( \left(\frac{\lambda_0 - \lambda(\phi)}{\lambda_0 - \lambda(\phi)} \partial_t^2 \phi_1 \right) \right)
\]

\[
- \lambda'(\phi) \partial_t^2 \phi_1 - \lambda''(\phi) \partial_t^2 \phi_2,
\]

\[
\partial_t^4 \phi_2 + \lambda(\phi) \partial_x (\partial_t^3 \phi_2) = (\lambda_0 - \lambda(\phi)) \partial_x \partial_t^3 \phi_1 - \lambda'(\phi) \partial_x \partial_t^3 \phi_2 - \lambda''(\phi) \partial_x \partial_t^3 \phi_2 - \lambda''(\phi) \partial_x \partial_t^3 \phi_2
\]

\[
= \lambda''(\phi) \partial_x \partial_t^3 \phi_2 - \lambda'(\phi) \partial_x \partial_t^3 \phi_2 - \lambda''(\phi) \partial_x \partial_t^3 \phi_2
\]

\[
= \lambda''(\phi) \partial_x \partial_t^3 \phi_2 - \lambda'(\phi) \partial_x \partial_t^3 \phi_2 - \lambda''(\phi) \partial_x \partial_t^3 \phi_2.
\]

(8.5.47)

and

\[
\partial_t \phi_2(t, 0) = -k_1 I_2(t), \quad \partial_t^2 \phi_2(t, 0) = -k_1 \tilde{I}_2(t),
\]

\[
I_2 = \partial_t \phi_2(t, L), \quad \tilde{I}_2 = \partial_t^2 \phi_2(t, L).
\]

(8.5.48)
From (8.5.41), \( p_1(\lambda_0 - \lambda(\phi))\partial_{xx}^2 \phi \) can be bounded by \( (|\phi|_{H^2}^2 + |\dot{I}|^2 + (\partial_t^2 \phi(t, L))^2) \) and, from (8.4.5), \( \partial_{tx} \phi_1 \) is proportional to \( \partial_{tt} \phi_1 \). Thus, as all the other terms in the right hand sides are quadratic perturbations and include at most a second order derivative, the \( L \) bound of the right-hand sides can be bounded by \( (|\phi_{,1}|_{H^2}^2 + |\phi|_{H^2}^2 + |I_1|^2 + |\dot{I}_1|^2 + |\dot{I}_2|^2 + \phi^2(t, L) + (\partial_t \phi(t, L))^2 + (\partial_t^2 \phi(t, L))^2) \), which is small compared to the first-order term in the left-hand sides. Therefore we have, as previously

\[
\frac{dV_{2,k}}{dt} \leq -\mu_1 C_3 V_{2,k} - (\lambda(\phi(t, L)) f(L)) e^{-\frac{4kL}{\rho}} - C_{2,k} |\phi_1| \left| \partial_t^{k-1} \phi_2(t, L) \right|^2 + O \left( \left( |\phi_{,1}|_{H^2} + |I_1| + |\phi|_{H^2} + |I_2| + |\dot{I}_2| \right) \right), \quad \text{for } k = 2, 3,
\]

Thus we can perform exactly as for \( V_{2,1} \) and consequently

\[
\frac{dV_{2,k}}{dt} \leq -\mu_1 C_3 V_{2,k} - \sum_{k=1}^{3} (\lambda(\phi(t, L)) f(L)) e^{-\frac{4kL}{\rho}} - C_{2,k} |\phi_1| \left| \partial_t^{k-1} \phi_2(t, L) \right|^2 + O \left( \left( |\phi_{,1}|_{H^2} + |I_1| + |\phi|_{H^2} + |I_2| + |\dot{I}_2| \right) \right), \quad \text{for } k = 2, 3,
\]

Thus, from (8.5.32) and (8.5.45),

\[
\frac{dV}{dt} \leq -\min (2|\text{Re}(q_1)|, 2|\text{Re}(q_1)|\lambda_0, \mu_1 C_3) V - \left( \lambda(\phi(t, L)) f(L)) e^{-\frac{4L}{\rho}} - C_4 |\phi|_{H^2} \right) \left( |\phi_{,1}|_{H^2}^2 + |\partial_t \phi_2(t, L)|^2 + |\partial_t^2 \phi_2(t, L)|^2 \right) + O \left( \left( |\phi_{,1}|_{H^2} + |I_1| + |\phi|_{H^2} + |I_2| + |\dot{I}_2| + |\ddot{I}_2(t)| \right) \right).
\]

But from (8.3.4) and (8.5.35), \( V \) is equivalent to the norm \( \left( |\phi_{,1}|_{H^2} + |I_1| + |\dot{I}_1| + |\phi|_{H^2} + |\dot{I}_2(t)| + |\ddot{I}_2(t)| \right)^2 \). Using Cauchy-Schwarz inequality and (8.4.8)–(8.5.33) as previously,

\[
V_1(t) \leq \left( \int_0^L |\alpha_1 e^{-\frac{4kL}{\rho}}|^2 dx + \frac{\alpha_1 e^{-\frac{4kL}{\rho}}}{q_1} \right)^{\frac{1}{2}} \left( \int_0^L e^{2\alpha_1^2} dx \right)^{\frac{1}{2}} \left( \int_0^L \partial_t^{k-1} \phi_2 dx \right)^{\frac{1}{2}} \left( \int_0^L \partial_t^2 \phi_2 dx \right)^{\frac{1}{2}} + \partial_t^{k-1} I e^{\frac{4kL}{\rho}} \right)^2 \leq C_5 \left( |\phi(\cdot, t)|_{H^2}^2 + |\dot{I}(t)|^2 + |\ddot{I}(t)|^2 \right) + (8.5.54)
\]
where $C_5$ is a constant that does not depend on $I$ or $\phi$. Then using (8.3.4), (8.5.26), noting that $\phi_2 = \phi - \phi_1$ and $I_2 = I - I_1$,

$$V_2(t) \leq C_2 \left( (\phi_2(t, \cdot)|_{H^2} + |I_2(t)|^2 + |\dot{I}_2(t)|^2 \right)$$

$$\leq C_6 \left( (\phi(t, \cdot)|_{H^2} + |I(t)|^2 + |\dot{I}(t)|^2 \right),$$

which implies that

$$\left( |\phi_2|_{H^2} + |I_2| + |\dot{I}_2| \right) \leq O \left( (\phi|_{H^2}) \right).$$

But from (8.2.2)–(8.2.3) and Sobolev inequality,

$$\left( |\phi|_{H^2} + |I| + |\dot{I}| \right) = O \left( (\phi|_{H^2}) \right).$$

Therefore, from (8.5.25), (8.5.26), (8.5.27), and (8.5.31), there exists $\gamma > 0$ and $\varepsilon_2 \in (0, \varepsilon_1]$ such that for any $\varepsilon \in (0, \varepsilon_2)$, one has

$$\frac{dV}{dt} \leq -\gamma V.$$

This shows the exponential decay for $V$. As in the linear case, it remains now only to show that it also implies the exponential decay for $\phi$ in the $H^2$ norm. Note that, compared to the linear case where we showed the exponential decay of $(\phi, I)$ in the $L^2$ norm, here we can actually show the exponential decay of $\phi$ in the $H^2$ norm as it is implied by the exponential decay of $(\phi, I)$ in the $H^2$ norm from (8.5.57). Observe first that from (8.5.26)–(8.5.27) and (8.5.57) there exists $C_7 > 0$ independent of $\phi$ and $I$ such that

$$V(t) \leq C_7|\phi(t, \cdot)|_{H^2}, \forall \ t \in [0, T].$$

And from (8.3.4), (8.5.2), and (8.5.6),

$$|\phi(t, \cdot)|_{H^2} \leq \max(1, \lambda_0^2) V_1(t) + C_2 V_2(t)$$

$$\leq \max(4, 4\lambda_0^2, C_2) e^{-\gamma t} V(0).$$

Thus, there exists $C_8 > 0$ independent of $\phi$ and $I$ such that

$$|\phi(t, \cdot)|_{H^2} \leq C_8 e^{-\gamma t} (|\phi(0, \cdot)|_{H^2}).$$

So far $\phi$ is assumed to be of class $C^2$, however since this inequality only involves the $H^2$ norm of $\phi$, this can be extended to any solution $(\phi, I) \in C^0([0, T], H^2(0, L)) \times C^1([0, T])$ of the system (8.2.1)–(8.2.3) (see for instance [20] for more details). This concludes the proof of Theorem 8.2.3. We now prove Proposition 8.2.4, which follows rapidly from the proof of Theorem 8.2.3.

Proof of Proposition 8.2.4. From (8.8.15) in the Appendix, one can see that (8.5.21) still holds with $k_1 = \pi \lambda_0/2L$. Thus by continuity there exists $k_1 > \pi \lambda_0/2L$ such that for any $k_1 \in (\pi \lambda_0/2L, k_1)$ (8.5.21) still holds and consequently the quadratic form $Q$ given by (8.5.14) is still definite positive. Suppose now by contradiction that the system is stable for the $H^2$ norm. Then for any $\varepsilon > 0$, there exists $\delta_1 > 0$ such that for any initial condition $(\phi_0, I_0) \in H^2(0, L) \times \mathbb{R}$ such that $(|\phi_0|_{H^2} + |I_0|) \leq \delta_1$ and satisfying the compatibility condition $I_0 = -k_1^{-1} \phi_0(0)$ and $\phi(L) = k_1^{-1} \lambda(\phi_0(0)) \phi_0(L)$, the associated solution $(\phi, I)$ is defined on $[0, +\infty)$ and

$$\left( |\phi|_{H^2} + |I| \right) \leq \varepsilon, \forall \ t \in [0, +\infty).$$

(8.5.62)
Let $\Theta > 0$, from (8.5.11) and (8.5.52), using that $Q > 0$,
\[
\frac{dV_1 - \Theta V_2}{dt} \geq 2\text{Re}(\eta_1) \min(\lambda_0, 1)V_1 + \mu \Theta C_3 V_2 \\
+ \left( \Theta f(L)\lambda_0 e^{-\mu \frac{|k_0|}{\varepsilon}} - C_3 (1 + \Theta) \left( \|\phi\|_{H^2} + |I| + |I| + |I| + |I| \right) \right) \left( \sum_{k=1}^{k=3} |(\partial_t^{k-1}\phi_k)(t, L)|^2 \right) \\
+ O \left( \left( \|\phi_{g_1}\|_{H^2} + |I_{g_1}| + |\phi_2|_{H^2} + |I_2(t)| + |I_2(t)| + |I_2(t)| \right)^3 \right),
\]  
(8.5.63)

where $C_3$ is a constant independent of $\phi$ and $I$. We can choose $(\phi_0, I_0)$ satisfying the compatibility conditions and $\Theta > 0$ such that $c := (V_1 - \Theta V_2)(0) > 0$, and $(\|\phi_0\|_{H^2} + |I_0|) \leq \delta$ with $\delta$ to be chosen. Actually $\Theta$ only depends on the ratio between $V_1$ and $V_2$ thus it can be made independent of $\delta$ by simply rescaling $|\phi_0|_{H^2}$ and $|I_0|$. Using (8.5.63) and (8.5.57) there exists $\gamma_2 > 0$ and $\varepsilon > 0$ such that, if $(\|\phi\|_{H^2} + |I|) \leq \varepsilon$, then
\[
\frac{dV_1 - \Theta V_2}{dt} \geq \gamma_2 (V_1 - \Theta V_2).
\]  
(8.5.64)

Thus, from (8.5.62) and the stability hypothesis, we can choose $\delta > 0$ such that (8.5.64) holds. This implies that
\[
(V_1 - \Theta V_2)(t) \geq ce^{\gamma_2 t}, \quad \forall \ t \in [0, +\infty),
\]  
(8.5.65)

which contradicts (8.5.62). This ends the proof of Proposition 8.2.4.

**Remark 8.5.2.** This last proof is limited by the limit value of $k_1$ for which $Q$ is not positive definite anymore. This is due to the fact that we have only extracted the first limiting eigenvalues from the solution. It is natural to think that we could apply the same method to extract a finite number of eigenvalues instead and separate $(\phi, I)$ in $(\phi_1, I_1)$, its projection on a n-dimensional space, and $(\phi_2, I_2)$. Then we would deduce more constraints like (8.4.22) on $(\phi_2, I_2)$, which would increase the upper bound of $k_1$ for which $Q$ defined in (8.5.14) is definite positive, and thus the bound $k_1$ for which Proposition 8.2.4 holds, and maybe, by increasing this number of eigenvalues, prove that this proposition holds for arbitrary large $k_1$.

### 8.6 Numerical simulations

In this section we give a numerical simulation that illustrates Theorem 8.2.3 and Proposition 8.2.4. In this example we use $\lambda(z) = 1 + z$, this corresponds to the study of the Burgers equation around the constant steady-state with value 1, as in this case $y = 1 + z$ is solution to the Burgers equation $\partial_t y + y \partial_x y = 0$.

### 8.7 Conclusion

In this article we studied the exponential stability of a general nonlinear transport equation with integral boundary controllers and we introduced a method to obtain an optimal stability condition through a Lyapunov approach, by extracting first the limiting part of the stability from the solution using a projector on a finite-dimension space. We believe that this method could be used for many other systems and could be useful in the future as, for many nonlinear systems governed by partial differential equations, the stability conditions that are known today are only sufficient and may still be improved.

### 8.8 Appendix

In this section we prove (8.5.21) under assumption (8.2.10). Note that this is equivalent to
\[
\left( \int_0^L \theta(x)dx \right)^2 \geq 1 - \frac{\lambda_0^2}{k^2 L^2}.
\]  
(8.8.1)

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Figure 8.1: Example of numerical simulations of $\phi(t,0)$ with respect to $t$ varying between 0 and 10 for various values of $k_I$ between $0.1k_{I,c}$ to $2k_{I,c}$, where $k_{I,c} = \pi \lambda_0 / 2L$ is the critical value of Theorem 8.2.3 and Proposition 8.2.4. The black line represents the trajectory for $k_I = k_{I,c}$. On the left $k_I$ is larger and the system is unstable, and on the right $k_I$ is smaller and the system is stable. As expected, the exponential decay observed in the stable region is the same as the exponential decay given by the real part of $\rho_1$. The system parameters are chosen such that $\lambda(z) = 1 + z$, $\lambda_0 = L = 1$, and $\phi_0(x) = 0.1$ on $[0,L/2]$ and $\phi_0(L) = 0$ so that $\phi_0$ satisfies the compatibility conditions (8.2.4) for any $k_I \in [0.1k_{I,c},2k_{I,c}]$. The simulations are obtained by a finite-difference method.

By definition of $\varrho_1$ (see Section 8.4) and (8.4.3), we have

$$\sigma = -\frac{\omega}{\tan(\omega L)}, \tag{8.8.2}$$

and using (8.4.3) and (8.8.2)

$$\frac{\lambda_0}{k_I L} = -\frac{\sin(\omega L)}{\omega L} e^{-\frac{\omega L}{\tan(\omega L)}}. \tag{8.8.3}$$

Condition (8.8.1) thus becomes

$$\left(\frac{\int_0^L \theta(x) dx}{L (\int_0^L \theta^2(x) dx)}\right)^2 + \frac{\sin^2(\omega L)}{(\omega L)^2} e^{2 \frac{\omega L}{\tan(\omega L)}} - 1 > 0. \tag{8.8.4}$$

From (8.2.8) and the definition of $\theta$ given by (8.5.15), we have

$$\int_0^L \theta(x) dx = \frac{\omega}{\sigma^2 + \omega^2}. \tag{8.8.5}$$

Using (8.8.2),

$$\left(\int_0^L \theta(x) dx\right)^2 = \frac{\sin^4(\omega L)}{\omega^2}. \tag{8.8.6}$$

Similarly we have

$$\int_0^L \theta^2(x) dx = \frac{\sigma e^{-2\sigma L} (\sigma \cos(2\omega L) - \omega \sin(2\omega L)) + (\omega^2 + \sigma^2)}{4\sigma (\sigma^2 + \omega^2)}. \tag{8.8.7}$$
Therefore, using again (8.8.2) and the fact that \(1 + \tan^{-2}(\omega L) = \sin^{-2}(\omega L)\),
\[
\int_0^L \theta^2(x)dx = \left(\frac{\cos^2(\omega L)}{\sin^2(\omega L)} e^{2\frac{\omega L}{\gamma(L)}} (\cos^2(\omega L) + \sin^2(\omega L)) - e^{2\frac{\omega L}{\gamma(L)}} \frac{1}{\sin^2(\omega L)} + 1\right) \tan(\omega L)
\]
\[
= \frac{\left(\frac{e^2 \frac{\omega L}{\gamma(L)}}{\sin^2(\omega L)} - 1\right)}{4\omega} \sin^2(\omega L) \tan(\omega L).
\]

Therefore using (8.8.6) and (8.8.8), condition (8.8.4) becomes
\[
\frac{4\sin^2(\omega L)}{(\omega L) \tan(\omega L)} + \frac{\sin^2(\omega L)}{(\omega L)^2} e^{2\frac{\omega L}{\gamma(L)}} - 1 > 0,
\]
which is equivalent to
\[
\frac{2}{\left(\frac{e^{2\frac{\omega L}{\gamma(L)}}}{\sin^2(\omega L)} - 1\right)} + e^{2\frac{\omega L}{\gamma(L)}} \frac{\sin^2(\omega L)}{(\omega L)^2} = 1 > 0.
\]

Note that, under assumption (8.2.10) and from the definition of \(g_1\), \(\omega L \in (-\pi/2, \pi/2)\), which implies that \(2(\omega L)/\tan(\omega L) \in (0, 2)\). Hence, let us study the function \(g: X \rightarrow (2X/(e^X - 1) + e^X)\) on (0, 2). Taking its derivative one has
\[
g'(X) = \frac{(e^X - 1)(2 + e^X(e^X - 1) - 2Xe^X)}{(e^X - 1)^2}.
\]

Taking again the derivative of the numerator of the right-hand side of (8.8.11), one has
\[
((e^X - 1)(2 + e^X(e^X - 1) - 2Xe^X))' = (e^X - 1)(e^X(e^X - 1) + e^{2X})
\]
\[
\quad + e^X(2 + e^X(e^X - 1)) - 2e^X - 2Xe^X.
\]

Thus using that \(X < e^X - 1\) on (0, +\(\infty\)) and in particular on (0, 2), we get
\[
((e^X - 1)(2 + e^X(e^X - 1) - 2Xe^X))' > (e^X - 1)(e^X(e^X - 1) + 2e^X - 2e^X) > 0.
\]

Hence \(g'\) is non-decreasing on (0, 2). But, from (8.8.11), \(g'(0) = 0\), therefore \(g\) is non-decreasing on (0, 2). As \(\lim_{X \to 0} g(X) = 3\), we have
\[
\left(\frac{2}{\left(\frac{e^{2\frac{\omega L}{\gamma(L)}}}{\sin^2(\omega L)} - 1\right)} + e^{2\frac{\omega L}{\gamma(L)}} \frac{\sin^2(\omega L)}{(\omega L)^2}\right) - 1 \geq \frac{12}{\pi^2} - 1 > 0.
\]

and, as \(x \to \sin(x)/x\) is positive and decreasing on \([0, \pi/2]\), we have
\[
\left(\frac{2}{\left(\frac{e^{2\frac{\omega L}{\gamma(L)}}}{\sin^2(\omega L)} - 1\right)} + e^{2\frac{\omega L}{\gamma(L)}} \frac{\sin^2(\omega L)}{(\omega L)^2}\right) - 1 \geq \frac{12}{\pi^2} - 1 > 0.
\]

Hence (8.8.10) holds and therefore condition (8.8.1) holds as well. This ends the proof of (8.5.21) under assumption (8.2.10).

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Chapter 9

PI controllers for the general Saint-Venant equations

This chapter is taken from the following article (also referred to as [108]):


Abstract. We study the exponential stability in the $H^2$ norm of the nonlinear Saint-Venant (or shallow water) equations with arbitrary friction and slope using a single Proportional-Integral (PI) control at one end of the channel. Using a local dissipative entropy we find a simple and explicit condition on the gain of the PI control to ensure the exponential stability of any steady-states. This condition is independent of the slope, the friction coefficient, the length of the river, the inflow disturbance and, more surprisingly, can be made independent of the steady-state considered. When the inflow disturbance is time-dependant and no steady-state exist, we still have the Input-to-State stability of the system, and we show that changing slightly the PI control enables to recover the exponential stability of slowly varying trajectories.

9.1 Introduction

Discovered in 1871, the Saint-Venant equations [10] (or 1-D shallow water equations) are among the most famous equations in fluid dynamics and have been investigated in hundreds of studies. Their richness, although being quite simple, has made them become a major tool in practice for many industrial goal, the most famous being probably the regulation of navigable rivers. They are the ground model for such purpose in France and Belgium. Regulation of rivers is a major issue, for navigation, freight transport, renewable energy production, but also for safety reasons, especially as several nuclear plants all around the world are implanted close to rivers. For these reasons, the stability of the steady-states of the Saint-Venant equations has been, and is still, a major issue.

Many results were obtained in the last decades. In 1999, the robust stability of the homogeneous linearized Saint-Venant equations was shown using a Lyapunov approach and proportional feedback controllers [52]. Later the stability of the homogeneous nonlinear Saint-Venant equations was achieved, still using proportional feedback controllers. In 2008, through a semi-group approach [83], the stability of the inhomogeneous nonlinear Saint-Venant equation was shown for sufficiently small friction and slope (or equivalently sufficiently small canal), and these results were successfully applied to real data sets from the Sambre river in Belgium. More recently, in [16] the authors have given sufficient conditions to stabilize the nonlinear Saint-Venant equations with arbitrary friction for the $H^2$ norm but no slope using again proportional feedback controllers, and in [110] with both arbitrary friction and slope. This last result being proved by exhibiting an explicit local entropy for the nonlinear inhomogeneous Saint-Venant equations.

It is worth mentioning that other stability results have also been obtained in less classical cases or with less
classical feedbacks. For instance, in [19] was shown the rapid stabilization of the homogeneous nonlinear Saint-Venant equations when a shock (e.g. a hydraulic jump) occurs in the target steady-state. Such shock induces new difficulties and the presence of shocks can limit in general the controllability and the stability in weaker norms of hyperbolic systems with boundary controls [2] [44]. Also, several results (e.g. [65]) were obtained using a backstepping approach, a very powerful method based on a Volterra transformation, developed mainly for PDE in [128], and generalized recently with a Fredholm transformation for hyperbolic systems [60] [200] [201]. One may look at [110] for a more detailed survey about this method and its use for the Saint-Venant equations. However, backstepping gives rise to non-local and non-static feedback laws that are likely to be harder to implement, and, to our knowledge, have not been implemented yet.

Most of the previous results were performed with static proportional feedback controllers. When it comes to industrial applications, however, the proportional integral (PI) control is by far the most popular regulator. It is used for instance for the regulation of the Sambre and Meuse river in Belgium [13, Chapter 8]. The reason behind such preference is the robustness of the PI control with off-set errors [6, Chap. 11.3]. An example can be found in [84] where the authors show the interest of adding an integral term to a proportional control on a linear and homogeneous system, and exhibit coherent experimental result.

For these reasons, the PI controller has fed a wide literature, at least when used on finite dimensional systems. However, despite their indisputable practical interest, PI controllers for nonlinear infinite dimensional systems have shown hard to handle mathematically and even studying simple systems give sometimes rise to lengthy proofs with relatively sophisticated tools [59]. While the behaviour and the stability of linearized equations with PI controller has been well understood in the past, partly thanks to spectral tools like the spectral mapping theorem (e.g. [141], [159] for hyperbolic systems), no such tools exist for nonlinear systems and the stability of the nonlinear Saint-Venant equations has remained a challenge until today. Among the existing linear result using a spectral approach one can refer to [198], [199] where the authors find a sufficient condition for the stabilization of the linearized inhomogeneous Saint-Venant equations. Necessary and sufficient conditions for the linearized homogeneous Saint-Venant equations are given in [13, Section 2.2.4.1, 3.4.4]. In [61] the authors find a necessary and sufficient condition for a linear scalar equation and show the difficulty of finding good conditions for the nonlinear equation, while in [22] the authors deal with 2 × 2 systems. Among the existing nonlinear results one can refer to [186] in the case where the operator without PI control generates an exponentially stable semi-group, [191] where the authors find a sufficient condition for the nonlinear homogeneous Saint-Venant equations. [13, 2.2.4.2] where the authors find a necessary and sufficient condition also for the nonlinear homogeneous Saint-Venant equations, while [13, 5.4.4,5.5] give a sufficient condition for the inhomogeneous Saint-Venant equations for a single channel or a network, but in the particular case of constant steady-states only, which simplifies their analysis [106]. Strictly speaking this last result was derived for the linearized system but with a Lyapunov approach which can easily be generalized to the nonlinear system. More recently, and to our knowledge this is the most advanced result, [17] gave a sufficient condition of stability for the inhomogeneous Saint-Venant equations with an arbitrary friction and river length but only in the absence of slope, using a Lyapunov approach.

In this chapter, we consider the stabilization of the general nonlinear Saint-Venant equations with a single boundary PI control. We give a simple and explicit condition on the parameters of the PI controller such that any steady-state is exponentially stable for the $H^2$ norm. While stability results in inhomogeneous and nonlinear systems often imply a limit length for the domain, depending on the source term, above with we are unable to guarantee any stability ([111], [83], [106], [107] or [13, Chap. 6]), this result holds whatever the friction, the slope, and the length of the channel. Besides, our condition is independent of the slope, the friction coefficient, the river length, and, more surprisingly, can be made independent of the steady-state considered. Finally, when there is no slope this condition is less restrictive that the condition obtained in [17] and when there is no friction or slope this condition coincides with the necessary and sufficient spectral condition of stability for the linearized system given in [22] and [13, Theorem 2.7].

The case where the inflow disturbances are time dependent and no steady-states exists was seldom considered in the literature. However, it is in fact unlikely that the industrial target state is a real steady-state as the inflow disturbance often depends on time in practice, even though only slowly. Therefore, in the more general framework of slowly time-varying target states, we show the Input-to-State Stability (ISS) of the system with
respect to the variation of the inflow disturbance. Finally, we show that if we allow the controller to depend on the target state, by changing slightly the PI controller, we can ensure the exponential stability of slowly-varying target trajectories that are the natural target trajectories to consider when there is no steady-state of the system.

This paper is organised as follows: in Section 9.2 we give a description of the nonlinear Saint-Venant equations, we introduce the time-varying target trajectories together with some definitions and existence results, then we state our main results. In Section 9.3 we prove our main result, Theorem 9.2.3, that deals with the exponential stability of time-varying state. In the Appendix, we show that Corollary 2 dealing with the exponential stability of steady-states, and Theorem 9.2.4 showing the ISS of the system with respect to the variation of the inflow disturbance, are both deduced from the proof of Theorem 9.2.3.

9.2 Model description

We consider the following nonlinear Saint-Venant equations for a rectangular channel with arbitrary slope and friction.

\[
\begin{align*}
\partial_t H + \partial_x (HV) &= 0, \\
\partial_t V + V \partial_x V + g \partial_x H + \left( \frac{kV^2}{H} - C(x) \right) &= 0.
\end{align*}
\] (9.2.1)

Here, \( k \) is an arbitrary nonnegative friction coefficient and \( C(x) \) denotes the slope, which is assumed to be a \( C^2 \) function, with \( C(x) := -gdB/dx \) where \( B \) is the bathymetry and \( g \) the acceleration of gravity. We are interested in systems where the water flow uphill is a given function, unknown and imposed by external conditions, for instance a flow coming from another country, while the water flow downhill is controlled through a hydraulic installation. Therefore we have the following boundary conditions,

\[
\begin{align*}
H(t,0)V(t,0) &= Q_0(t), \\
H(t,L)V(t,L) &= U(t),
\end{align*}
\] (9.2.2)

where \( U(t) \) is a control feedback and \( Q_0(t) \) is the incoming flow, which is a given (and unknown) function. Here \( L \) denotes the length of the water channel. In practical situations, the formal control \( U(t) \) can be expressed by a simple linear model

\[
U(t) = v_G (H(t,L) - U_1(t)),
\] (9.2.3)

where \( U_1(t) \) is the elevation of the gate of the dam, which is the real control input that can be chosen, while \( v_G \) is a constant depending on the parameters of the gate (potentially unknown as well).

Usually, the industrial goal of such system is to stabilize the level of the water at the end point \( H(t,L) \), called control point, to a target value \( H_c > 0 \). On the other hand, the usual mathematical goal in such problem is to stabilize a target steady-state \((H^*, V^*)\), potentially nonuniform [13][Preface]. However, in the present problem (9.2.1)–(9.2.2), it is clear that, when \( Q_0 \) is not constant, it is impossible to aim at stabilizing any steady-state and one needs to aim at stabilizing other target trajectories. Therefore, we define the following target trajectory \((H_1, V_1)\) that we aim stabilizing as the solution of

\[
\begin{align*}
\partial_t H_1 + \partial_x (H_1 V_1) &= 0, \\
\partial_t V_1 + V_1 \partial_x V_1 + g \partial_x H_1 + \left( \frac{kV_1^2}{H_1} - C(x) \right) &= 0, \\
H_1(t,0)V_1(t,0) &= Q_0(t), \\
H_1(t,L) &= H_c,
\end{align*}
\] (9.2.4)

with the initial condition

\[
H_1(0,\cdot) = H^*(\cdot) \text{ and } V_1(0,\cdot) = V^*(\cdot),
\] (9.2.5)

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where \((H^*, V^*)\) is the (unique) steady-state solution of the system when \(Q_0\) is constant, equal to \(Q_0(0)\). Namely \((H^*, V^*)\) is the solution of

\[
\partial_x(HV) = 0, \\
V\partial_x V + g\partial_x H + \left( \frac{kV^2}{H} - C(x) \right) = 0, \\
H(L) = H_c,
\]

with condition at \(x = 0\)

\[
H^*(0)V^*(0) = Q_0(0).
\]

We are now going to show that the trajectory \((H_1, V_1)\) exists for any time and satisfies some bounds.

**Existence and bounds of the target trajectory** Instead of studying directly our target trajectory \((H_1, V_1)\) we first construct an intermediary family of functions \((H_0, V_0)\). We defined previously \((H^*, V^*)\) as the steady-state associated to a constant flow rate \(Q_0 \equiv Q_0(0)\), i.e. \((H^*, V^*)\) is the solution of the ODE problem \((9.2.6)\) with initial condition \(H^*(0)V^*(0) = Q_0(0)\). But in fact at each time \(t^* \in \mathbb{R}_+\), we can define a steady-state \((H^*_t, V^*_t)\) associated to a constant flow rate \(Q_0 \equiv Q_0(t^*)\), i.e. \((H^*_t, V^*_t)\) is the solution of the ODE problem \((9.2.6)\) with initial condition satisfying

\[
H^*_t(0)V^*_t(0) = Q_0(t^*). 
\]

This problem could seem peculiar as all conditions should be imposed exclusively in 0 or in \(L\) to ensure the well-posedness. However looking at the first equation of \((9.2.6)\), the problem \((9.2.6)\), \((9.2.8)\) is in fact equivalent to a single ODE on \(H^*_t\) with boundary condition \(H^*_t(L) = H_c\) and \(V^*_t\) defined by \(V^*_t = Q_0(t)/H^*_t\). Thus for each \(t^* \in [0, +\infty)\) such function exists on \([0, L]\), is unique and \(C^3\) provided that the state stays in the fluvial regime (or subcritical regime), i.e. \(gH^*_t > V^*_t^2\) on \([0, L]\), which, for a given \(H_c\), is equivalent to a bound on \(Q_0(t^*)\) (see \([110]\) for more details). As we are interested in stabilizing physical trajectories in the fluvial regime, we assume that this assumption is satisfied in the following and that there exist \(\alpha > 0\) and \(H_\infty > 0\) independent of \(t^* \in [0, \infty)\) such that

\[
H^*_t < \frac{1}{2}H_\infty \text{ on } [0, L], \\
gH^*_t - V^*_t^2 > 2\alpha \text{ on } [0, L].
\]

For a given \(H_c\), this is again equivalent to imposing a bound \(Q_\infty\) on \(\|Q_0\|_{L_\infty([0, \infty))}\), from \((9.2.6)\) and \((9.2.8)\), which would be more logical. However, for convenience, we will still use \(H_\infty\) and \(\alpha\) in the following. This assumption is quite physical, especially as in practical situation the river is in fluvial regime and \(Q_0(t)\) is often periodic or quasi-periodic. This gives a family of one-variable functions indexed by a parameter \(t^*\), which can also be seen as the two-variable functions \((H_0, V_0) : (t, x) \rightarrow (H^*_t(x), V^*_t(x))\). Besides, from \((9.2.7)\), as \((H^*_t, V^*_t)\) is the solution of a system of ODE with a parameter \(t\), the two variable functions \((H_0, V_0)\) therefore belongs to \(C^3([0, +\infty) \times (0, L))\) (see \([105]\) Chap. 5, Cor. 4.1). And from its definition, one can note that \((H_0(0, \cdot), V_0(0, \cdot)) = (H^*, V^*)\) Now that we have introduced this intermediary family of functions, we can show the existence of the target trajectory \((H_1, V_1)\) and we have the following Input-to-State Stability (ISS) result (see \([181]\) for a definition of ISS for finite dimensional systems, \([117]\) Chap 1, Chap 3) for a generalization to first-order hyperbolic PDE and \([168]\) for the use of Lyapunov function to achieve ISS on time-varying hyperbolic systems).

**Proposition 9.2.1.** There exist positive constants \(c_1, c_2\) such that if \(\partial_t Q_0 \in C^2([0, \infty))\), there exist \(\mu > 0\), \(\nu > 0\) and \(\delta > 0\) such that if \(\|\partial_t Q_0\|_{C^2([0, +\infty))} \leq \delta\), then for any \((H^0, V^0)\) in \(H^2((0, L), \mathbb{R}^2)\) such that

\[
\|H^0_1 - H^*\|_{L^2(0, L)} + \|V^0_1 - V^*\|_{L^2(0, L)} \leq \nu,
\]

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the system \((9.2.4)\) with initial condition \((H_0^0, V_1^0)\) has a unique solution \((H_1, V_1) \in C^0([0, +\infty), H^2(0, L))\) which satisfies the following ISS inequality
\[
\|H_1(t, \cdot) - H_0(t, \cdot)\|_{H^2(0,L)} + \|V_1(t, \cdot) - V_0(t, \cdot)\|_{H^2(0,L)} \\
\leq c_1(\|H_0^0 - H^*\|_{H^2(0, L)} + \|V_1^0 - V^*\|_{H^2(0, L)}) e^{-\frac{\alpha t}{2}} + c_2 \left( \int_0^t |\partial_t Q_0(s) + \partial_{tt}^2 Q_0(s) + \partial_{ttt}^3 Q_0(s)| e^{\frac{\alpha s}{2}} ds \right) e^{-\frac{\alpha t}{2}}.
\] (9.2.10)

This result is shown in Appendix \(9.5.1\) and a definition of the \(C^2\) norm is recalled in Remark \(9.2.1\). Note that \(Q_0\) is supposed to be bounded, which is quite physical, but there is no additional requirement on this bound besides the physical assumption given by \(Q_\infty\) of remaining in the fluvial regime. This is important as in practical situations the value of the incoming flow can change a lot, even though slowly.

Here, we choose to stabilize the trajectory \((H_1, V_1)\) associated to \(H_0^0 = H^*\) and \(V_1^0 = V^*\). As we will see, this target trajectory can be seen as the natural trajectory to stabilize as it satisfies the industrial goal \(H(t, L) = H_c\) and it coincides with the steady-state solution when \(Q_0\) is a constant. In this last case \(Q_0\) and \(H_c\) are imposed and \(H^*\) and \(V^* = Q_0/H^*\) are thus fully determined using \((9.2.6)\). But one can note from \((9.2.10)\) that, in fact, the behaviour of \((H_1, V_1)\) at large time does not depend on the initial condition \((H_0^0, V_1^0)\) in \((9.2.5)\), provided that it is close in \(H^2\) norm to \((H^*, V^*)\).

**Remark 9.2.1.** The same ISS result can be shown replacing the \(H^2\) norm in Proposition \(9.2.1\) by the \(H^p\) norm where \(p \in \mathbb{N} \setminus \{1\}\), with the condition \(\|\partial_t Q_0\|_{C^p([0, +\infty))} \leq \delta\) instead of \(\|\partial_t Q_0\|_{C^2([0, +\infty))} \leq \delta\). This is shown in Appendix \(9.5.7\). We define here the \(C^p\) norm for a function \(U \in C^p(I)\), where \(I\) is an interval, as
\[
\|U\|_{C^p(I)} := \max_{i \in [0, p]} \|\partial_t^i U\|_{L^\infty(I)}
\] (9.2.11)

Thus, from Proposition \(9.2.1\) and \(9.2.9\), there exists a constant \(\delta > 0\) such that, if \(\|\partial_t Q_0\|_{C^2([0, +\infty))} < \delta\), then \((H_1, V_1) \in C^0([0, +\infty), H^2(0, L))\) and
\[
H_1(t, x) < H_\infty, \forall (t, x) \in [0, +\infty) \times [0, L],
\] (9.2.12)
\[
gH_1(t, x) - V_1^0(t, x) > \alpha, \forall (t, x) \in [0, +\infty) \times [0, L].
\] (9.2.13)

Besides, when \(Q_0\) is a constant, it is easy to check that \((H_0, V_0) = (H^*, V^*)\) is also solution of \((9.2.4)\)\-\((9.2.5)\). Thus, from the uniqueness of the solution of \((9.2.4)\)\-\((9.2.5)\), \((H_1, V_1) = (H^*, V^*)\) and therefore we recover a steady-state. This illustrates that \((H_1, V_1)\) can be seen as the natural target state when \(Q_0\) is not a constant anymore. Moreover, from \((9.2.4)\), stabilizing \((H_1, V_1)\) also satisfies the industrial goal by stabilizing \(H(t, L)\) on the value \(H_c\).

**Control design and main result** As mentioned in the introduction, a usual type of controller used in practice to reach this aim is the proportional-integral (PI) controller. It has the advantage of eliminating the offset coming from constant load disturbances, which can usually appear in these systems as the command on the gate’s level are only known up to some constant uncertainties. A generic PI controller is given by
\[
U_1(t) = k_p(H_c - H(t, L)) + k_i Z,
\] (9.2.14)
where \(k_p\) and \(k_i\) are coefficients that can be designed and \(Z\) accounts for the integral term, i.e.
\[
\dot{Z} = H_c - H(t, L).
\] (9.2.15)

With such controller, and using \((9.2.3)\), the boundary conditions \((9.2.2)\) become \((9.2.15)\) and
\[
H(t, 0)V(t, 0) = Q_0(t),
\]
\[
H(t, L)V(t, L) = v_G(1 + k_p)H(t, L) - v_G k_p H_c - v_G k_i Z,
\] (9.2.16)
In Corollary [2] we show that this boundary control can be used to stabilize exponentially a steady-state when \(Q_0\) is a constant. In Theorem 9.2.4 we show that this control can also provide an Input-to-State Stability property with respect to \(\partial_t Q_0\). However, this control (9.2.14) cannot be used to stabilize a dynamic target trajectory \((H_1, V_1)\), as there is no function \(Z_1 \in C^1(0, +\infty)\) such that \((H_1, V_1, Z_1)\) is a solution of (9.2.1), (9.2.15), (9.2.16) while \((H_1, V_1)\) is a solution of (9.2.1). Therefore, when stabilizing a dynamic target trajectory, one has to add an additional term and use

\[
U_1(t) = k_p(H_c - H(t, L)) + k_lZ - f(t),
\]

where \(f(t) := H_1(t,L)V_1(t,L)/v_G\). The boundary conditions (9.2.2) become then

\[
\begin{align*}
H(t, 0)V(t, 0) &= Q_0(t), \\
H(t, L)V(t, L) &= H_1V_1(t, L) + v_G(1 + k_p)(H(t, L) - H_c) - v_Gk_lZ,
\end{align*}
\]

where we have actually changed \(Z\) and re-define \(Z := Z - k_p/k_l\), which still satisfies the equation (9.2.15). This new control (9.2.17) assumes that \(V_1(t, L)\) is known at least up to a constant, as \(H_1(t, L) = H_c\) and additional constants can be incorporated into \(Z\). When no knowledge on the target state is available besides \(H_c\), it is impossible to stabilize exponentially the system, and the best one can get is the Input-to-State Stability which is given by Theorem 9.2.4. However in the following we will keep working with (9.2.17) and (9.2.18) to show Theorem 9.2.3 and the exponential stability of the system, as the proof of Theorem 9.2.4 and Corollary 2 which uses only the control (9.2.14) and (9.2.16) are easily deduced from the proof of Theorem 9.2.3.

We introduce the first-order compatibility conditions associated to the boundary conditions (9.2.18) for an initial condition \((H^0, V^0, Z^0)\).

\[
\begin{align*}
H^0(0)V^0(0) &= Q_0(0), \\
H^0(L)V^0(L) &= H_1V_1(0, L) + v_G(1 + k_p)(H^0(L) - H_c) - k_lZ^0, \\
-\partial_x(H^0(0)V^0(0) + g\frac{H^0(0)^2}{2} - (k(V^0)^2(0) - CH^0(0))) &= Q_0(0), \\
-\partial_x(H^0(L)V^0(L) + g\frac{H^0(L)^2}{2} - (k(V^0)^2(L) - CH^0(L))) &= \partial_t(H_1V_1)(0, L) \\
&- v_G(1 + k_p)\partial_x(H^0(L)V^0(L)) + k_l(H^0(L) - H_c).
\end{align*}
\]

With such compatibility conditions the system (9.2.1), (9.2.15), (9.2.18) is well-posed and we have the following theorem due to Wang [196] [Theorem 2.1]:

**Theorem 9.2.2.** Let \(T > 0\), and assume that \(\|\partial_t Q_0\|_{C^0([0,T],H^3(0,L))} \leq \delta(T), \) such that \((H_1, V_1)\) is well-defined and belongs to \(C^0([0,T],H^3(0,L))\). There exists \(\nu(T) > 0\) such that for any \((H^0, V^0, Z^0) \in (H^2((0,L)), C^1(0,T)) \times \mathbb{R}\) satisfying

\[
\|H^0(\cdot) - H_1(0,\cdot)\|_{H^2(0,L)} + \|V^0(\cdot) - V_1(0,\cdot)\|_{H^2(0,L)} + |Z^0| \leq \nu(T),
\]

and satisfying the compatibility conditions (9.2.19), the system (9.2.1), (9.2.15), (9.2.18) has a unique solution \((H, V, Z) \in (C^0([0,T],H^2((0,L)))) \times (C^1(0,T))\). Moreover there exists a positive constant \(C(T)\) such that

\[
\|H(t, \cdot) - H_1(t, \cdot)\|_{H^2(0,L)} + \|V(t, \cdot) - V_1(t, \cdot)\|_{H^2(0,L)} + |Z|
\leq C(T) \left(\|H^0(\cdot) - H_1(0,\cdot)\|_{H^2(0,L)} + \|V^0(\cdot) - V_1(0,\cdot)\|_{H^2(0,L)} + |Z^0|\right).
\]

To apply the result from [196], note that \(Z\) can be seen as a third component of the hyperbolic system with a null propagation speed, a constant initial condition \(Z^0\) and \(Z(t)\) being thus its value everywhere on \([0,L]\) including at the boundaries.
Remark 9.2.2. If, in addition, \((H^0, V^0) \in H^3((0, L); \mathbb{R}^2)\), then the unique solution \((H, V, Z)\) given by Theorem 9.2.2 belongs to \(C^1([0, T], H^3((0, L); \mathbb{R}^2)) \times C^1([0, T])\) and there exists a constant \(C(T)\) such that
\[
\|H(t, \cdot) - H_1(t, \cdot)\|_{H^3(0, L)} + \|V(t, \cdot) - V_1(t, \cdot)\|_{H^2(0, L)} + |Z| \leq C(T) (\|H^0(\cdot) - H_1(0, \cdot)\|_{H^3(0, L)} + \|V^0(\cdot) - V_1(0, \cdot)\|_{H^2(0, L)} + |Z_0|). \tag{9.2.22}
\]
We recall the definition of exponential stability.

Definition 9.2.1. We say that a trajectory \((H_1, V_1)\) is exponentially stable for the \(H^2\) norm if there exists \(\nu > 0, C > 0\) and \(\gamma > 0\) such that for any \(T > t_0 \geq 0\) and any \((H^0, V^0, Z^0)\) satisfying
\[
\|H^0(\cdot) - H_1(t_0, \cdot)\|_{H^2(0, L)} + \|V^0(\cdot) - V_1(t_0, \cdot)\|_{H^2(0, L)} + |Z^0| \leq \nu, \tag{9.2.23}
\]
and the compatibility conditions \((9.2.19)\), the system \((9.2.1), (9.2.15), (9.2.18)\) with initial condition \((H^0, V^0, Z^0)\) at \(t_0\) has a unique solution \((H, V, Z) \in (C^0([t_0, T], H^2((0, L)))) \times C^1([t_0, T])\) and,
\[
\|H(t, \cdot) - H_1(t, \cdot)\|_{H^2(0, L)} + \|V(t, \cdot) - V_1(t, \cdot)\|_{H^2(0, L)} + |Z| \leq C e^{-\gamma t} (\|H^0(\cdot) - H_1(t, \cdot)\|_{H^2(0, L)} + \|V^0(\cdot) - V_1(t, \cdot)\|_{H^2(0, L)} + |Z^0|), \tag{9.2.24}
\forall t \in [t_0, +\infty).
\]

Remark 9.2.3. From (9.2.4) and Sobolev inequality, this exponential stability implies in particular the exponential convergence of \(H(t, L)\) to \(H_c\).

We can now state the main results of this article.

Theorem 9.2.3. There exists \(\delta > 0\) such that if \(\|\partial_t Q_0\|_{C^1((0, +\infty))} \leq \delta\), then the trajectory \((H_1, V_1)\) given by (9.2.4) of system (9.2.1), (9.2.15), (9.2.18) is exponentially stable for the \(H^2\) norm if :
\[
k_p > -1 \quad \text{and} \quad k_1 > 0, \quad \text{or} \quad k_p < -1 - \frac{g H_1(t, L) - V_1^2(t, L)}{v_G V_1(t, L)} \quad \text{and} \quad k_1 < 0. \tag{9.2.25}
\]
This result is proved in Section 9.3. The main idea of the proof consist in finding a local convex and dissipative entropy for the system (9.2.1), (9.2.15), (9.2.18).

In particular, in the case where \(Q_0\) is constant, we can use the static boundary control (9.2.14), and we have the following corollary :

Corollary 2. If \(Q_0\) is constant, then the steady-state \((H^*, V^*)\) of the system (9.2.1), (9.2.15), (9.2.16) given by (9.2.6)–(9.2.7) is exponentially stable for the \(H^2\) norm if :
\[
k_p > -1 \quad \text{and} \quad k_1 > 0, \quad \text{or} \quad k_p < -1 - \frac{g H^*(L) - V^*2(L)}{v_G V^*(L)} \quad \text{and} \quad k_1 < 0. \tag{9.2.26}
\]
Proof. This is a particular case of Theorem 9.2.3. To see this, note, as mentioned earlier, that when \(Q_0\) is constant, then \((H_1, V_1) = (H^*, V^*)\). Then, observe that \(f(t)\) given in (9.2.17) is a constant that can be added in \(Z\) (i.e. we can re-define \(Z := Z - f(t)\), which still satisfies (9.2.15)).

Remark 9.2.4. In the literature, results about PI control of the Saint-Venant equations sometimes leave the step of modeling the spillway and use a generic formulation of the PI control on the outflow rate of the form
\[
H(t, L) V(t, L) = k_1 (H(t, L) - H_c) - k_2 Z, \tag{9.2.27}
\]
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where $Z$ is the integral term, still given by (9.2.15). Note that, with these notations, the sufficient condition of Corollary 3 becomes

$$k_p > 0 \text{ and } k_I > 0,$$

or

$$k_p < -\frac{g H^*(L) - V^2(L)}{V^*(L)} \text{ and } k_I < 0. \quad (9.2.28)$$

which is a known result in the linear case using a spectral approach. Theorem 9.2.3 and Corollary 3 show that this result remains true when the system is nonlinear, using a Lyapunov approach.

**Remark 9.2.5.** When the system is homogeneous, conditions (9.2.26) are optimal (necessary and sufficient) [22, Section 2.2.4.1].

This approach uses very little knowledge of the state of the system, as we only measure the height at the boundary $x = L$. In practical situation, however, we may have also little knowledge of the target trajectory $(H_1, V_1)$ or the input disturbance $Q_0(t)$ and we only know $H_c$. In this case we cannot use a controller of the form (9.2.18), but only a static controller of the form (9.2.16), namely

$$H(t, L)V(t, L) = v_G(1 + k_p)H(t, L) - v_Gk_pH_c - v_Gk_IZ. \quad (9.2.29)$$

In this case, it is impossible to aim at stabilizing the target trajectory $(H_1, V_1)$, but we still have the Input-to-state Stability with respect to the input disturbance $\partial_tQ_0$.

**Theorem 9.2.4.** There exists $\nu > 0$, $\delta > 0$, $\gamma > 0$ and $C$, such that if $\|\partial_tQ_0\|_{C^2([0, +\infty))} \leq \delta$, then for any $T > 0$ and $(H^0, V^0) \in (H^2(0, L))^2$ such that

$$\|H^0 - H^*\|_{H^2(0, L)} + \|V^0 - V^*\|_{H^2(0, L)} \leq \nu,$$

the system (9.2.1), (9.2.15), (9.2.16) with initial condition $(H^0, V^0)$ has a unique solution $(H, V) \in C^0([0, T], H^2(0, L))$ which satisfies the following ISS inequality

$$\|H(t, \cdot) - H_0(t, \cdot)\|_{H^2(0, L)} + \|V(t, \cdot) - V_0(t, \cdot)\|_{H^2(0, L)} \leq C e^{-\gamma t} \left( \|H^0 - H^*, V^0 - V^*\|_{H^2(0, L)} + \int_0^t \|\partial_tQ_0(s) + \partial_t^2Q_0(s) + \partial_{tt}^3Q_0(s)e^{-s}ds \right). \quad (9.2.30)$$

The proof is given in Appendix 9.5.2 and is a consequence from the proof of Theorem 9.2.3.

In Section 9.3 we prove Theorem 9.2.3.

**9.3 Exponential stability for the $H^2$ norm**

This section is divided in three parts. First we transform the system through a change of variables. Then we state three lemma, useful for the analysis. Finally we prove Theorem 9.2.3.

**9.3.1 A change of variables**

For any solution of (9.2.1), (9.2.15), (9.2.18) we define the perturbation as

$$\begin{pmatrix} h \\ v \end{pmatrix} = \begin{pmatrix} H - H_1 \\ V - V_1 \end{pmatrix}. \quad (9.3.1)$$

Let us assume that there exists $\nu \in (0, \nu_0)$ to be selected later on, such that

$$\|H^0(\cdot) - H_1(0, \cdot)\|_{H^2(0, L)} + \|V^0(\cdot) - V_1(0, \cdot)\|_{H^2(0, L)} + |Z^0| \leq \nu. \quad (9.3.2)$$
The boundary conditions (9.2.18) can be written in the following form

\[
v(t, 0) = B_1(h(t, 0), t), \\
v(t, L) = B_2(h(t, L), z, t),
\]

with

\[
\partial_t B_1(0, t) = -\frac{v_1(t, 0)}{H_1(t, 0)}, \\
\partial_t B_2(0, 0, t) = \frac{v_2(1 + k_v) - v_1(t, L)}{H_1(t, L)}, \\
\partial_t B_2(0, 0, t) = -\frac{v_2 k_v}{H_2(t, L)}.
\]

We introduce the following change of variables:

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix} = \begin{pmatrix}
  v + \sqrt{\frac{g}{H_1}} h \\
  v - \sqrt{\frac{g}{H_1}} h
\end{pmatrix}.
\]

Note that this change of variables is very similar to the change of variables used in [11] with the only difference that \((H_1, V_1)\) is not a steady-state anymore. It corresponds to the transformation in Riemann coordinates for the perturbations. Indeed, denoting \(S, F\) and \(G\) by

\[
S(x, t) = \begin{pmatrix}
  \sqrt{\frac{g}{H_1(t, x)}} & 1 \\
  -\sqrt{\frac{g}{H_1(t, x)}} & 1
\end{pmatrix}, \\
F \begin{pmatrix}
  H \\
  V
\end{pmatrix} = \begin{pmatrix}
  V \\
  g \ V
\end{pmatrix}, \\
G \begin{pmatrix}
  H \\
  V
\end{pmatrix} = \begin{pmatrix} \frac{1}{2}v^2 & 0 \\
  \gamma(1 - v^2) - C(x) \end{pmatrix},
\]

and using (9.2.1), (9.2.15), (9.2.18), (9.2.4), (9.3.1) – (9.3.5), one has

\[
\partial_t u_1 + \Lambda_1(u, x, t)\partial_x u_1 + l_1(u, x, t)\partial_x u_2 + B_1(u, x, t) = 0, \\
\partial_t u_1 - \Lambda_2(u, x, t)\partial_x u_2 + l_2(u, x, t)\partial_x u_1 + B_2(u, x, t) = 0,
\]

where,

\[
A(u, x, t) = \begin{pmatrix} A_1(u, x, t) & l_1(u, x, t) \\
  l_2(u, x, t) & A_2(u, x, t) \end{pmatrix}, \\
B(u, x, t) = \begin{pmatrix} B_1(u, x, t) \\
  B_2(u, x, t) \end{pmatrix},
\]

and thus

\[
\Lambda_1(0, x, t) = V_1 + \sqrt{\frac{g}{H_1}}, \quad \Lambda_2(0, x, t) = V_1 - \sqrt{\frac{g}{H_1}}, \\
l_1(0, x, t) = B_1(0, x, t) = 0, \quad l_2(0, x, t) = B_2(0, x, t) = 0, \\
\frac{\partial B_1}{\partial u}(0, x, t) = \gamma_1(t, x)u_1(t, x) + \gamma_2(t, x)u_2(t, x), \\
\frac{\partial B_2}{\partial u}(0, x, t) = \delta_1(t, x)u_1(t, x) + \delta_2(t, x)u_2(t, x).
\]
where

\[
\begin{align*}
\gamma_1 &= \frac{3}{4} \sqrt{g} H_1 x_1 + \frac{3}{4} V_1 x_1 - \frac{k V_2^2}{2 H_1^2} \sqrt{\frac{H_1}{g}}, \\
\gamma_2 &= \frac{1}{4} \sqrt{g} H_1 x_1 + \frac{1}{4} V_1 x_1 + \frac{k V_2^2}{2 H_1^2} \sqrt{\frac{H_1}{g}}, \\
\delta_1 &= -\frac{1}{4} \sqrt{g} H_1 x_1 + \frac{1}{4} V_1 x_1 - \frac{k V_2^2}{2 H_1^2} \sqrt{\frac{H_1}{g}}, \\
\delta_2 &= -\frac{3}{4} \sqrt{g} H_1 x_1 + \frac{3}{4} V_1 x_1 + \frac{k V_2^2}{2 H_1^2} \sqrt{\frac{H_1}{g}}.
\end{align*}
\tag{9.3.14}
\]

And for the boundary conditions, there exists \( \nu_1 \in (0, \nu_0) \) such that for any \( \nu \in (0, \nu_1) \), one has:

\[
\begin{align*}
u_1(t, 0) &= \mathcal{D}_1(u_2(t, 0), t), \\
u_2(t, L) &= \mathcal{D}_2(u_1(t, L), Z, t), \\
\check{Z} &= \frac{(u_1(t, L) - u_2(t, L))}{2} \sqrt{\frac{H_1(t, L)}{g}},
\end{align*}
\tag{9.3.15}
\]

where \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are \( C^2 \) functions and

\[
\begin{align*}
\partial_1 \mathcal{D}_1(0, t) &= \frac{\lambda_2(0)}{\lambda_1(0)}, \\
\partial_1 \mathcal{D}_2(0, 0, t) &= -\frac{\lambda_1(L) - \nu G(1 + k_p)}{\lambda_2(L) + \nu G(1 + k_p)}, \\
\partial_2 \mathcal{D}_2(0, 0, t) &= -2 \frac{\nu G k_1}{\nu G (1 + k_p) + \lambda_2(t, L)}.
\end{align*}
\tag{9.3.16}
\]

Expression (9.3.14) is simply a computation, very similar to what it done in [110] for instance, while the derivation of (9.3.15) and (9.3.16) are detailed in the appendix.

**Remark 9.3.1.** Obviously, from the change of variables (9.3.1)–(9.3.5), the exponential stability of the system (9.2.1), (9.2.15), (9.2.18) is equivalent to the exponential stability of the steady-state \( \mathbf{u}^* = 0 \) for the system (9.3.8), (9.3.15).

As the operator \( A \), given by (9.3.9), is a \( C^2 \) function in \( \mathbf{u}, t \) and \( x \) (and in particular \( C^1 \)) and as, from (9.3.13) and (9.2.13), \( \lambda_1(0, x, t) > 0 > \lambda_2(0, x, t) \), there exists \( \nu_2 \in (0, \nu_1) \) and \( E \in C^1(B_{\nu_2} \times (0, L) \times [0, +\infty); \mathcal{M}_2(\mathbb{R})) \), where \( B_{\nu_2} \subset \mathbb{R}^2 \) is the disc of radius \( \nu_2 \), such that for any \( \| \mathbf{u}(\cdot, \cdot) \|_{H^2(0, L)} \leq \nu_2 \),

\[
E(\mathbf{u}(t, x), x, t) A(\mathbf{u}(t, x), x, t) = D(\mathbf{u}(t, x), x, t) E(\mathbf{u}(t, x), x, t),
\tag{9.3.17}
\]

where \( D(\mathbf{u}(t, x), x, t) = (D_i(\mathbf{u}(t, x), x, t))_{i \in 1,2} \) is a diagonal matrix and \( Id \) is the identity matrix. Before going any further, let us note a few useful properties of these functions. For simplicity in the following we will denote for any \( n \in \mathbb{N}^* \) and any function \( U \in L^\infty((0, T) \times (0, L); \mathbb{R}^n) \) (resp. \( L^\infty((0, L); \mathbb{R}^n) \))

\[
\|U\|_\infty := \|U\|_{L^\infty((0, T) \times (0, L); \mathbb{R}^n)}, \\
\text{ (resp.} \|U\|_\infty := \|U\|_{L^\infty((0, L); \mathbb{R}^n)}). 
\tag{9.3.18}
\]

We may also denote \( \|\mathbf{u}\|_{H^2(0, L)} \) instead of \( \|\mathbf{u}(\cdot, \cdot)\|_{H^2(0, L)} \) to lighten the expressions. From the definition of \( A \) given in (9.3.9) and from (9.2.13), for \( \|\mathbf{u}\|_{H^2(0, L)} \leq \nu_2 \), there exists a constant \( C_1 \) depending only on \( H_\infty \),
α and ν2 such that we have the following estimates
\[
\max (\|\partial_1(A(u(t, x), x, t) - A(0, x, t))\|_\infty, \|\partial_2(D(u(t, x), x, t) - D(0, x, t))\|_\infty, \|\partial_1(E(u(t, x), x, t))\|_\infty) \\
\leq C_1 (\|u\|_\infty (\|\partial_1 H_1\|_\infty + \|\partial_2 V_1\|_\infty) + \|\partial_1 u\|_\infty), \\
\max (\|\partial_1(A(u(t, x), x, t) - A(0, x, t))\|_\infty, \|\partial_2(D(u(t, x), x, t) - D(0, x, t))\|_\infty, \|\partial_1(E(u(t, x), x, t))\|_\infty) \\
\leq C_1 (\|u\|_\infty (\|\partial_1 H_1\|_\infty + \|\partial_2 V_1\|_\infty) + \|\partial_2 u\|_\infty).
\]
(9.3.19)

For E and D, this comes from the fact that E and D are C∞ functions with respect to the coefficients of A (note that D is the matrix of eigenvalues of A), and that A ∈ C²(B₀; C¹([0, +∞) × [0, L])).

### 9.3.2 Three useful lemma

We introduce now three lemma, which will be useful in the following analysis. The first one is a classical result about Lyapunov functions,

**Lemme 9.3.1.** Let V : (H²(0, L))² × ℝ × ℝ₊ → ℝ₊ such that there exists a constant c > 0 such that
\[
c (\|U\|_{H^2(0,L)} + |z|) \leq V(U, z, t) \leq \frac{1}{c} (\|U\|_{H^2(0,L)} + |z|), \forall (U, z, t) \in (H^2(0, L))^2 × ℝ × ℝ₊.
\]
(9.3.20)

If there exists γ > 0 and δ > 0 such that, for any solution (u, Z) of the system \((9.3.8), (9.3.15)\) with initial conditions satisfying \(\|u(0,\cdot)\|_{H^2(0,L)} + |Z(0)| \leq \delta\),
\[
\frac{d}{dt} [V(u(\cdot, t), t)] \leq -\gamma V(u(\cdot, t), t)
\]
(9.3.21)
in a distribution sense, then the system \((9.3.8), (9.3.15)\) is exponentially stable for the \(H^2\) norm and V is called a Lyapunov function for the system \((9.3.8), (9.3.15)\).

This first lemma reduces the problem of proving the exponential stability to finding a Lyapunov function V for the system \((9.3.8), (9.3.15)\). A proper definition of a differential inequality in a distribution sense as in \((9.3.21)\) can be found in [106]. To lighten this article we do not give a proof of this classical lemma, although a proof for a very similar case (Lyapunov function that does not depend explicitly on time and for the \(C^1\) norm instead) can be found for instance in [106][Proposition 2.1], and is easily extended to this case.

The second Lemma is a variation of a result shown in [110] that gives a local entropy of the Saint-Venant equations. Let us first introduce the following function \(\varphi\) defined by
\[
\varphi_1(t, x) = \exp \left( \int_0^x \frac{\gamma_1}{\lambda_1} dx \right), \\
\varphi_2(t, x) = \exp \left( -\int_0^x \frac{\gamma_2}{\lambda_2} dx \right), \\
\varphi(t, x) = \frac{\varphi_1(t, x)}{\varphi_2(t, x)},
\]
(9.3.22)
where \(\lambda_1\) and \(\lambda_2\) are defined by
\[
\lambda_1(t, x) := \Lambda_1(0, x, t) > 0, \quad \lambda_2(t, x) := -\Lambda_2(0, x, t) > 0.
\]
(9.3.23)

We can now state the following lemma

**Lemme 9.3.2.** There exists \(\delta_0 > 0\) such that if \(\|\partial_1 H_1\|_{L^\infty([0, +\infty) × [0, L])} \leq \delta_0\), the function \(\lambda_2 \varphi / \lambda_1\) is solution on \([0, L]\) to the following equation
\[
\partial_x f = \left| \frac{\varphi_2}{\lambda_1} + \frac{\varphi^{-1} \delta_1}{\lambda_2} f^2 + \sqrt{\frac{\delta}{H_1}} \partial_1 H_1 \right|, \forall x \in [0, L], \ t \in [0, +\infty),
\]
(9.3.24)
and for any \( x \in [0, L] \) and any \( t \in [0, +\infty) \),
\[
\left( \frac{\varphi_2}{\lambda_1} + \frac{\varphi_1}{\lambda_2} f^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) > 0. \tag{9.3.25}
\]
The proof is given in the Appendix.

Eventually, we introduce our last Lemma, which seems very natural and is stated here to lighten the proof of Theorem 9.2.3

**Lemma 9.3.3.** There exists \( l > 0 \) and \( C > 0 \) such that if \( \| \partial_t Q_0 \|_{C^1([0, +\infty))} \leq l \), then
\[
\max \left( \| \partial_t H_1 \|_{C^1([0, +\infty), L^\infty(0, L))}, \| \partial_t V_1 \|_{C^1([0, +\infty), L^\infty(0, L))} \right) < C \| \partial_t Q_0 \|_{C^1([0, +\infty))}. \tag{9.3.26}
\]

This is a consequence of the ISS property (Proposition 9.2.1) and Remark 9.2.1 with \( p = 3 \), the relations (9.2.4), and Sobolev inequality. Thanks to this Lemma, we now only need to find a bound on \( \partial_t H_1 \) and \( \partial_t V_1 \) instead of a bound on \( \partial_t Q_0 \) in the proof of Theorem 9.2.3.

### 9.3.3 Proof of Theorem 9.2.3

We can now prove Theorem 9.2.3

**Proof of Theorem 9.2.3** From Theorem 9.2.2, Remark 9.3.1 and Lemma 9.3.1, one only needs to find a Lyapunov function \( V : (H^2(0, L))^2 \times \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+ \) satisfying (9.3.20) and (9.3.21). We define the following candidate:
\[
V_a(U, Z, t) := \int_0^L f_1(t, x)e^{-\mu t}(E(U(x), x, t)U^2_1(t, x) + f_2(t, x)e^{\mu t}(E(U(x), x, t)U^2_2(t, x)dx + qz^2, \tag{9.3.27}
\]
where \( f_1, f_2 \) are positive and bounded functions which will be defined later on, and \( \mu \) and \( q \) are positives constants which will also be defined later on. Recall that \( E \) is still given by (9.2.15), \( T > t_0 \geq 0 \) and \((u^0, Z^0) \in H^2(0, L) \times \mathbb{R} \) satisfying the compatibility condition (9.2.19) and such that
\[
\left( \| u^0 \|_{H^2(0, L)} + |Z^0| \right) < \nu, \tag{9.3.28}
\]
where \( \nu \) is a constant to be chosen later on but such that \( \nu < \min(\nu_2, \nu(T)) \). Recall that \( \nu(T) \) is given by Theorem 9.2.2. From Theorem 9.2.2 there exists a unique solution \( u \in C^0([t_0, T], H^2(0, L)) \). We suppose in addition that \((u^0, Z^0) \in H^3(0, L) \) and that (9.3.28) also hold for the \( H^3 \) norm instead of the \( H^2 \) norm in \( u \). From Remark 9.2.2 \((u, Z) \in C^0([t_0, T] \times H^2(0, L) \times C^3([t_0, T]) \). This assumption is here to allow us to compute easily the derivative of \( u \) but will be relaxed later on by density.

Let \( \delta \in (0, \delta_0) \) to be chosen later on, with \( \delta_0 \) is given by Lemma 9.3.2 and assume that
\[
\max(\| \partial_t H_1 \|_{C^1([t_0, \infty), L^\infty(0, L))}, \| \partial_t V_1 \|_{C^1([t_0, \infty), L^\infty(0, L))}) < \delta. \tag{9.3.29}
\]
As this is the only assumption on \( H_1 \) and \( V_1 \), we can assume from now on that \( t_0 = 0 \) without loss of generality.

Looking at (9.3.27), \( V_a \) is indeed a function defined on \( H^2(0, L) \times \mathbb{R} \times \mathbb{R}_+ \), but for notational ease we will denote \( V_a(t) := V_a(u(t, \cdot), Z(t), t) \), where \( Z(t) \) is given by (9.2.19), and \( E := E(u(t, x), x, t) \). Similarly we
introduce

\[
V_0(U, t) := \int_0^L f_1 e^{-\mu z} (E(U(x), x, t) I(U(x, x, t)))^2 + f_2 e^{\mu z} (E(U(x), x, t) J(U(x, x, t)))^2 dx + \frac{q H_1(t, L)}{4g} (U_1(L) - U_2(L))^2,
\]

\[
V_{\varepsilon}(U, t) := \int_0^L f_1 e^{-\mu z} (E(U(x), x, t) J(U(x, x, t)))^2 + f_2 e^{\mu z} (E(U(x), x, t) J(U(x, x, t)))^2 dx + q \left( \sqrt{H_1(t, L)} \left(I_1(t, L) - I_2(t, L)\right) + \frac{\partial_t H_1(t, L)}{4} \sqrt{1 \over g H_1(t, L)} (U_1(L) - U_2(L)) \right)^2, \]

(9.3.30)

where

\[
I(U, x, t) := (A(U, x, t) \partial_t (\partial_t U) + (\partial_t A(U, x, t, t) + \partial U A(U, x, t, \partial_t U) \partial_t U + \partial_t B(U, x, t)) (\partial_t U),
\]

\[
J(U, x, t) := A(U, x, t) \partial_x (\partial_t^2 U) + (\partial_t A(U, x, \partial_t U) \partial_x U + (\partial_t B(U, x))(\partial_t^2 U)
\]

\[
+ (\partial_t^2 A(U, x, t) + 2 \partial U (\partial_t A(U, x, t)) \partial_t U) \partial_x U
\]

\[
+ 2 \partial_x A(U, x, t) \partial_x (\partial_t U) + 2 \partial U A(U, x, x) \partial_t U \partial_x U + ((\partial_t^2 A(U, x) \partial_t U) \partial_x U
\]

\[
+ \partial_t^2 B(U, x) + 2 \partial U (\partial_t B(U, x, x)) \partial_t U + (\partial_t^2 B(U, x) \partial_t U) \partial_t U).
\]

(9.3.31)

Observe that for a solution \(u\) of (9.3.8), and using the expression of \(Z\) given by (9.2.15), the expressions of \(V_0(u(t, \cdot), t)\) and \(V_{\varepsilon}(u(t, \cdot), t)\) become

\[
V_0(u(t, \cdot), t) := \int_0^L f_1(t, x) e^{-\mu z} (E(\partial_t u))_1^2 + f_2(t, x) e^{\mu z} (E(\partial_t u))_2^2 dx + q(\tilde{Z}(t))^2,
\]

\[
V_{\varepsilon}(u(t, \cdot), t) := \int_0^L f_1(t, x) e^{-\mu z} (E(\partial_t^2 u))_1^2 + f_2(t, x) e^{\mu z} (E(\partial_t^2 u))_2^2 dx + q(\tilde{Z}(t))^2,
\]

(9.3.32)

which justifies the expression chosen for (9.3.30) and (9.3.31). We also note for notational ease \(V_0(t) := V_0(t, u(t, \cdot))\) and \(V_{\varepsilon}(t) := V_{\varepsilon}(t, u(t, \cdot))\). Finally we denote \(V := V_0 + V_0 + V_{\varepsilon}\). We start now by dealing with \(V_0\). Differentiating \(t \to V_0(t)\) with respect to time, using (9.3.8), (9.3.17) and integrating by parts, one has

\[
\dot{V}_0 = -2 \int_0^L \left[ (E A(u, x, t) \partial_t u)_1 + (E B)_1(u, x, t) \right]
\]

\[
+ f_2(t, x) e^{\mu z} (E(\partial_t^2 u))_2^2 + f_2(t, x) e^{\mu z} (E(\partial_t^2 u))_2^2 dx
\]

\[
+ \int_0^L \partial_t f_1(t, x) e^{-\mu z} (E\partial_t u)_1^2 + \partial_t f_2(t, x) e^{\mu z} (E\partial_t u)_2^2 dx
\]

\[
+ 2 \int_0^L f_1(t, x) e^{-\mu z} (E\partial_t U) \partial_t (\partial_t U, E, \partial_t u) \partial_t u dx + f_2(t, x) e^{\mu z} (E\partial_t U) \partial_t (\partial_t U, E, \partial_t u) \partial_t u dx + 2 q Z(t) \tilde{Z}(t)
\]

\[
= -2 \int_0^L f_1(t, x) e^{-\mu z} (E\partial_t U) \partial_t (\partial_t U, E, \partial_t u) \partial_t u \partial_t u dx + f_2(t, x) e^{\mu z} (E\partial_t U) \partial_t (\partial_t U, E, \partial_t u) \partial_t u \partial_t u dx
\]

\[
+ \int_0^L \partial_t f_1(t, x) e^{-\mu z} (E\partial_t u)_1^2 + \partial_t f_2(t, x) e^{\mu z} (E\partial_t u)_2^2 - 2 \int_0^L f_1(t, x) e^{-\mu z} (E\partial_t U)_1(EB)_1(u, x, t) + f_2(t, x) e^{\mu z} (E\partial_t U)_2(EB)_2(u, x, t) dx
\]

\[
+ 2 \int_0^L f_1(t, x) e^{-\mu z} (E\partial_t U) \partial_t (\partial_t U, E, \partial_t u) \partial_t u + f_2(t, x) e^{\mu z} (E\partial_t U) \partial_t (\partial_t U, E, \partial_t u) \partial_t u dx + 2 q Z(t) \tilde{Z}(t),
\]

(9.3.33)
\[ \dot{V}_a = - \left[ f_1 e^{-\mu x} D_1(Eu)^2 + D_2 f_2 e^{\mu x} (Eu)^2 \right]^L \nabla_e + \int_0^L (Eu)_2 e^{-\mu x} \left( \partial_x (\partial_1 f_1) - f_1 \partial_0(D_1) \right) \partial_x (Eu_1) - 2 f_1 D_1 \left( \partial_x E + \partial_0 E \partial_x (Eu) \right) dx + \int_0^L \partial_1 (f_1) e^{-\mu x} (Eu)_1 + \partial_1 (f_2) e^{\mu x} (Eu)_2 dx - 2 \int_0^L f_1 e^{-\mu x} (Eu)_1 (\partial_x E + \partial_0 E \partial_x (Eu))_1 + f_2 e^{\mu x} (\partial_x E + \partial_0 E \partial_x (Eu))_2 dx - \mu \int_0^L D_1 f_1 e^{-\mu x} (Eu)_1^2 - D_2 f_2 e^{\mu x} (Eu)_2^2 dx + 2q Z(t) \dot{Z}(t). \]  

(9.3.34)

In order to simplify this expression, observe that from (9.3.9), (9.3.17) and (9.3.23), there exists a constant that only depends on \( \nu \) and \( \lambda \) such that

\[ \|D_1 - \text{sgn}(D_1(0, x, t)) \lambda_1\|_\infty \leq \|Cu\|_\infty, \]  

\[ \|\partial_x D_1 + \partial_0 D_1 \partial_x u - \text{sgn}(D_1(0, x, t)) \partial_x \lambda_1\|_\infty \leq C \left( \|\partial_x u\|_\infty + \|u\|_\infty \right), \]  

where \( i \in \{1, 2\} \),

and

\[ \|\partial_x E\|_\infty \leq C \left( \|u\|_\infty + \|\partial_x u\|_\infty \right). \]  

Thus, using this together with (9.3.34)

\[ \dot{V}_a \leq - \left[ f_1 e^{-\mu x} D_1(Eu)^2 + D_2 f_2 e^{\mu x} (Eu)^2 \right]^L \nabla_e + \int_0^L (Eu)_2 e^{-\mu x} \left( \partial_x (\lambda_1 f_1) - \partial_x (\lambda_2 f_2) - \partial_0 f_1 \right) dx \]  

\[ - 2 \int_0^L f_1 e^{-\mu x} (Eu)_1 (\partial_x E + \partial_0 E \partial_x (Eu))_1 + f_2 e^{\mu x} (Eu)_2 (\partial_x E + \partial_0 E \partial_x (Eu))_2 dx \]  

\[ - \mu \int_0^L \lambda_1 f_1 e^{-\mu x} (Eu)_1^2 + \lambda_2 f_2 e^{\mu x} (Eu)_2^2 dx + 2q Z(t) \dot{Z}(t) \]  

\[ + C \left( \|u\|_\infty + \|\partial_x u\|_\infty \right) \int_0^L (Eu)_1^2 + (Eu)_2^2 dx \]  

\[ + C \left( \|u\|_\infty + \|\partial_x u\|_\infty \right)^2 \int_0^L |(Eu)_1| + |(Eu)_2| dx, \]  

where \( C \) is a constant that may change between lines but only depends on \( \nu \), an upper bound of \( \delta \) (for instance \( \delta_0 \)), \( \mu \), \( H_\infty \) and \( \alpha \). Note that \( C \) is continuous in \( \mu \in [0, \infty) \), thus it can be made independent of \( \mu \) by imposing an upper bound on \( \mu \), for instance \( \mu \in (0, 1] \). Finally, from the second equation of (9.3.17), and the fact that \( E \) is \( C^1 \) in \( u \), there exists a continuous function \( r_1 \) such that, for any vector \( v \in \mathbb{R}^2 \)

\[ E(u(t, x), x, t) v - v = (u(t, x), r_1(u(t, x), x, t)) v, \quad \forall (t, x) \in [0, T] \times [0, L]. \]  

(9.3.40)

As \( E(u(t, x), x, t) \) is locally a \( C^\infty \) function of the coefficients of \( A \), \( r_1 \) is bounded on \( B_{2\alpha} \times [0, L] \times [0, T] \) by a bound that only depends on \( \nu_2 \), \( H_\infty \) and \( \alpha \). Thus there exists a constant \( C \) depending only on \( \nu_2 \), \( H_\infty \) and \( \alpha \) such that

\[ \frac{1}{C} \|v\|_{L^2((0,L);\mathbb{R}^2)} \leq \|Ev\|_{L^2((0,L);\mathbb{R}^2)} \leq C \|v\|_{L^2((0,L);\mathbb{R}^2)}. \]  

(9.3.41)
Thus, using this together with the fact that \(D_1\) and \(D_2\) are \(C^1\) with \(u_0\), and Young’s inequality and then Cauchy-Schwarz inequality on the last integral term,

\[
\dot{V}_a \leq - \left[ f_1 e^{-\mu x} \lambda_1 (E u_1^2) - \lambda_2 f_2 e^{\mu x} (E u_2^2) \right]_{0}^{L} \\
- \int_{0}^{L} (E u_2^2) e^{-\mu x} (-\partial_x (\lambda_1 f_1) + \partial_t (f_1)) + (E u_2^2) e^{\mu x} (\partial_x (\lambda_2 f_2) - \partial_t (f_2)) \, dx \\
- 2 \int_{0}^{L} f_1 e^{-\mu x} (E u_1) (E (u_1, x, t) + f_2 e^{\mu x} (E u_2 (E B) (u, x, t), x, t) \, dx \\
- \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) V_a + \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) q Z^2 (t) + 2 q Z (t) \dot{Z} (t) \\
+ C (\|u\|_{\infty} + \|\partial_x u\|_{\infty}) \|u\|_{L^2(0, L)}^2 + C (\|\partial_x u\|_{\infty})^3 + C \|u\|_{\infty} (|u(t, 0)^2| + |u(t, L)^2|) \tag{9.3.42}
\]

Now, as \(E\) and \(B\) are \(C^2\) with \(u\) and continuous with \(x\) and \(t\), and as \((0, 0, t) = 0\), there exists a continuous function \(r_2 \in C^0(B_{\nu_2} \times [0, T] \times [0, L]; R^{n \times n})\) such that

\[
(EB) (u (t, x), x, t) = \partial_u (EB) (0, x, t) u (t, x) + (r_2 (u, x, t), u (t, x), x, t) u (t, x), \quad \forall t \in [0, T] \times [0, L]. \tag{9.3.43}
\]

Note that from \(9.3.10\), \(r_2\) is bounded on \(B_{\nu_2} \times [0, L] \times [0, T]\) by a constant that only depends on \(\nu_2\), \(\delta\), \(H_\infty\), and \(\alpha\). From \(9.3.10\) and \(9.3.17\), \(\partial_u (EB) (0, x, t) = \partial_u B (0, x, t)\). Besides, \(9.3.17\) is invertible and \(C^1\), thus an inequality similar to \(9.3.17\) holds for \(E^2\), and \(u = E^{-1} (E u)\). Therefore, using \(9.3.43\) together with \(9.3.40\), the fact that \(r_1\) and \(r_2\) are bounded, and the expression of \(\partial_u B (0, x, t)\) given in \(9.3.13\)–\(9.3.14\), one has

\[
\dot{V}_a \leq - \left[ f_1 e^{-\mu x} \lambda_1 u_1^2 - \lambda_2 f_2 e^{\mu x} u_2^2 \right]_{0}^{L} \\
- \int_{0}^{L} (E u_2^2) e^{-\mu x} (-\partial_x (\lambda_1 f_1) + \partial_t (f_1)) + (E u_2^2) e^{\mu x} (\partial_x (\lambda_2 f_2) - \partial_t (f_2)) \, dx \\
- 2 \int_{0}^{L} f_1 e^{-\mu x} \gamma_1 (E u_1^2) + f_2 e^{\mu x} \delta_2 (E u_2^2) + \left( \gamma_2 f_1 e^{-\mu x} + \delta_1 f_2 e^{\mu x} \right) (E u_1) (E u_2) \, dx \\
- \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) V_a + \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) q Z^2 (t) + 2 q Z (t) \dot{Z} (t) \\
+ C (\|u\|_{\infty} + \|\partial_x u\|_{\infty}) \|u\|_{L^2(0, L)}^2 + C (\|\partial_x u\|_{\infty})^3 + C \|u\|_{\infty} (|u(t, 0)^2| + |u(t, L)^2|) \tag{9.3.44}
\]

As \(D_1\) and \(D_2\) are of class \(C^2\), denoting for simplicity \(k_2 := \partial_1 D_1 (0, 0, t), k_1 := \partial_1 D_2 (0, 0, t)\) and \(k_3 := - \partial_2 D_2 (0, 0, t)\), and using \(9.3.15\)

\[
\dot{V}_a \leq - \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) V_a + \left[ f_1 \lambda_1 k_2^2 - \lambda_2 f_2 \right] u_2^2 (t, 0) \\
- I_1 (u_1 (t, L), Z (t)) - \int_{0}^{L} I_2 ((E u_1), (E u_2)) \, dx \\
+ C (\|u\|_{\infty} + \|\partial_x u\|_{\infty}) \left( \|u\|_{L^2(0, L)}^2 + (\|u\|_{\infty} + \|\partial_x u\|_{\infty})^2 + (|u(t, 0)|^2 + |u(t, L)|^2) \right) \tag{9.3.45}
\]
where $I_1$ and $I_2$ denote the following quadratic forms

$$I_1(x, y) = (\lambda_1 f_1(L)e^{-\mu L} - \lambda_2 f_2(L)e^{\mu L}k_1^2) x^2 + \left(\sqrt{\frac{H_1}{g}}k_3 - \lambda_2 f_2(L)e^{\mu L}k_3^2 - \mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) q\right) y^2$$

$$+ (2\lambda_2 f_2(L)e^{\mu L}k_3k_1 - \sqrt{\frac{H_1}{g}}(k_1 - 1)) xy,$$

$$I_2(x, y) = (-(\lambda_1 f_1) + 2f_1 \gamma_1(t, x) - \partial_t f_1)e^{-\mu x} x^2 + ((\lambda_2 f_2) + 2f_2 \gamma_2(t, x) - \partial_t f_2)e^{\mu x} y^2$$

$$+ 2(\gamma_2 f_1 e^{-\mu x} + \delta f_2 e^{\mu x}) xy.$$  

(9.3.46)

We can perform similarly with $V_B$ and $V_c$, to do this observe that $\partial_t u$ and $\partial^2_t u$ are respectively solutions of

$$\partial_t(\partial_t u) + A(u, x, t)\partial_x(\partial_t u) + (\partial_u B(u, x, t))\partial_t u) + (\partial_t A(u, x, t) + \partial_u A(u, x, t)\partial_x u + \partial_t B(u, x, t) = 0$$

(9.3.47)

$$\partial_t(\partial^2_t u) + A(u, x, t)\partial_x(\partial^2_t u) + (\partial_u A(u, x, t))\partial^2_t u)\partial_x u + (\partial_u B(u, x, t))\partial^2_t u),$$

$$+ 2\partial_u(\partial_t A(u, x, t))\partial_x u + (\partial^2_u A(u, x, t))\partial_x u + 2\partial_u A(u, x, t)\partial_t (\partial_t u) + \partial_t A(u, x, t)\partial_x (\partial_t u)$$

(9.3.48)

$$+ ((\partial^2_u A(u, x, t))\partial_x u + \partial^2_u B(u, x) + \partial_u (\partial_t B(u, x)))\partial_x u + (\partial^2_u B(u, x))\partial_x u)(\partial_t u) = 0,$$

which are very similar to (9.3.3), as they only differ by quadratic perturbations or terms involving a time derivative of $(H_1, V_1)$. We get then

$$\dot{V} = \dot{V}_a + \dot{V}_b + \dot{V}_c \leq -\mu \min_{x \in [0, L]} (\lambda_1, \lambda_2) V + \left|f_1\lambda_1 k_2^2 - \lambda_2 f_2\right| \left(u_2^2(t, 0) + (\partial_t u_2(t, 0))^2 + (\partial^2_t u_2(t, 0))^2\right)$$

$$- I_1(u_1(t, L), Z) - I_1(\partial_t u_1(t, L), Z) - I_1(\partial^2_t u_1(t, L), Z)$$

$$- \int_0^L I_2((E u)_1, (E u)_2 + I_2((E \partial_t u)_1, (E \partial_t u)_2) + I_2((E \partial^2_t u)_1, (E \partial^2_t u)_2) dx$$

$$+ C \left(\|u\|_\infty + \|\partial_x u\|_\infty\right) \left(\|u_2\|^2_{L^2(0, L)} + \|\partial_t u_2\|^2_{L^2(0, L)} + \|\partial^2_t u_2\|^2_{L^2(0, L)} + (\|u\|_\infty + \|\partial_x u\|_\infty)^2 + \|u_2(t, 0)^2 + (\|u_1(t, L) + |Z|)^2 + \|\partial_t u_2(t, 0)^2 + (\|\partial_t u_1(t, L) + |Z|)^2 + \|\partial^2_t u_2(t, 0)^2 + (\|\partial^2_t u_1(t, L) + |Z|)^2\right)$$

$$+ C\delta \left(|u_2(t, 0)|^2 + (\|u_1(t, L) + |Z|)^2 + \|\partial_t u_2(t, 0)|^2 + (\|\partial_t u_1(t, L) + |Z|)^2\right) + C\delta V.$$  

(9.3.49)

The two last terms come from the successive differentiations of the boundary conditions (9.3.15), together with (9.3.29), or the terms in (9.3.47)–(9.3.48) involving a time derivative of $A$ or $B$. One can see that three identical quadratic form appears in the integral in $(E \partial_t u)_i, (E \partial^2_t u)_i, i = 0, 1, 2$, as well as three identical quadratic form at the boundaries in $(\partial^2_t u_1(t, L), \partial^2_t Z), i = 0, 1, 2$, and three identical terms proportional respectively to $(\partial^2_t u_2(t, 0), i = 0, 1, 2$. Thus a sufficient condition to have $V$ decreasing strictly would be that the square terms and the forms that appear at the boundaries are negative-definite and the quadratic form in the integral is negative, i.e. the three following conditions :

1. Condition at 0

$$\frac{\lambda_2 f_2(0)}{\lambda_1 f_1(0)} > k_2^2.$$  

(9.3.50)

2. Condition at $L$

$$\frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} > k_2^2.$$  

(9.3.51a)

$$\left(\lambda_1 f_1(L) - \lambda_2 f_2(L)k_2^2\right) \left(\sqrt{\frac{H_1}{g}} - \lambda_2 f_2(L)k_3\right) k_3 - \left(\lambda_2 f_2(L)k_3 k_1 - \frac{1}{2} q \left(\frac{H_1}{g} (k_1 - 1)\right)^2 > 0.\right)$$  

(9.3.51b)
3. Condition from the integral

\[
\begin{align*}
((\lambda_1 f_1)_t + 2 f_1 \gamma_1(t, x) - \partial_t f_1) & > 0, \quad (9.3.52a) \\
((\lambda_1 f_1)_t + 2 f_1 \gamma_1(t, x) - \partial_t f_1) ((\lambda_2 f_2)_t + 2 f_2 \delta_2(t, x) - \partial_t f_2) - (\gamma_2 f_1 + \delta_1 f_2)^2 & > 0, \quad \forall (t, x) \in [0, T] \times (0, L).
\end{align*}
\]

Let assume for the moment that (9.3.50)–(9.3.52) are satisfied for any \( \delta \in (0, \delta_3) \) where \( \delta_3 \) is a positive constant. Then, as the inequalities (9.3.50)–(9.3.52) are strict, by continuity there exist \( \mu > 0 \) such that the square terms and the quadratic forms \( I_1 \) at the boundaries and the quadratic forms \( I_2 \) in the integral are positive definite. And there exists \( \nu_3 \in (0, \nu_2) \) and \( \delta_4 \in (0, \delta_3) \) such that, for any \( \nu \in (0, \nu_3) \), and any \( \delta \in (0, \delta_4) \),

\[
\dot{V} \leq -\mu \min_{[0,L]}(\lambda_1, \lambda_2)V + C\delta V + C \left( \|u\|_{\infty} + \|\partial_x u\|_{\infty} \right)^2,
\]

where \( C \) is a positive constant depending only on the system. Note that here, the cubic boundary terms that appeared in (9.3.49) have been compensated by the strictly negative quadratic boundary terms, taking \( \nu \) sufficiently small and using (9.2.21). Choosing \( \delta_5 \in (0, \delta_4) \) such that \( \delta_5 < \mu \min_{[0,L]}(\lambda_1, \lambda_2)/4C \), for any \( \delta \in (0, \delta_5) \) one has

\[
\dot{V} \leq -\frac{3}{4} \mu \min_{[0,L]}(\lambda_1, \lambda_2)V + C \left( \|u\|_{\infty} + \|\partial_x u\|_{\infty} \right)^2.
\]

Now, if we assume in addition that (9.3.20) hold, using (9.2.21), and Sobolev inequality, there exists \( \nu_4 \in (0, \nu_3) \) such that, for any \( \nu \in (0, \nu_4) \),

\[
C \left( \|u\|_{\infty} + \|\partial_x u\|_{\infty} \right)^2 \leq \frac{\mu}{4} \min_{[0,L]}(\lambda_1, \lambda_2)V,
\]

thus, setting \( \gamma = \mu \min_{[0,L]}(\lambda_1, \lambda_2) \),

\[
\dot{V} \leq -\frac{\gamma}{2} V.
\]

which shows the exponential decay of \( V \) and ends the proof of Theorem 9.2.3.

In other words, all that remains to do is to find \( f_1, f_2 \) and \( q \) such that (9.3.50)–(9.3.52) are satisfied and such that \( V \) satisfies (9.3.20). In order to find such function we are now going to use Lemma 9.3.2. To understand the link between Lemma 9.3.2 and the three conditions (9.3.50)–(9.3.52), observe that the condition (9.3.52) give rise to a differential inequation, which, as it will appear later on, is linked to the differential equation solved by Lemma 9.3.2 Then (9.3.50) and (9.3.51) can be seen as boundary conditions/values of the solution of this differential inequation.

From Lemma 9.3.2 we know that there exists a solution on \([0, L]\) to equation (9.3.24), namely \( \lambda_2 \varphi/\lambda_1 \). Therefore, as \([0, L]\) is a compact set, there exists \( \epsilon_1 \) such that for any \( \epsilon \in [0, \epsilon_1] \) there exists a solution \( f_{\epsilon}(t, x) \) to the following system

\[
\begin{align*}
\partial_x f_{\epsilon}(t, x) &= \left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi \lambda_2} \right) f_{\epsilon}^2 + \frac{g}{H_1} \partial_t H_1 + \epsilon, \\
f_{\epsilon}(0) &= \frac{\lambda_2(t, 0)}{\lambda_1(t, 0)} + \epsilon,
\end{align*}
\]

and moreover \((t, x, \epsilon) \to f_{\epsilon}(t, x)\) is of class \( C^0 \) and \( \partial_x f_{\epsilon}(t, x) \) as well. This is a classical result on ODE due to Peano (see e.g. [105, Chap. 5, Th 3.1]). From (9.3.57), \( \partial_t f_{\epsilon} \) satisfies the following equation

\[
\partial_x \partial_t f_{\epsilon} = \frac{\delta_1}{\varphi \lambda_2} f_{\epsilon} \partial_t f_{\epsilon} + \left( \frac{\varphi \gamma_2}{\lambda_1} \right)_t f_{\epsilon}^2 + \frac{g}{H_1} \partial_t H_1 - \frac{1}{2} \frac{g}{H_1^2} (\partial_t H_1)^2.
\]
We used here that, from Proposition 9.2.1 and Remark 9.2.1, \((H_1, V_1) \in C^0([0, +\infty); H^1(0, L))\), and from 9.2.4, \(\partial_t \partial_x H_1 = -\partial_x^2 (HV)\) and \(\partial_t \partial_x V_1 = \partial_x (-V_1 \partial_x V_1 - g \partial_x H_1 - (kV_h^2 / H_1 - gC))\). Thus \(\partial_t^2 H_1\) belongs to \(C^0([0, T] \times H^1(0, L))\), and \((\gamma_1, \gamma_2, \delta_1, \delta_2)\) belong to \(C^1([0, T] \times H^1(0, L))\). Using (9.3.58), we have

\[
\partial_t f_\varepsilon(t, x) = \partial_t f_\varepsilon(t, 0) + \int_0^t \exp \left(\int_y^x 2 \frac{\delta_1}{\varphi\lambda_2} f_\varepsilon(t, \omega) d\omega\right) \partial_t (\partial_y (H_1 V_1)) dy dt
\]

Thus, using (9.3.23), (9.3.59), and similarly for \(\varepsilon\) and \(\lambda_1\), we have

\[
|\partial_t f_\varepsilon(t, x)| \leq (|\partial_t f_\varepsilon(t, 0)| + \max (|\partial_t H_1|_{C^1([0, +\infty); L^\infty(0, L))}, |\partial_t V_1|_{C^1([0, +\infty); L^\infty(0, L))}) C(\varepsilon).
\]

But, from (9.3.57) \(\partial_t f_\varepsilon(t, 0) = (\lambda_2 / \lambda_1)\), thus using (9.3.29) we obtain

\[
|\partial_t f_\varepsilon(t, x)| \leq \delta C_2(\varepsilon),
\]

where \(C_2\) is again a constant that only depends on \(\varepsilon, \alpha\) and \(H_\infty\) and is continuous with \(\varepsilon\) and \(\alpha\) continuous with \(\varepsilon\). We can now restrict ourselves to \(\varepsilon \in [0, \varepsilon_1/2]\) and then \(C_2\) can be chosen independent of \(\varepsilon\) by simply taking its maximum on \([0, \varepsilon_1/2]\). Recall that from Lemma 9.3.2 we have, \(f_0 = \varphi \lambda_2 / \lambda_1\), and

\[
\left(\frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi \lambda_2} f_0^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1\right) > 0.
\]

Recall that we still have not chosen the bound \(\delta \in (0, \delta_0)\) on \(|\partial_t H_1|_{C^1([0, +\infty); L^\infty(0, L))}\) and \(|\partial_t V_1|_{C^1([0, +\infty); L^\infty(0, L))}\) given in (9.3.29). From the assumptions on \(k_\rho\) and \(k_T\), i.e. 9.2.25, and 9.3.16, and recalling that \(k_1 = \partial_t D_2(0, 0, t)\) and \(k_3 = -\partial_t D_2(0, 0, t)\), one has

\[
k_1^2 < \left(\frac{\lambda_1(L)}{\lambda_2(L)}\right)^2, \quad k_3 > 0.
\]

Thus, using 9.3.23,

\[
\eta_1 := \min \left(\frac{1}{|k_1|}, \frac{\lambda_2(L)}{\lambda_1(L)}, 1 - \frac{\lambda_2(L)}{\lambda_1(L)}\right) > 0.
\]
As \( \varepsilon \to f_\varepsilon(t,x) \) is uniformly continuous with \( \varepsilon \) for \( (t,x) \in [0,\infty) \times [0,L] \), there exists \( \varepsilon_2 \in (0,\varepsilon_1/2) \) such that for any \( (t,x) \in [0,\infty) \times [0,L] \)
\[
|f_{\varepsilon_2}(t,x) - f_0(t,x)| \leq \varphi(t,L)\eta_1,
\]
(9.3.68)
and
\[
\left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi \lambda_2} f_{\varepsilon_2}^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) > 0.
\]
(9.3.69)

Note that \( \varepsilon_2 \) depends a priori on \( \delta \) from (9.3.69). However, from Lemma 9.3.2, we can in fact choose \( \varepsilon_2 \) independent of \( \delta \) and depending only on an upper bound of \( \delta \) (for instance \( \delta_0 \) given by Lemma 9.3.2). This is important as, in the following, we will choose a \( \delta \) that may depend on \( \varepsilon \).

We select \( f_1 \) and \( f_2 \) in the following way:
\[
f_1(t,x) = \frac{\varphi^2}{\lambda_1 f_{\varepsilon_2}(t,x)} > 0,
\]
(9.3.70)
\[
f_2(t,x) = \varphi_2^2 \frac{f_{\varepsilon_2}(t,x)}{\lambda_2} > 0,
\]
and we can now check that the condition (9.3.52) is verified for \( \delta \) small enough as
\[
(-\lambda_1 f_1)_x = -2 \frac{(\varphi_1)_x \lambda_1 f_1}{\varphi_1} + \varphi_1^2 \frac{\partial_x f_{\varepsilon_2}(t,x)}{f_{\varepsilon_2}^2(t,x)}.
\]
(9.3.71)

Thus from (9.3.22)
\[
- (\lambda_1 f_1)_x + 2\gamma_1 f_1 = \varphi_1^2 \frac{\partial_x f_{\varepsilon_2}}{f_{\varepsilon_2}^2}.
\]
(9.3.72)
and similarly
\[
(\lambda_2 f_2)_x + 2\delta_2 f_2 = (\varphi_2^2 f_{\varepsilon_2}(t,x))_x - (\varphi_2^2) f_{\varepsilon_2}(t,x)
= \varphi_2^2 \partial_t f_{\varepsilon_2}.
\]
(9.3.73)
Therefore, from (9.3.57), (9.3.72), and (9.3.73), one has
\[
(-\lambda(f_1)_x + 2\gamma_1 f_1 - \partial_t f_1)(\lambda_2 f_2)_x + 2\delta_2 f_2 - \partial_t f_2 = \left( \frac{\varphi_1 \varphi_2}{f_{\varepsilon_2}} \right)^2 \left( \left( \frac{\varphi_2^2}{\lambda_1} + \frac{\delta_1}{\varphi_2 \varphi_1} f_{\varepsilon_2}^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) + \varepsilon_2 \right)^2
- \partial_x f_{\varepsilon_2} \left( \frac{\varphi_1^2}{f_{\varepsilon_2}^2} \partial_x f_1 + \varphi_2^2 \partial_t f_1 \right) + (\partial_t f_1)(\partial_t f_2).
\]
(9.3.74)
But we have
\[
\partial_t f_1 = \frac{2 \partial_t \varphi_1 \varphi_1}{\lambda_1 f_{\varepsilon_2}} - \frac{\partial_t \lambda_1}{\lambda_1^2 f_{\varepsilon_2}^2} + \frac{\partial_t \lambda_2}{\lambda_2^2 f_{\varepsilon_2}^2} \varphi_1^2,
\]
(9.3.75)
and besides, from (9.2.4) and (9.3.29), there exists \( C_3 > 0 \) depending only on \( \alpha \) and \( H_\infty \), and an upper bound of \( \delta \) (for instance \( \delta_0 \)), such that
\[
\max(\|H_1\|_{L^\infty((0,\infty) \times (0,L))}, \|V_1\|_{L^\infty((0,\infty) \times (0,L))}) \leq C_3.
\]
(9.3.76)
Thus, using (9.3.14) and (9.3.29), there exists \( C_4 > 0 \) depending only on \( \alpha \) and \( H_\infty \), and \( \delta_0 \) (but not on \( \delta \)) such that
\[
\max(\|\varphi_1\|_{L^\infty((0,\infty) \times (0,L))}, \|\varphi^{-1}\|_{L^\infty((0,\infty) \times (0,L))}) < C_4,
\]
(9.3.77)
and similarly for \( \varphi_2 \). Observe now that, from \( f_0 = \lambda_2 f_1 \lambda_1 \) and (9.3.77), \( |f_0| \) and \( 1/|f_0| \) can be bounded by a constant depending only on \( \alpha \), \( H_\infty \), and \( \delta_0 \). Thus from (9.3.68)
\[
1/C_5 \leq \|f_{\varepsilon_2}\|_{L^\infty((0,\infty) \times (0,L))} \leq C_5,
\]
(9.3.78)
where $C_5$ only depends on $\alpha$, $H_\infty$ and $\delta_0$. And therefore, from (9.3.23), (9.3.64), (9.3.62), and (9.3.78) one has

$$|\partial_t f_1| \leq C_6 \delta,$$  

(9.3.79)

and similarly

$$|\partial_t f_2| \leq C_7 \delta,$$  

(9.3.80)

where $C_6$ and $C_7$ are constants that only depend on $\alpha$, $H_\infty$ (and $\delta_0$). We now select the bound on $\max (|\partial_t H_1|, |\partial_t V_1|)$: we select $\delta_3 \in (0, \delta_0)$ such that, for any $\delta \in [0, \delta_3]$ and any $(t,x) \in [0, \infty) \times [0, L],$

$$C_6 C_5^2 C_7^2 \delta < \varepsilon_2,$$  

(9.3.81)

and

$$\varepsilon_2^2 + 2 \varepsilon_2 \inf_{x \in [0, L], t \in [0, +\infty), \varepsilon \in (0, \varepsilon_2)} \left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi \lambda_2} f^2_x + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right)$$

$$> \left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi \lambda_2} X^2 + \sqrt{\frac{g}{H_1}} \delta + \varepsilon_2 \right) \left( C_6 \frac{\varphi^2}{X^2} + C_7 \varphi_2^2 \right) (\frac{X}{\varphi_1 \varphi_2})^2 \delta$$

$$+ 2 \sqrt{\frac{g}{H_1}} \left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\varphi^{-1} \delta_1}{\lambda_2} X^2 \right) \delta + (\frac{X}{\varphi_1 \varphi_2})^2 C_7 C_6 \delta^2,$$

(9.3.82)

for any $x \in [0, L]$ and any $X \in [1/C_5, C_5]$. Observe that this is obviously possible as $\varepsilon_2 > 0$ and, when $\delta_3 = 0$, (9.3.82) is verified and the inequality is strict. Then, from (9.3.22), (9.3.77), (9.3.74), (9.3.78), and (9.3.82),

$$(-\lambda_1 f_1)_x + 2 \gamma_1 f_1 - \partial_t f_1) (\lambda_2 f_2)_x + 2 \delta_2 f_2 - \partial_t f_2) > \left( \frac{\varphi_1 \varphi_2}{f_x^2} \right)^2 \left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi \lambda_2} f^2_x \right)^2$$

$$= \left( \frac{\gamma_2}{\lambda_1} + \frac{\delta_1}{\lambda_2} \right)^2,$$

(9.3.83)

which is exactly the second inequality of (9.3.52). Besides, from (9.3.25) and (9.3.81),

$$(-\lambda_1 f_1)_x + 2 \gamma_1 f_1 - \partial_t f_1 = \varphi_1 \frac{\partial_t f_x}{f_x^2} - \partial_t f_1$$

$$= \frac{\varphi_1^2}{f_x^2} \left( \left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\varphi \lambda_2} f^2_x + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) + \varepsilon_2 - \frac{\partial_t f_1 f^2_x}{\varphi_1^2} \right)$$

$$> 0.$$

We can now check that (9.3.50) and (9.3.51) are also verified thanks to the choice of $\varepsilon_2$ and $\eta_1$. Indeed, using (9.3.57) and (9.3.16), one has

$$\frac{\lambda_2(0) f_2(t, 0)}{\lambda_1(0) f_1(t, 0)} = f_{x_0}(t, 0) > \left( \frac{\lambda_2(0)}{\lambda_1(0)} \right)^2 > \left( \frac{\lambda_2(0)}{\lambda_1(0)} \right)^2 = k^2_2.$$  

(9.3.85)

This explains our choice of initial condition for $f_{x_0}$. Now, from (9.3.68), one has

$$\frac{\lambda_1(t, L) f_1(t, L)}{\lambda_2(t, L) f_2(t, L)} = \frac{\varphi^2(t, L)}{f_x^2(L)} > \frac{1}{\left( \frac{\lambda_2(t, L)}{\lambda_1(t, L)} + \eta_1 \right)^2},$$  

(9.3.86)

and from the definition of $\eta_1$ given by (9.3.67),

$$\eta_1 + \frac{\lambda_2(L)}{\lambda_1(L)} = \min \left( \frac{1}{|k_1|}, 1 \right).$$  

(9.3.87)

Therefore

$$\frac{\lambda_1(t, L) f_1(t, L)}{\lambda_2(t, L) f_2(t, L)} > \max(k^2_1, 1),$$  

(9.3.88)

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and in particular the condition \([9.3.51a]\) is verified. Let us now look at condition \([9.3.51b]\). So far we have not selected the positive constant \(q\). We want to show that there exists \(q > 0\) such that the condition \([9.3.51b]\) is satisfied. Observe that the left-hand side of \([9.3.51b]\) can be seen as a polynomial in \(q\), and the condition \([9.3.51b]\) can be rewritten as

\[
P(q) := -\frac{q^2}{4} \frac{H_1}{g} (k_1 - 1)^2 + q \sqrt{H_1/g} k_3 (\lambda_1 f_1(L) - \lambda_2 f_2(L) (k_1^2 - k_1(k_1 - 1))) - (\lambda_1 f_1(L)) (\lambda_2 f_2(L)) k_3^2
\]

\[
= -\frac{q^2}{4} \frac{H_1}{g} (k_1 - 1)^2 + q \sqrt{H_1/g} k_3 (\lambda_1 f_1(L) - \lambda_2 f_2(L) k_1)) - (\lambda_1 f_1(L)) (\lambda_2 f_2(L)) k_3^2 > 0.
\]

\[(9.3.89)\]

From \([9.3.88]\) \(\lambda_1 f_1(t, L) > \lambda_2 f_2(t, L) k_1 \) and from \([9.3.66]\) \(k_3 > 0\). Thus the real roots of \(P\) are positive if they exist. This implies that there exists a positive constant \(q\) such that \([9.3.51b]\) is satisfied if the discriminant of \(P\) is positive. Denoting its discriminant by \(\Delta\),

\[
\Delta = \frac{H_1}{g} k_3^2 \lambda_2^2 f_2(t, L)^2 \left[ \left( \frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} - k_1 \right)^2 - \left( \frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} \right) (k_1 - 1) \right].
\]

\[(9.3.90)\]

Let us introduce \(h : X \to (X - k_1)^2 - X(k_1 - 1)^2\). The function \(h\) is a second order polynomial with a positive dominant coefficient and observe that its roots are \(k_1^2\) and 1. Thus \(h\) is increasing strictly on \([\max(k_1^2, 1), +\infty)\). Hence, using \([9.3.88]\),

\[
\Delta = \frac{H_1}{g} k_3^2 \lambda_2^2 f_2(t, L)^2 h\left( \frac{\lambda_1 f_1(L)}{\lambda_2 f_2(L)} \right)
\]

\[
> \frac{H_1}{g} k_3^2 \lambda_2^2 f_2(t, L)^2 h(\max(k_1^2, 1)) = 0.
\]

\[(9.3.91)\]

This proves that there exists \(q > 0\) such that \([9.3.51b]\) is satisfied, and we select such \(q\). All it remains to do now is to show that the function \((U, z) \to V(t, U, z)\), which is now entirely selected, satisfies \([9.3.20]\).

From \([9.2.13]\) and \([9.2.12]\) we know that for any \((t, x) \in [0, \infty) \times [0, L] , \)

\[
\sqrt{g H_\infty} > \lambda_2 > \alpha, \ 2\sqrt{g H_\infty} > \lambda_1 > \alpha.
\]

\[(9.3.92)\]

Besides, from the definition of \(\varphi_1\) and \(\varphi_2\) given by \([9.3.22]\), \([9.3.14]\) and the bound \([9.2.13], [9.2.12]\), there exists a constant \(C_8\) that only depends on \(\delta, \alpha\) and \(H_\infty\) such that

\[
1 \leq \|\varphi_1\|_\infty \leq C_8, \ \frac{1}{C_8} \leq \|\varphi_2\|_\infty \leq C_8.
\]

\[(9.3.93)\]

Thus, using that \(f_0 = \lambda_2 \varphi/\lambda_1\) \([9.3.70], [9.3.68], [9.3.93]\), and \([9.3.92]\), there exists \(c_1 > 0\) constant independent of \(U\) and \(z\) such that, for any \((U, z) \in H^2(0, L) \times \mathbb{R} , \)

\[
c_1 (\|U\|_{H^2(0, L)} + |z|) \leq V(t, (U, z)) \leq \frac{1}{c_1} (\|U\|_{H^2(0, L)} + |Z|) \ \forall \ t \in [0, +\infty),
\]

\[(9.3.94)\]

which is exactly \([9.3.20]\). This concludes the proof of Theorem \([9.2.3]\). \(\square\)

### 9.4 Conclusion

In this chapter we gave simple conditions on the design of a single PI controller to ensure the exponential stability of the nonlinear Saint-Venant equations with arbitrary friction and slope in the \(H^2\) norm. These conditions apply when the inflow is an unknown constant, that case the system has steady-states and any of them are stable. But they also apply when the inflow is time-dependant and slowly variable. In that
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9.5 Appendix

9.5.1 Proof of Proposition 9.2.10

This appendix uses many computations that are very similar to the computations in Section 9.3, but in a simpler way. Thus, in order to avoid writing two times the same thing and to keep the proof relatively short, some steps might be quicker in this appendix. Let \( T_1 > 0 \) and to be chosen later on. As \((H_0(0), V_0(0))\) satisfies (9.2.9), there exists \( \nu_a > 0 \) such that for \( \nu \in (0, \nu_a) \), \( F((H_0^1, V_0^1)^T) \) has two distinct nonzero eigenvalues. Recall that \( F \) is given by (9.3.7) and that that \( \nu \) is the bound on \( \|H_1^0 - H_0(0), V_1^0 - V_0(0)\|_{H^2(0,L)} \). Besides, from (9.2.8) \((H_0(t, \cdot), V_0(t, \cdot))\) can be seen as the solution of a system of ODE with a parameter \( t \) in the initial condition. Thus, as \( \delta \), \( 0 < \delta \in C^2([0, +\infty)) \) and the slope \( C \) satisfies \( C \in C^2([0, L]) \), using (9.2.6) and (10.3) [Chap. 5, Theorem 3.1], \((H_0, V_0) \in C^1([0, T_1] \times C^4([0, L]))\) and there exists a constant \( C \) depending only on \( H_\infty \), \( \alpha \) and an upper bound of \( \delta \), such that,

\[
\|\partial^1 H_0, \partial^1 V_0\|_{C^2([0, L])} \leq C \sum_{i=1}^1 \|H_0, V_0\|_{C^2([0, L])}, \quad \forall \ i \in [1, 3], \tag{9.5.1}
\]

and in particular

\[
\|\partial^1 H_0, \partial^1 V_0\|_{C^2([0, T_1], C^2([0, L]))} \leq C \|\delta Q_0\|_{C^2([0, +\infty)} \tag{9.5.2}
\]

Thus [105][Theorem 2.1] can still be used on \((H_1 - H_0)\) and there exist \( \delta_0(T_1) > 0 \) and \( \nu_0(T_1) \in (0, \nu_a) \) such that, if \( \nu \in (0, \nu_0(T_1)) \) and \( \delta \in (0, \delta_0(T_1)) \), there exists a unique solution \((H_1, V_1) \in C^0([0, T_1]; H^2(0, L))^2\) to the system (9.2.4)–(9.2.5). Besides, \((H_1, V_1)\) satisfies an estimate as \( (9.2.2) \) but with \((H_1, V_1)\) instead of \((H, V)\) and \((H_0, V_0)\) instead of \((H_1, V_1)\). We denote by \( C(T_1) \) the associated constant. Let us define \( h_1 := H_1 - H_0 \) and \( v_1 := V_1 - V_0 \). We transform \((h_1, v_1)^T\) into \( w = (w_1, w_2)^T\) using the change of variables defined by (9.3.1)–(9.3.5) with \( H_0 \) and \( V_0 \) instead of \( H_1 \) and \( V_1 \). Thus we obtain

\[
\partial_t w + A_0(w, \cdot) \partial_x w + B_0(w, x) + S_0 \left( \frac{\partial H_0}{\partial V_0} \right) = 0,
\]

\[
w_1(t, 0) = H_1(w_2(t, 0), Q_0(t) - Q_0(0)),
\]

\[
w_2(t, L) = H_2(w_2(t, L)), \tag{9.5.3}
\]

where

\[
A_0(w, x) = \left( \begin{array}{c}
H_0(0, x) \nu_0(T_1) \\
\frac{\partial H_0}{\partial V_0}(0, x)
\end{array} \right),
\]

\[
B_0(w, x) = \left( \begin{array}{c}
H_0(x) \\
\frac{\partial H_0}{\partial V_0}(x)
\end{array} \right),
\]

\[
S_0 = \left( \begin{array}{c}
1 \\
0
\end{array} \right).
\]
where \( A_0, B_0 \) and \( S_0 \) have the same expression as \( A, B \) and \( S \) (given by (9.3.9), (9.3.10), (9.3.6)) but with \((H_0, V_0)\) instead of \((H_1, V_1)\). Similarly we define
\[
\lambda_1^0 = V_0 + \sqrt{gH_0}, \quad \lambda_2^0 = \sqrt{gH_0} - V_0, \tag{9.5.4}
\]
and \( \varphi^0 \), defined as \( \varphi \) but with \((H_0, V_0)\) instead of \((H_1, V_1)\). Similarly as in Appendix 9.3.3
\[
\mathcal{H}_2'(0) = -\lambda_1^0(L)/\lambda_2^0(L), \quad \mathcal{H}_1'(0) = -\lambda_1^0(0)/\lambda_2^0(0), \tag{9.5.5}
\]
which is of the form \([9.3.15]\) with \( \nu_c = 0 \) and \( Z = 0 \). Before going any further, note that we can perform the same computations as in Section 2 with no problem, as the proof in Section 9.3 only used Proposition 9.2.1 to get that \((H_1, V_1)\) exists for any time and that \([9.2.13]\) and Lemma 9.3.3 hold, but we will see now that such claims are true for \( H_0 \) and \( V_0 \). The existence of \((H_0, V_0)\) was already shown in section 9.2 and \((9.2.9)\) is exactly \([9.2.13]\) with \((H_1, V_1)\). Finally, \((9.2.8)\) is exactly the equivalent of Lemma 9.3.3 for \((H_0, V_0)\). We define now the Lyapunov function candidate \( V := V_0(w(t, x), t) + V_0(w(t, x), t) + V_c(w(t, x), t) + V_c(w(t, x), t) \) where \( V_c, V_0 \) and \( V_c \) are defined in \([9.3.27]\), \([9.3.30]\), with \( f_1 \) and \( f_2 \) chosen as \( f_1 := (\varphi(t))^2/\lambda_1^0(t) \) and \( f_2 := (\varphi(t))^2/\lambda_2^0(t) \), where \( \eta \) is a function such that there exists a constant \( \varepsilon > 0 \) independent of \( w \) such that
\[
\eta' = \left[ \frac{\gamma_0}{\lambda_1^0} + \frac{\delta_0}{\lambda_2^0} \right] \eta^2 + \varepsilon, \forall x \in [0, L], \tag{9.5.6}
\]
\[
\eta(0) = \frac{\lambda_1^0(0)}{\lambda_2^0(0)} \varphi^0(0) + \varepsilon.
\]
Note that \( \eta \) exists as, for any \( t \in [0, +\infty) \), \((\varphi(t)^0 \lambda_1^0(t) / \lambda_1^0(t))\) is a solution of
\[
\partial_x f = \left[ \frac{\gamma_0}{\lambda_1^0} + \frac{\delta_0}{\lambda_2^0} \right] \eta^2, \forall x \in [0, L], \tag{9.5.7}
\]
this can be proved as in Lemma 9.3.2, and this case was actually shown in \([110]\). Note that from \((9.2.6), (9.2.8) \) and \((9.2.9)\), \((H_0)_x\) and \((V_0)_x\) can be bounded by above and by below by constants that only depend on \( H_\infty, \alpha \) and an upper bound of \( Q_0 \) (which can also be expressed only with \( H_\infty, \alpha \) and \((9.2.9)\)). Therefore, looking at their definition, the function \( f_1 \) and \( f_2 \) can also be bounded by above and by below by constants that only depend on \( H_\infty, \alpha \) and \( \varepsilon \). Thus there exist \( c_1 > 0 \) and \( c_2 > 0 \) depending only on \( H_\infty \) and \( \alpha, \varepsilon \) and \( \mu \) such that
\[
c_1 \left\| h_1(t, \cdot), v_1(t, \cdot) \right\|_{H^3(0, L)} + V(t) \leq c_2 \left\| h_1(t, \cdot), v_1(t, \cdot) \right\|_{H^3(0, L)}, \forall t \in [0, T_1], \tag{9.5.8}
\]
Consequently, by differentiating \( V \) exactly as in \([9.3.33], \tag{9.3.49}\) and from \([9.5.3]\), we obtain that there exists \( \mu > 0, v_1 \in (0, V_0(T_1)) \) and \( \delta_3 > 0 \) such that, for any \( \left\| h_1(0, \cdot), v_1(0, \cdot) \right\|_{H^3(0, L)} \leq v_1 \), and \( \left\| \partial_\tau Q_0 \right\|_{C^2(0, \infty)} \leq \delta \), where \( \delta \in (0, \delta_3) \),
\[
\dot{V} \leq -\mu V + \int_0^L 2f_1w_1(S_0) (\partial_\tau H_0 \partial_\tau V_0) dx + 2f_2w_2(S_0) (\partial_\tau H_0 \partial_\tau V_0) dx + \int_0^L 2f_1\partial_\tau w_1(S_0) (\partial_\tau H_0 \partial_\tau V_0) dx + 2f_2\partial_\tau w_2(S_0) (\partial_\tau H_0 \partial_\tau V_0) dx, \tag{9.5.9}
\]
Thus, using Cauchy-Schwarz inequality, \((9.5.8), \tag{9.5.1}\) there exists \( C_4 > 0 \) depending only on \( H_\infty, \alpha \) and an upper bound of \( \mu \) such that
\[
\dot{V}(t) \leq -\mu V(t) + C_4 \left\| \partial_\tau Q_0(t) \right\|_{C^2(0, \alpha)} V(t)^{1/2}, \forall t \in [0, T_1]. \tag{9.5.10}
\]
and in particular
\[
\dot{V}(t) \leq -\mu V(t) + C_4 \left\| \partial_\tau Q_0 \right\|_{C^2(0, \alpha)} V(t)^{1/2}, \forall t \in [0, T_1]. \tag{9.5.11}
\]
Let us define \( V_{eq} := (C_1\delta/\mu)^2 \). From (9.5.11), if \( V(t) > 2V_{eq} \), then there exists a constant \( k > 0 \) such that \( \dot{V}(t) < -kV^{1/2}(t) \). We now choose \( \delta \) such that \( \sqrt{2}C_1\delta/(\mu\sqrt{c_1}) < \nu_1 \). Thus, from (9.5.11) and as \( c_1, c_2, C_1 \) and \( \mu \) do not depend on \( T_1 \), we can choose \( \delta \) large enough such that

\[
V(T_1) \leq 2V_{eq} \leq c_1\nu_1^2, \tag{9.5.12}
\]

which implies that

\[
\|h_1(T_1, \cdot), v_1(T_1, \cdot)\|_{C^2[0, L]} \leq \nu_1 \tag{9.5.13}
\]

and therefore there exists a unique solution \( (h_1, v_1) \in C^0([T_1, 2T_1], H^2(0, L)) \), with initial condition \( (h_1(T_1, \cdot), v_1(T_1, \cdot)) \) (we use the same existence Theorem ([196] Theorem 2)) and, noting that \( V(T_1) \leq 2V_{eq} \) implies \( V(2T_1) \leq 2V_{eq} \), this analysis still hold. We can do similarly for any \( [nT_1, (n + 1)T_1] \) with \( n \in \mathbb{N} \), thus, as \( (H_0, V_0) \in C^0([0, \infty), H^2(0, L)) \), there exists a unique solution \( (H_1, V_1) \in C^0([0, \infty), H^2(0, L)) \) and (9.5.10) holds for any \( t \in [0, \infty) \). Therefore denoting \( g(t) = V(t)e^\mu t \), we deduce from (9.5.10) that

\[
g'(t) \leq C_1|\partial_t Q_0(t) + \partial^2_t Q_0(t) + \partial^3_{ttt} Q_0(t)|e^{\mu t} \sqrt{g(t)}. \tag{9.5.14}
\]

Thus

\[
V^{1/2}(t) \leq V^{1/2}(0)e^{-\mu t} + \frac{C_1}{2} \left( \int_0^t |\partial_t Q_0(s) + \partial^2_t Q_0(s) + \partial^3_{ttt} Q_0(s)|e^{\mu s} ds \right) e^{-\mu t}. \tag{9.5.15}
\]

This implies the ISS property

\[
\|h_1(t, \cdot), v_1(t, \cdot)\|_{H^2((0, L); \mathbb{R}^2)} \leq \frac{C_2}{\sqrt{c_1}} \|h_1(0, \cdot), v_1(0, \cdot)\|_{H^2((0, L); \mathbb{R}^2)} e^{-\mu t} + \frac{C_1}{2\sqrt{c_1}} \left( \int_0^t |\partial_t Q_0(s) + \partial^2_t Q_0(s) + \partial^3_{ttt} Q_0(s)|e^{\mu s} ds \right) e^{-\mu t}. \tag{9.5.16}
\]

This ends the proof of Proposition 9.2.1. To extend this proof to the \( H^p \) norm for \( p > 2 \), note that using the same argument \( \|h_1, v_1\|_{C^0([0, L]; H^4(0, L))} \) holds with the \( C^0([0, L]; C^4(0, L)) \) norm in the left-hand side and the \( C^p \) norm in the right-hand side. We can define \( V_{eq} \) on \( H^p(0, L) \times \mathbb{R} \times \mathbb{R}^+ \) as in (9.3.30) such that \( V_k(w(t, x), t) = V_0(\partial^k w(t, x), t) \), for any \( k \in [3, p] \). Then (9.5.10) holds with \( V := V_a + V_b + V_c + V_3 + \ldots V_p \) and the \( H^p \) norm, and the rest can done done identically.

### 9.5.2 Proof of Theorem 9.2.4

Theorem 9.2.4 result from the proof of Theorem 9.2.3. Note that the boundary conditions (9.2.16) can be written under the form (9.2.18) with \( (H_0, V_0) \) instead of \( (H_1, V_1) \) where the only difference is that \( Z \) satisfies now

\[
Z = H_\varepsilon - H(t, L) + \frac{f(t)}{tvG^k}, \tag{9.5.17}
\]

where \( f(t) = H_\varepsilon \partial_\varepsilon V_0(t, L) \). The rest of the proof can be conducted as in Appendix 9.5.1 for \( (H_1, V_1) \), with \textit{a priori} two differences : \( (H, V) \) satisfies the boundary conditions of the form (9.2.18) and not of the form given in (9.2.4), and \( Z \) satisfies (9.5.17) instead of (9.2.15). However, note that in Appendix 9.5.1 the only assumption used on the boundary conditions of the transformed system is that they are of the form (9.3.3), which is still the case here. Thus, the only difference with Appendix 9.5.1 are some additional terms when \( Z \) is used, which is in the boundary terms in the derivative of the Lyapunov function. There exists therefore \( \delta_4 > 0 \) and \( \nu_2 > 0 \) such that, for any \( \|h_1(0, \cdot), v_1(0, \cdot)\|_{H^2(0, L)} \leq \nu_2 \), and \( \|\partial_t Q_0\|_{C^2([0, \infty))} \leq \delta_4 \), where \( \delta \in (0, \delta_4) \),

\[
\dot{V}(t) \leq -\frac{\gamma}{2}V(t) + C_1|\partial_t Q_0(t) + \partial^2_t Q_0(t) + \partial^3_{ttt} Q_0(t)|V^{1/2} + 2qZf(t) + 2q\dot{Z}f'(t) + 2q\ddot{Z}f''(t), \tag{9.5.18}
\]

where \( C_1 \) is a constant only depending on \( H_\infty, \alpha, \nu_2 \) and \( \delta_4 \). Using Lemma 9.3.3 there exists a constant \( C > 0 \) depending only on \( H_\infty, \alpha, \nu_2 \) and \( \delta_4 \) such that

\[
\dot{V} \leq -\frac{\gamma}{2}V + CV^{1/2}|\partial_t Q_0(t) + \partial^2_t Q_0(t) + \partial^3_{ttt} Q_0(t)|. \tag{9.5.19}
\]

The same argument as in Appendix 9.5.1 (9.5.14) - (9.5.16), implies directly the ISS property (9.2.30).
9.5.3 Boundary conditions (9.3.15) and (9.3.16)  

In this appendix we justify the boundary conditions (9.3.15) with (9.3.16) after the change of variables. From the boundary conditions (9.3.3) in the physical coordinate \((h, v)\), together with the definition of \(u_1\) and \(u_2\) given in (9.3.5), one has at \(x = L\)

\[
\begin{align*}
\frac{\partial}{\partial x}B_2(h(t, L), Z(t), t) & + \frac{\partial}{\partial t}F_2(h(t, L), Z(t), t, x) = - \frac{g}{H_1}h(t, L) = : \mathcal{F}_2(h(t, L), Z(t), x, t), \\
\frac{\partial}{\partial x}B_2(h(t, L), Z(t), t) & - \frac{\partial}{\partial t}F_2(h(t, L), Z(t), t, x) = - \frac{g}{H_1}h(t, L) = : \mathcal{F}_2(h(t, L), Z(t), x, t).
\end{align*}
\]  

(9.5.20)

From its definition, \(\mathcal{F}_1\) is \(C^1\) and, from \(9.3.3\), \(9.2.21\), there exists \(\nu_1 \in (0, \nu_0)\) such that, for any \(t \in [0, \infty)\), \(\partial_t \mathcal{F}_0(0, Z(t), t) \neq 0\). Thus \(\mathcal{F}_1\) is locally invertible with respect to its first variable, thus there exists \(\nu_2 \in (0, \nu_1)\) such that \(h(t, L) = \mathcal{F}_1^{-1}(u_1(t, L), Z(t), t)\), where \(\mathcal{F}_1^{-1}\) denotes the inverse with respect to the first variable. Besides, as \(\mathcal{F}_1\) is of class \(C^2\) with respect to the two first variables, \(\mathcal{F}_1^{-1}\) is also of class \(C^2\). Then, using (9.5.20)

\[
\begin{align*}
\frac{\partial}{\partial x}B_2(h(t, L), Z(t), t) & = \mathcal{F}_2(\mathcal{F}_1^{-1}(u_1(t, L), Z(t), t), Z(t), t) = : \mathcal{D}_2(u_1(t, L), Z(t), t).
\end{align*}
\]  

(9.5.21)

and, using (9.3.4),

\[
\begin{align*}
\frac{\partial}{\partial t}D_2(u_1, 0, 0, t) & = \frac{\partial}{\partial t} \mathcal{F}_2(0, 0, t) \partial_t(\mathcal{F}_1^{-1})(0, 0, t) \\
& = \frac{\partial}{\partial t} \mathcal{F}_2(0, 0, t) \left( \frac{\partial}{\partial \mathcal{F}_1}(0, 0, t) \right) - \frac{g}{H_1} \left( \frac{\partial}{\partial \mathcal{F}_1}(0, 0, t) \right) \\
& = - \frac{\lambda_1(L) - \nu_G(1 + k_p)}{\lambda_2(L) + \nu_G(1 + k_p)}.
\end{align*}
\]  

(9.5.22)

Now, as \(\partial_t \mathcal{F}_1^{-1}(0, 0, t) = -\partial_t \mathcal{F}_1(0, 0, t)/\partial_t \mathcal{F}_1(0, 0, t)\), using (9.3.4),

\[
\begin{align*}
\frac{\partial}{\partial t}D_2(u_1, 0, 0, t) & = \frac{\partial}{\partial t} \mathcal{F}_2(0, 0, t) \partial_t(\mathcal{F}_1^{-1})(0, 0, t) + \partial_t \mathcal{F}_2(0, 0, t) \\
& = \frac{\partial}{\partial t} \mathcal{F}_2(0, 0, t) \left( \frac{\partial}{\partial \mathcal{F}_1}(0, 0, t) \right) + \partial_t \mathcal{F}_2(0, 0, t) \\
& = \partial_t \mathcal{F}_2(0, 0, t) \left( 1 - \frac{\partial}{\partial \mathcal{F}_1}(0, 0, t) \right) \\
& = - \frac{\nu_G k_1}{H_1(t, L)} \left( \frac{2 \lambda_2(L)}{\nu_G(1 + k_p) + \lambda_2(L)} \right).
\end{align*}
\]  

(9.5.23)

The same can be done in \(x = 0\) in a slightly easier way, as \(\mathcal{B}_1\) does not depends on \(Z\). This gives (9.3.15) and (9.3.16).

9.5.4 Proof of Lemma 9.3.2  

In this appendix we prove Lemma 9.3.2. The proof is very similar to the proof given in [110] in the special case where \((H_1, V_1)\) is a steady state. However, it happens that the proof actually does not need the relation \((H_1 V_1)_x = 0\) which is no longer true when \((H_1, V_1)\) is not a steady-state. Let \(f = (\lambda_2 \varphi/\lambda_1)\), we have from
\[ \partial_x f = \frac{\varphi}{\lambda_1} (\lambda_1 \partial_x \lambda_2 - \lambda_2 \partial_x \lambda_1 + \lambda_2 \gamma_1 + \lambda_1 \delta_2) \]

\[ = \frac{\varphi}{\lambda_1} \left( (V_1 + \sqrt{gH_1})(-V_{1x} + \frac{\sqrt{gH_1}}{2H_1} H_{1x}) - (V_1 + \sqrt{gH_1})(V_{1x} + \frac{\sqrt{gH_1}}{2H_1} H_{1x}) \right) \]

\[ + (\sqrt{gH_1} - V_1) \left( \frac{3}{4} \sqrt{\frac{g}{H_1}} H_{1x} + \frac{3}{4} V_{1x} + \frac{kV_1^2}{H_1} - \frac{kV_1^2}{2H_1^2} \sqrt{\frac{H_1}{g}} \right) \]

\[ + (V_1 + \sqrt{gH_1}) \left( -\frac{3}{4} \sqrt{\frac{g}{H_1}} H_{1x} + \frac{3}{4} V_{1x} + \frac{2kV_1}{H_1} + \frac{kV_1^2}{2H_1^2} \sqrt{\frac{H_1}{g}} \right) \] (9.5.24)

\[ = \frac{\varphi}{\lambda_1} \left( \sqrt{gH_1} \left( -2V_{1x} + \frac{3}{2} V_{1x} + \frac{2kV_1}{H_1} \right) - V_1 \left( \frac{3}{2} \sqrt{\frac{g}{H_1}} H_{1x} - \frac{kV_1^2}{H_1^2} \sqrt{\frac{H_1}{g}} - \sqrt{\frac{g}{H_1}} H_{1x} \right) \right) \]

\[ = \frac{\varphi}{\lambda_1} \left( \frac{2kV_1}{H_1} \sqrt{gH_1} + \frac{kV_1^2}{H_1^2} \sqrt{\frac{H_1}{g}} V_1 + \frac{1}{2} \sqrt{\frac{g}{H_1}} \partial_t H_1 \right). \]

And on the other hand:

\[ \left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\lambda_2 \varphi} f^2 \right) = \frac{\varphi}{\lambda_1} (\lambda_1 \gamma_2 + \lambda_2 \delta_1) \]

\[ = \frac{\varphi}{\lambda_1} \left( \frac{2kV_1}{H_1} \sqrt{gH_1} + \frac{kV_1^2}{H_1^2} \sqrt{\frac{H_1}{g}} V_1 + V_1 \sqrt{\frac{g}{H_1}} H_{1x} + \frac{1}{2} \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) \] (9.5.25)

Thus from (9.5.24) and (9.5.25)

\[ \partial_x f = \left( \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\lambda_2 \varphi} f^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right). \] (9.5.26)

And there exists \( \delta_0 \) such that, if \( \| \partial_t H_1 \|_{L^\infty((0,+\infty)\times[0,L])} \delta_0, \)

\[ \frac{\varphi}{\lambda_1} \left( \frac{2kV_1}{H_1} \sqrt{gH_1} + \frac{kV_1^2}{H_1^2} \sqrt{\frac{H_1}{g}} V_1 + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right) > 0, \ \forall \ x \in [0,L], \ t \in [0,+\infty), \] (9.5.27)

and, from (9.5.24) and (9.5.26),

\[ \partial_x f = \left| \frac{\varphi \gamma_2}{\lambda_1} + \frac{\delta_1}{\lambda_2 \varphi} f^2 + \sqrt{\frac{g}{H_1}} \partial_t H_1 \right|, \] (9.5.28)

this ends the proof of Lemma 9.3.2.
Bibliography


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References


Abstract:
This thesis is devoted to study the stabilization of nonlinear hyperbolic systems of partial differential equations. The main goal is to find boundary conditions ensuring the exponential stability of the system. In a first part, we study general systems that we aim at stabilizing in the $C^1$ norm by introducing a certain type of Lyapunov functions. Then we take a closer look at systems of two equations and we compare the results with the stabilization in the $H^2$ norm. In a second part we study a few physical equations: Burgers’ equation and the density-velocity systems, which include the Saint-Venant equations and the Euler isentropic equations. Using a local dissipative entropy, we show that these systems can be stabilized with very simple boundary controls which, remarkably, do not depend directly on the parameters of the system, provided some physical admissibility condition. Besides, we develop a way to stabilize shock steady-states in the case of Burgers’ and Saint-Venant equations. Finally, in a third part, we study proportional-integral (PI) controllers, which are very popular in practice but seldom understood mathematically for nonlinear infinite dimensional systems. For scalar systems we introduce an extraction method to find optimal conditions on the parameters of the controller ensuring the stability. Finally, we deal with the Saint-Venant equations with a single PI control.

Keywords: stabilization, partial differential equations, inhomogeneous, nonlinear, Boundary feedback control, Lyapunov functions, entropy, Saint-Venant equations, Burgers’ equation.

Stabilisation de systèmes hyperboliques non-linéaires en dimension un d’espace

Résumé:
Cette thèse est consacrée à l’étude de la stabilisation des systèmes d’équations aux dérivées partielles hyperboliques non-linéaires. L’objectif principal est de trouver des conditions de bords garantissant la stabilité exponentielle du système. Dans une première partie on s’intéresse à des systèmes généraux qu’on cherche à stabiliser en norme $C^1$ en introduisant un certain type de fonctions de Lyapunov, puis on regarde plus précisément les systèmes de deux équations pour lesquels on peut comparer nos résultats avec la stabilisation en norme $H^2$. On s’intéresse ensuite à quelques équations physiques: l’équation de Burgers et les systèmes densité-vitesse, dont font partie les équations de Saint-Venant et les équations d’Euler isentropiques. À l’aide d’une entropie locale dissipative, on montre qu’on peut stabiliser les systèmes densité-vitesse par des contrôles aux bords simples et, étonnamment, ces contrôles ne dépendent pas explicitement des paramètres du système, pourvu qu’ils soient physiquement admissibles. Par ailleurs, on développe une méthode pour stabiliser les états-stationnaires avec un choc dans le cas de l’équation de Burgers et des équations de Saint-Venant. Enfin, dans une troisième partie on s’intéresse aux contrôles proportionnels-intégraux (PI), très utilisés en pratique mais mal compris mathématiquement dans le cas des systèmes non-linéaires de dimension infinie. Pour les systèmes d’une seule équation on introduit une méthode d’extraction pour trouver des conditions optimales de stabilité sur les paramètres du contrôle. Finalement on traite le cas des équations de Saint-Venant avec un unique contrôle PI.

Mot clés: stabilisation, équations aux dérivées partielles, inhomogénéités, non-linéaire, contrôles aux bords, fonctions de Lyapunov, entropie, équations de Saint-Venant, équation de Burgers.