Null controllability of the Burgers system with distributed controls

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Abstract

This paper is devoted to present some positive and negative controllability results for the viscous Burgers equation. More precisely, in the context of null controllability with distributed controls, we present sharp estimates of the minimal time of controllability $T(r)$ of initial data of $L^2$-norm less or equal to $r$. In particular, we see that (global) null controllability does not hold in general (unless the control is exerted everywhere). The same results apply to similar boundary controllability systems with one boundary control.

Keywords: Burgers equation; Controllability

1. Introduction and main results

Let $T > 0$ be an arbitrary positive time and let us assume that $\omega \subseteq (0, 1)$ is a nonempty open set, with $0 \notin \overline{\omega}$. In this paper, we will be concerned with the null controllability of the following system for the Burgers equation:

$$
\begin{align*}
\frac{\partial y}{\partial t} - \gamma_{xx} + \gamma v \gamma_x &= \gamma I_{\omega}, & (x, t) &\in (0, 1) \times (0, T), \\
y(0, t) &= y(1, t) = 0, & t &\in (0, T), \\
y(x, 0) &= y^0(x), & x &\in (0, 1).
\end{align*}
$$

(1)

Here, $1_\omega$ stands for the characteristic function associated to $\omega$, $v = v(x, t)$ denotes the control and $y = y(x, t)$ denotes the state.

It will be said that (1) is null controllable at time $T$ if, for every $y^0 \in L^2(0, 1)$, there exists $v \in L^2((0, 1) \times (0, T))$ such that the solution $y \in L^2(0, T; H^1_0(0, 1)) \cap C^0([0, T]; L^2(0, 1))$ of (1) satisfies

$$
y(x, T) = 0 \quad \text{in} \quad (0, 1).
$$

(2)

System (1) arises as a 1-D simplification of the control system associated to the Navier–Stokes system, which has been extensively studied these last years (see, for instance, [9,4]). As it is well known, the solution of this kind of systems decay exponentially in the absence of control, that is why one is usually interested in the null controllability.

Some controllability properties of (1) have been studied in [10] (see Chapter 1, Theorems 6.3 and 6.4). There, it is shown that one cannot reach (even approximately) stationary solutions of (1) with large $L^2$-norm at any time $T$. Another work where a similar result is proved is [6].

In the more recent work [5] the author proves that, with the help of two boundary control forces, one can drive the solution of the corresponding Burgers equation from the null state to large constants $M$ at any time. A negative result concerning the same control system has recently been proved in [11]; more details are given in Remark 2.

Some results on the attainable set of states of the nonviscous Burgers equation were proved in [12]; further results in this sense are given in [1]. Both papers are concerned with the so-called ‘entropy solutions’.

For each $y^0 \in L^2(0, 1)$, let us introduce

$$
T(y^0) = \inf \{ T > 0 : (1) \text{ is null controllable at time } T \}.
$$

Then, for each $r > 0$, let us set

$$
T(r) = \sup \{ T(y^0) : \| y^0 \|_{L^2(0, 1)} \leq r \}.
$$

Our main purpose in this paper is to prove that $T(r) > 0$, with an explicit sharp estimate in terms of $r$ as $r \to 0^+$. In
Theorem 1. There exist positive constants \( C_0 \) and \( C_1 \) independent of \( r \) such that
\[
C_0 \phi(r) \leq T(r) \leq C_1 \phi(r) \quad \text{as} \quad r \to 0^+.
\] (3)

Remark 1. The same estimates hold when the control \( v \) acts on system (1) through the boundary only at \( x = 1 \) (or only at \( x = 0 \)). Indeed, it is very easy to transform the boundary controlled system
\[
\begin{align*}
y_t - y_{xx} + y_x &= 0 \quad (x, t) \in (0, 1) \times (0, T), \\
y(0, t) &= 0, \quad y(1, t) = w(t), \quad t \in (0, T), \\
y(x, 0) &= y_0(x), \quad x \in (0, 1)
\end{align*}
\]
into a system of the kind (1).

Remark 2. The boundary controllability of the Burgers equation with two controls (at \( x = 0 \) and \( x = 1 \)) has been analyzed in [11]. There, it is shown that even in this more favorable situation null controllability does not hold for small time. It is also proved in this paper that exact controllability does not hold for large time. Let us remark that the results in [11] do not allow to estimate \( T(r) \); moreover, the proofs of the results in [11] are based in contradiction arguments.

Remark 3. We know that there exists \( r_0 > 0 \) such that (3) holds for all \( r \in (0, r_0) \). Now, let us assume that \( y^0 \) is an arbitrary (possibly large) state in \( L^2(0, 1) \), let us set \( R = \|y^0\|_{L^2(0,1)} \) and let us assume that \( T \geq T_s(R) \), where
\[
T_s(R) = \frac{1}{\pi} \log R + \frac{1}{\pi^2 \phi(\min(e^{-\sqrt{C_1 r^2}}, r_0))} + C_1 \phi\left(\min\left(e^{-\sqrt{C_1 r^2}}, r_0\right)\right),
\]
and thus
\[
\|y(\cdot, T_0(R))\|_{L^2(0,1)} \leq r.
\]

Let us now apply Theorem 1 in the time interval \([T_0(R), T_0(R) + C_1 \phi(r)]\), with initial data \( y(\cdot, T_0(R)) \). We deduce that there exists a control such that
\[
y(x, T_0(R) + C_1 \phi(r)) = 0 \quad \text{in} \quad (0, 1).
\]

Observe that this \( r \) given by \( e^{-\sqrt{C_1 r^2}} \) does not tend to zero, but using the fact that the function \( \phi(\cdot) \) is increasing, the last identity holds true. Finally, since \( T_0(R) + C_1 \phi(r) = T_s(R) \), our assertion follows.

Therefore, in order to get null controllability for large \( \|y^0\|_{L^2(0,1)} = R \), we need at most a time of the form \( C_3 + (1/\pi) \log R \).

The plan of this paper is the following. In Section 2, we give the proof of Theorem 1. The proof of the ‘negative’ part, i.e. the first inequality in (3), relies on the construction of a particular solution of the Burgers equation. This has been inspired by the results in [2]. On the other hand, the second estimate in (3) is a consequence of the arguments usually employed to establish the controllability of semi-linear parabolic equations; see, for instance, [10,8].

The results in this paper were announced in [7]. In the sequel, \( Q \) denotes the space–time domain \( Q = (0, 1) \times (0, T) \).

2. Proof of Theorem 1

2.1. Proof of the estimate from below

In this subsection, we will establish the first estimate in (3). More precisely, we will prove that there exists a positive constant \( C_0 \) such that, for any sufficiently small \( r > 0 \), we can find initial data \( y^0 \) satisfying \( \|y^0\|_{L^2(0,1)} \leq r \) and the following property: the solution \( y \) associated to \( y^0 \) satisfies
\[
|y(x, t)| > 0 \quad \text{for some} \quad x \in (0, 1) \quad \text{and} \quad \text{any} \quad t : 0 < t < C_0 \phi(r).
\]

Thus, let \( \rho_0 \in (0, 1) \) be such that \((0, \rho_0) \cap \partial = \emptyset \). In the sequel, we assume that \( \rho_0 \) is as small as desired but satisfying \( 0 < (\log(\log(1/r)))^{-1} < \rho_0 \). Let us choose \( y^0 \in L^2(0, 1) \) such that \( y^0(x) = -r \) for all \( x \in (0, \rho_0) \) and let us denote by \( y \) the associated solution of (1).

In a first step, using the maximum principle, we are going to bound \( y \) from above by an exponential function \( Z = Z(x, t) \) that is given explicitly; see (13). Then, we will introduce an auxiliary Burgers system (14) and we will show that \( y \) is also bounded from above by its solution \( u \); see (17). Finally, we will check that \( u(x, t) \leq -C_0^0 r \) at some point \( x \in (0, 1) \) for all \( t < C_0 \phi(r) \), where \( C_0^0 \), and \( C_0 \) are independent of \( r \). This will end the proof.

We first remark that the values of \( y \) at \( x = \rho_0 \) are not known, so that \( Z \) has to diverge as \( x \to \rho_0 \). Let us introduce the function
\[
\left\{ \begin{array}{ll}
Z(x, t) = \exp \left\{ -(1 - e^{-\rho_0^2/(\rho_0 - x)^2})^2 \frac{1}{t} + \frac{1}{\rho_0 - x} \right\} \\
\forall (x, t) \in (0, \rho_0) \times (0, C_0 \phi(r)).
\end{array} \right.
\]
We compute each term separately:

\[ Z_t = (1 - e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2}) \frac{2}{(\rho_1-x)^2} Z, \]

\[ Z_x = \left( \frac{6(\rho_0-x)^2(\rho_1-x) - 4(\rho_0-x)^3}{(\rho_1-x)^3} \right) \frac{2\rho_0^2}{t} e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2} \frac{2\rho_0^2}{t} + \frac{1}{(\rho_0-x)^2} Z \]

and

\[ Z_{xx} = \left[ \frac{-(\rho_1-x)(6\rho_0-x)(\rho_1-x) - 12(\rho_0-x)^2 - 6(\rho_0-x)^3}{(\rho_1-x)^4} \right] \frac{2\rho_0^2}{t} e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2} \frac{2\rho_0^2}{t} + \frac{1}{(\rho_0-x)^2} Z \]

Notice that \( AZ \) can be bounded by \( \frac{1}{3} Z_t \), using that \( t < C_0 \phi(r) \) and that \( \rho_0 > (\log(\log(1/r)))^{-1} \) is small. Thus, it suffices to check that

\[ \frac{3}{(\rho_0-x)^4} \left( \frac{3(\rho_0-x)^2(\rho_1-x) - 2(\rho_0-x)^3}{(\rho_1-x)^3} \right) \frac{2}{t} e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2} \frac{2}{t} + \frac{1}{(\rho_0-x)^2} \]

\[ \leq Z_t + ZZ_x. \quad (5) \]

The first term in the left-hand side can also be bounded by \( \frac{1}{3} Z_t \). Indeed, the quantity

\[ \frac{3}{(\rho_0-x)^4} \left( \frac{3(\rho_0-x)^2(\rho_1-x) - 2(\rho_0-x)^3}{(\rho_1-x)^3} \right) \frac{2}{t} e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2} \]

vanishes exponentially as \( x \to \rho_1 \), vanishes like \( (\rho_0-x)^4 \) as \( x \to \rho_0 \) and behaves like \( \rho_0^4 \) elsewhere, while \( 1 - e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2} \) vanishes like \( (\rho_0-x)^3 \) as \( x \to \rho_0 \) and behaves like \( \rho_0^3 \) elsewhere (recall that \( \rho_0 \) can be taken as small as we want).

Hence, it suffices to prove that

\[ \frac{3}{(\rho_0-x)^4} Z \leq \frac{1}{3} Z_t + ZZ_x \quad \text{in} \ (0, \rho_0) \times (0, C_0 \phi(r)). \quad (6) \]

Notice that only at the points \((x, t)\) such that

\[ \frac{1}{(\rho_0-x)^2} > \frac{\sqrt{2}}{3} (1 - e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2})^{1/2} \frac{1}{t} \quad (7) \]

the left-hand side of (6) can be greater than the first one in the right. Observe that here we did not mention the term \( Z_{xx} \); this term is nonpositive only in the spatial-interval \((3\rho_1 - 2\rho_0, \rho_1)\), where it behaves like \( Z^2 \) (very small compared to \( Z_t \)), so it can be neglected. Consequently, we will finally simply check that, at any \((x, t)\) satisfying (7), we have

\[ \frac{1}{\rho_0-x} \geq \frac{\sqrt{2}}{3} (1 - e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2})^{1/2} \frac{1}{t}. \]

In order to prove this, we realize that from (7) we have

\[ \frac{1}{\rho_0-x} > \frac{\sqrt{2}}{3} (1 - e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2})^{1/2} \frac{1}{t}. \]

Then, combining this inequality and the expression of \( Z_x \), we obtain that

\[ Z_x \geq \left( \frac{6(\rho_0-x)^2(\rho_1-x) - 4(\rho_0-x)^3}{(\rho_1-x)^3} \right) \frac{2\rho_0^2}{t} e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2} \frac{2\rho_0^2}{t} + \frac{1}{(\rho_0-x)^2} \]

\[ \times \exp \left( \frac{1}{2(\rho_0-x)^2} \right) \exp \left[ \left( -1 - e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2} \right) \frac{1}{(\rho_0-x)^2} \right] \frac{2\rho_0^2}{t}. \quad (9) \]

Here, we have decomposed

\[ \exp \left( \frac{1}{\rho_0-x} \right) = \exp \left( \frac{1}{2(\rho_0-x)} \right) \exp \left( \frac{1}{2(\rho_0-x)} \right) \]

and we have used the fact that

\[ \exp \left( \frac{1}{2(\rho_0-x)} \right) \geq \exp \left( \frac{1}{6\sqrt{2}} (1 - e^{-\rho_0^2(\rho_0-x)^2/(\rho_1-x)^2})^{1/2} \frac{1}{(\rho_0-x)^2} \right). \]

Estimate (7) indicates that only the points \((x, t)\) with \( x \) close to \( \rho_0 \) are being considered (observe that (7) implies that
\((\rho_0 - x)^2 < (3/(\sqrt{2}\rho_0^{3/2}))t\) and that \(t < C_0 \phi(r)\), so that the argument of the last exponential in (9) is positive (observe that, when \(x \to \rho_0\), the negative term in the exponential goes to zero like \((\rho_0 - x)^3\), while the positive term goes to zero like \((\rho_0 - x)^{3/2}\).

Consequently, for \((x, t)\) satisfying (7), we have
\[
Z_x \geq \frac{1}{(\rho_0 - x)^2} \exp \left( \frac{1}{2(\rho_0 - x)} \right)
\]
and so (8) holds.

As a conclusion, we obtain (4).

Let us now set \(w(x, t) = Z(x, t) - y(x, t)\). It is immediate that
\[
\begin{align*}
&|u_t - u_{xx} + ZZ_t - yy_x| \geq 0 \\
&|w(0, t)| \geq 0, \quad w(\rho_0, t) = +\infty, \quad t \in (0, C_0 \phi(r)), \\
&w(x, 0) = r, \quad x \in (0, \rho_0).
\end{align*}
\]
(10)

We are going to deduce that \(w^-(x, t) \equiv 0\) from (10) and the properties of \(Z\).

Indeed, let us multiply the differential equation in (10) by \(-w^-\) and let us integrate with respect to \(x\) in \((0, \rho_0)\). Since \(w^-\) vanishes at \(x = 0\) and \(\rho_0\), after some computations we find that
\[
\frac{1}{2} \int_0^{\rho_0} |w^-|^2 \, dx + \int_0^{\rho_0} |w_x^-|^2 \, dx = \int_0^{\rho_0} w^- (Z_x - yy_x) \, dx.
\]
(11)

Then, we integrate by parts several times and we use again that \(w^-\) vanishes on the boundary. This yields
\[
\frac{1}{2} \int_0^{\rho_0} |w^-|^2 \, dx + \int_0^{\rho_0} |w_x^-|^2 \, dx = - \frac{1}{2} \int_0^{\rho_0} Z_x |w^-|^2 \, dx.
\]
(12)

Looking at the expression of \(Z_x\), we see that \(Z_x \geq 0\) for \(x \in \{\rho_1, \rho_0\}\). Furthermore, in \((0, \rho_1) \times (0, C_0 \phi(r))\) \(Z_x\) is bounded by a constant only depending on \(\rho_0\) and \(\rho_1\). Consequently, after a standard application of Gronwall’s lemma, we deduce that \(w^- \equiv 0\) and the following is found:
\[
y \leq Z \quad \text{in} \quad (0, \rho_0) \times (0, C_0 \phi(r)).
\]
(13)

Let us now introduce the auxiliary system
\[
\begin{align*}
&u_t - u_{xx} + u_{tx} = 0, \quad (x, t) \in (0, \rho_1) \times (0, C_0 \phi(r)), \\
&u(0, t) = u(\rho_1, t) = Z(\rho_1, t), \quad t \in (0, \phi(r)), \\
&u(x, 0) = -\tilde{r}(x), \quad x \in (0, \rho_1),
\end{align*}
\]
(14)

where \(\tilde{r}\) is any regular function satisfying
\[
\tilde{r}(x) = r \quad \text{for} \quad x \in (\delta \rho_1, 1 - \delta \rho_1) \quad \text{and} \quad \delta \in (0, 1/4),
\]
\[
\tilde{r}(0) = \tilde{r}(\rho_1) = 0 = \tilde{r}_{xx}(0) = \tilde{r}_{xx}(\rho_1)
\]
(observe that \(Z(\rho_1, 0) = 0 = Z_t(\rho_1, 0)\) and
\[
-r \leq -\tilde{r} \leq 0, \quad |\tilde{r}| \leq C \quad \text{and} \quad |\tilde{r}_{xx}| \leq C \quad \text{in} \quad (0, \rho_1),
\]
(15)

where \(C = C(\rho_1)\) is independent of \(r\).

Taking into account that
\[
Z(\rho_1, t) = \exp \left\{ -\frac{2}{r} + \frac{1}{\rho_0 - \rho_1} \right\} \quad \forall t \in (0, C_0 \phi(r)),
\]
we see that \(Z(\rho_1, t) \leq C r^2\), where \(C\) depends on \(\rho_0\) and \(\rho_1\) but is independent of \(r\). In particular, \(Z(\rho_1, \cdot)\) is bounded.

In view of (13) and the fact that \(u_x, y \in L^\infty((0, \rho_1) \times (0, C_0 \phi(r)))\) (see Lemma 1), a new application of Gronwall’s lemma shows that
\[
y \leq u \quad \text{in} \quad (0, \rho_1) \times (0, C_0 \phi(r)).
\]
(17)

As mentioned above, we will now prove that, for some appropriate choices of \(C_0\) and \(C_0^*\), \(u(\rho_1/2, t)\) remains below \(\frac{1}{2} r^2\) at any time \(t < C_0 \phi(r)\). This, together with (17), will serve to conclude.

We will need the following lemma, whose proof is postponed to the end of this subsection:

**Lemma 1.** One has
\[
|u| \leq Cr \quad \text{and} \quad |u_x| \leq C r^{1/2} \quad \text{in} \quad (0, \rho_1) \times (0, C_0 \phi(r)),
\]
(18)

where \(C\) is independent of \(r\).

A consequence of (18) is that \(u_t - u_{xx} \leq C^* r^{3/2} \quad \text{in} \quad (0, \rho_1) \times (0, C_0 \phi(r))\) for some \(C^* > 0\) independent of \(r\). Let us consider the functions \(p\) and \(q\), respectively, given by
\[
p(t) = C^* r^{3/2} t - r
\]
and
\[
q(x, t) = c(e^{-(t-(\rho_1/4))^2/4t} + e^{-(t-(\rho_1/4))^2/4t})
\]
for \((x, t) \in (\rho_1/4, 3 \rho_1/4) \times (0, C_0 \phi(r))\), where \(c\) is fixed below.

It is then clear that \(b = u - p - q\) satisfies
\[
\begin{align*}
&b_t - b_{xx} \leq 0, \quad (x, t) \in (\rho_1/4, 3 \rho_1/4) \times (0, C_0 \phi(r)), \\
&b(\rho_1/4, t) \leq b(1, t), \quad t \in (0, C_0 \phi(r)), \\
&b(3 \rho_1/4, t) \leq b(1, t), \quad t \in (0, C_0 \phi(r)), \\
&b(x, 0) = 0, \quad x \in (\rho_1/4, 3 \rho_1/4),
\end{align*}
\]
(19)

where
\[
b_1(t) = Z(\rho_1, t) - C^* r^{3/2} t + r - c(1 + e^{-\rho_1^2/(16t)})
\]

Obviously, the constant \(c\) can be chosen large enough in order to have
\[
Z(\rho_1, t) - C^* r^{3/2} t + r - c(1 + e^{-\rho_1^2/(16t)}) < 0
\]
\(\forall t \in (0, C_0 \phi(r))\).

If this is the case, we get \(u \leq p + q\) and, in particular,
\[
\begin{align*}
u \left( \frac{\rho_1}{2}, t \right) &\leq p \left( \frac{\rho_1}{2}, t \right) + q \left( \frac{\rho_1}{2}, t \right) \\
&= 2c \exp \left( \frac{-\rho_1^2}{64t} \right) + C^* r^{3/2} t - r.
\end{align*}
\]
Therefore, we see that there exist \( C_0 \) and \( C_1 \) such that
\[
\|u(\rho_1/2, t)\| < -C_0 t\quad \text{for any } t \in (0, C_0\phi(t)).
\]
This proves (3) and, consequently, ends the proof of Theorem 1.

**Proof of Lemma 1.** The first estimate in (18) can be obtained in a classical way, using arguments based on the maximum principle for the heat equation and the facts that
\[
|x| \leq r, \quad Z(\rho_1, t) \leq C r^2 \quad \text{and} \quad Z_t(\rho_1, t) \leq C r^2 \phi(r)^{-2}
\]
for \( x \in (0, \rho_1) \) and \( t \in (0, C_0\phi(t)) \).

Let us explain how the second estimate in (18) can be deduced. Thus, let us set \( \tilde{u}(x, t) = u(x, t) - Z(\rho_1, t) \). This function satisfies
\[
\begin{align*}
\tilde{u}_t - \tilde{u}_{xx} + (\tilde{u} + Z(\rho_1, t))\tilde{u}_x &= -Z_t(\rho_1, t) \\
\tilde{u}(0, t) &= 0, \quad \tilde{u}(\rho_1, t) = 0, \quad x \in (0, \rho_1) \times (0, C_0\phi(t)), \\
\tilde{u}(x, 0) &= -\tilde{r}(x), \quad x \in (0, \rho_1).
\end{align*}
\]
(20)

In a standard way, we can deduce energy estimates for \( \tilde{u} \):
\[
\begin{align*}
\|\tilde{u}\|_{L^\infty(0, C_0\phi(t); L^2(0, \rho_1))}^2 + \|	ilde{u}\|_{L^2(0, \rho_1)}^2 + C \int_0^\rho \int_0^t \|\tilde{u} Z_t(\rho_1, t)\| dx dt \\
&\leq C \|\tilde{r}\|_{L^2(0, \rho_1)}^2 + C \int_0^\rho \int_0^t \|\tilde{u} Z_t(\rho_1, t)\| dx dt.
\end{align*}
\]
Since \( |\tilde{u}| \leq C r \), we obtain that
\[
\begin{align*}
\|\tilde{u}\|_{L^\infty(0, C_0\phi(t); L^2(0, \rho_1))}^2 + \|	ilde{u}\|_{L^2(0, \rho_1)}^2 &\leq C r^2.
\end{align*}
\]
(21)

From the definition of \( \tilde{u} \), a similar estimate holds for \( u \).

Multiplying the equation satisfied by \( \tilde{u} \) by \( \tilde{u} \tilde{r} \), we also get that \( \tilde{u}_{xx} \in L^2(0, \rho_1) \times (0, C_0\phi(t)) \) and
\[
\begin{align*}
\|\tilde{u}_{xx}\|_{L^2(0, \rho_1)}^2 + \|	ilde{u}_{xx}\|_{L^2(0, \rho_1)}^2 &\leq C \|\tilde{u}\|_{L^2(0, \rho_1)}^2 \\
&\quad + C \|\tilde{u}_{xx}\|_{L^2(0, \rho_1)}^2 \\
&\quad + C \|\tilde{r}\|_{L^2(0, \rho_1)}^2
\end{align*}
\]
Taking into account (15), (21) and the fact that \( |\tilde{u}| \leq C r \), we deduce that
\[
\|\tilde{u}\|_{L^\infty(0, C_0\phi(t); L^2(0, \rho_1))}^2 + \|	ilde{u}_{xx}\|_{L^\infty(0, C_0\phi(t); L^2(0, \rho_1))}^2 \leq C r^2.
\]
(22)

Obviously, this also holds for the norm of \( \tilde{u}_{xx} \) in \( L^2(0, \rho_1) \times (0, C_0\phi(t)) \). Again, these estimates are satisfied by \( u \).

Next, by multiplying the equation satisfied by \( \tilde{u} \) by \( -\tilde{u}_{xx} \) and by integrating with respect to \( x \) in \( (0, \rho_1) \), we have
\[
\int_0^\rho \int_0^t |\tilde{u}_{xx}|^2 dx dt + \frac{1}{2} \int_0^t \int_0^\rho |\tilde{u}_{xx}|^2 dx dt
\]
\[
= \int_0^\rho \int_0^t \tilde{u}_{xx}(\tilde{u} + Z(\rho_1, t))\tilde{u}_x dx dt + \int_0^\rho \int_0^t \tilde{u}_{xx} Z_t(\rho_1, t) dx dt.
\]
Integrating with respect to the time variable in \( (0, t) \), we obtain the following after several integration by parts:
\[
\int_0^t \int_0^\rho |\tilde{u}_{xx}|^2 dx dt + \int_0^\rho |\tilde{u}_{xx}|^2 dx dt
\]
\[
\leq \frac{1}{2} \left( \int_0^\rho |\tilde{u}_{xx}|^2 dx dt \right) + \frac{1}{2} \int_0^t \int_0^\rho |\tilde{u}_{xx}|^2 dx dt ds
\]
\[
+ C \left( \int_0^\rho \int_0^t |\tilde{u} + Z(\rho_1, s)|^2 |\tilde{u}_x|^2 dx dt \right) ds
\]
\[
+ \int_0^t \int_0^\rho |\tilde{r}\tilde{r}_{xx}| dx dt + \int_0^t \int_0^\rho |\tilde{r}_{xx}|^2 dx dt
\]
\[
+ \int_0^t \int_0^\rho |\tilde{u}_{xx}|^2 |\tilde{u} + Z(\rho_1, s)|^2 dx ds + \frac{1}{2} \int_0^t \int_0^\rho |\tilde{u}_{xx}|^2 dx ds
\]
\[
+ |Z_t(\rho_1, t)|^2 \int_0^t |Z_x(\rho_1, s)|^2 dx ds.
\]
Using again that \( |\tilde{u}| \leq C r \) and (22), we deduce that
\[
\begin{align*}
\|\tilde{u}\|_{L^\infty(0, \rho_1)}^2 + \|\tilde{u}_{xx}\|_{L^\infty(0, C_0\phi(t); L^2(0, \rho_1))}^2 \\
&\leq C(r^2 + r^2 + 1 + r^4 \phi(r)^{-4} + r^4 \phi(r)^{-8}).
\end{align*}
\]
(23)
As a consequence, (23) implies that
\[
\|\tilde{u}\|_{L^\infty(0, \rho_1)}^2 + \|\tilde{u}_{xx}\|_{L^\infty(0, C_0\phi(t); L^2(0, \rho_1))}^2 \leq C.
\]
(24)
Finally, in order to estimate \( \tilde{u} \) in \( L^\infty((0, \rho_1) \times (0, C_0\phi(t))) \), we observe that for each \( t \in (0, C_0\phi(t)) \) there exists \( a(t) \in (0, \rho_1) \) such that \( \tilde{u}_x(a(t), t) = 0 \). Using this fact, we obtain
\[
|\tilde{u}_x(x, t)|^2 = \frac{1}{2} \int_0^t \tilde{u}_x(\xi, t)\tilde{u}_{xx}(\xi, t) d\xi.
\]
(25)
Applying the estimates (22) and (24) to the functions \( \tilde{u}_x \) and \( \tilde{u}_{xx} \), which, respectively, belong to \( L^\infty(0, C_0\phi(t); L^2(0, \rho_1)) \) and \( \tilde{u}_{xx} \in L^\infty(0, C_0\phi(t); L^2(0, \rho_1)) \), we deduce at once that
\[
\|\tilde{u}_x\|_{L^\infty(0, \rho_1)}^2 \leq C r
\]
which in particular implies the second estimate in (18).

2.2. **Proof of the estimate from above**

In this paragraph, we will prove the second estimate in (3). In other words, we will see that, for the Burgers equation, the minimal \( L^2 \)-controllability time for initial data satisfying \( \|y_0\|_{L^2(Q)} \leq r \) is not greater than \( C_1 (\log(1/r))^{-1} \) for a suitable positive constant \( C_1 \).

First, we recall that, as long as \( a \in L^\infty(Q) \), we can find controls \( v \) such that the solution of the linear system
\[
\begin{align*}
y_t - y_{xx} + a(x, t) y_x &= v_1(x) \quad (x, t) \in (0, 1) \times (0, T), \\
y(0, t) &= y(1, t) = 0, \quad t \in (0, T), \\
y(x, 0) &= y_0(x), \quad x \in (0, 1)
\end{align*}
\]
(26)
for some $C^* = C^*(\|a\|_{\infty})$ (see [8]); recall that we defined $Q = (0, 1) \times (0, T)$ in the Introduction. In order to prove this, we use the well-known property that the cost of the null controllability of (25) coincides with the best observability constant. More precisely, let us consider the adjoint system

$$
\begin{align*}
-\varphi_x - a(x,t)\varphi_x &= 0 \quad (x,t) \in Q, \\
\varphi(0,t) &= \varphi(1,t) = 0, \quad t \in (0,T), \\
\varphi(x,T) &= \varphi^0(x), \quad x \in (0,1),
\end{align*}
$$

where $\varphi^0 \in L^2(0,1)$. Then, it is known that the cost of the null controllability of (25) with $L^\infty$-controls is the best constant $\tilde{C}$ for which we have

$$
\|\varphi(\cdot,0)\|_{L^2(0,1)} \leq \tilde{C} \left| \int \int_{Q} \varphi \, dx \, dt \right| \quad \forall \varphi^0 \in L^2(0,1).
$$

From the arguments in [8], it follows that $\tilde{C} \leq \exp(C^*/T)$, where $C^*$ is a positive constant only depending on $\omega$ and $\|a\|_{\infty}$.

We are now going to see that this provides a local controllability result for the nonlinear system (1):

**Lemma 2.** Assume that

$$
y^0 \in H^1_0(0,1), \quad \|y^0\|_\infty \leq \frac{1}{2} \quad \text{and} \quad \|y^0\|_{L^2(0,1)} \leq \frac{1}{2T} e^{-C^*/T},
$$

where $C^*$ corresponds to the constant $C^*$ in (26) for $\|a\|_{\infty} = 1$.

Then there exist controls $v \in L^\infty(\omega \times (0, T))$ such that the associated solution to (1) satisfies (2).

**Proof (Sketch).** The proof of this lemma relies on well-known arguments, but we present a sketch of it for the sake of completeness.

Let $s \in (\frac{1}{2}, 1)$ and let us introduce the set-valued mapping $\mathcal{A} : H^s(Q) \mapsto H^s(Q)$, given as follows: for each $z \in H^s(Q)$, we first denote by $\mathcal{A}(z)$ the set of all controls $v \in L^\infty(\omega \times (0, T))$ such that (26) is satisfied and the associated solution of

$$
\begin{align*}
y_t - y_{xx} + z(x,t)y_x &= v \chi_\omega \quad (x,t) \in Q, \\
y(0,t) &= y(1,t) = 0, \quad t \in (0,T), \\
y(x,0) &= y^0(x), \quad x \in (0,1),
\end{align*}
$$

fulfills (2); then, $\mathcal{A}(z)$ is by definition the family of these associated solutions.

Let $K$ be the closed convex set $K = \{ z \in H^s(Q) : \|z\|_{H^s} \leq 1 \}$. We will check that the hypotheses of Kakutani’s fixed-point theorem are satisfied by $\mathcal{A}$ in $K$ (for the statement of this theorem see, for instance, [3]):

First, we note that the solution of (29) belongs to the space $X := L^2(0,T;H^2(0,1)) \cap H^1(0,T;L^2(0,1))$ so, in particular, $y \in H^s(Q)$. Then, an application of the classical maximum principle yields

$$
\|y\|_\infty \leq T \|v\|_\infty + \|y^0\|_{\infty}.
$$

Now, from (26) and (28), we deduce that

$$
\|y\|_\infty \leq 1.
$$

Consequently, $\mathcal{A}$ maps $K$ into $K$.

It is not difficult to prove that, for each $z \in K$, $\mathcal{A}(z)$ is a nonempty compact convex set of $H^s(Q)$, since $X \mapsto H^s(Q)$ with compact embedding.

Furthermore, $\mathcal{A}$ is upper hemicontinuous in $H^s(Q)$, i.e. for each $\mu \in (H^s(Q))^\prime$, the single-valued mapping $z \mapsto \sup_{y \in \mathcal{A}(z)} \langle \mu, y \rangle$ is upper semi-continuous. Indeed, let us assume that $z_n \in K$ for all $n$ and $z_n \to z_0$ in $H^s(Q)$. For each $n$, there exists $y_n \in \mathcal{A}(z_n)$ such that

$$
\sup_{y \in \mathcal{A}(z_n)} \langle \mu, y \rangle = \langle \mu, y_n \rangle.
$$

Then, from classical regularity estimates for the linear heat equation (see, for instance, [13]), we see that, at least for a subsequence, one has $y_n \to y^\ast$ weakly in $L^2(0,T;H^2(0,1))$ and

$$
y_n \to y^\ast \quad \text{strongly in } L^2(0,T;H^1_0(0,1)).
$$

Consequently, $y_n y_n |_{\omega \times (0, T)}$ converges weakly in $L^2(Q)$ to $0 y^\ast |_{\omega \times (0, T)}$ and the limit function $y^\ast$ satisfies (29) and $y^\ast \in \mathcal{A}(z_0)$. This shows that

$$
\limsup_{n \to \infty} \sup_{y \in \mathcal{A}(z_n)} \langle \mu, y \rangle \leq \sup_{y \in \mathcal{A}(z_0)} \langle \mu, y \rangle,
$$

as desired.

In view of Kakutani’s theorem, there exists $\hat{y} \in K$ such that $\hat{y} \in \mathcal{A}(\hat{y})$. This proves the lemma. $\square$

Let us finish the proof of the right inequality in (3).

Assume that $y^0 \in L^2(0,1)$ and $y^0 \|_{L^2(0,1)} \leq r$. In a first step, we take $v(x,t) \equiv 0$. Then from parabolic regularity, we know that $y(\cdot, t) \in H^1_0(0,1)$ for all $t > 0$ and there exist constants $r$ and $M$ such that the solution of (1) satisfies

$$
\|y(\cdot, t)\|_{H^s} \leq M r^{-1/4} \|y^0\|_{L^2(0,1)} \quad \forall t \in (0, T)
$$

(see, for instance, [13]).

We leave the solution evolve freely, until it reaches a set of the form

$$
\{ w : \|w\|_\infty \leq \frac{1}{2}, \quad \|w\|_{L^2(0,1)} \leq r \}.
$$

In other words, we take $v(x,t) \equiv 0$ for $t \in (0, t_0)$, where $t_0 = (2M)^4 r^4$.

Let us set $y^0 = y^0(\cdot, t_0)$. Then $y^0 \in H^1_0(0,1)$ and

$$
\|y^0\| \leq \frac{1}{2} \quad \text{and} \quad \|y^0\|_{L^2(0,1)} \leq r.
$$

Let us now consider the system

$$
\begin{align*}
y_t - y_{xx} + yy_x &= v \chi_\omega \quad (x,t) \in (0,1) \times (t_0, t_0 + t_1), \\
y(0,t) &= y(1,t) = 0, \quad t \in (t_0, t_0 + t_1), \\
y(x,t_0) &= y^0(x), \quad x \in (0,1),
\end{align*}
$$

(30)
where
\[ t_1 = \frac{C^*}{\log 1/r}. \]

In view of Lemma 2, since one can assume that \((1/2C^*) \log 1/r \geq 1\), we have that there exist controls \(\hat{v} \in L^\infty(\omega \times (t_0, t_0 + t_1))\) such that the associated solution of (30) satisfies
\[ y(x, t_0 + t_1) = 0 \quad \text{in} \quad (0, 1). \]

We then set \(v(x, t) \equiv \hat{v}(x, t)\) for \(t \in (t_0, t_0 + t_1)\).

In this way, we have shown that we can drive the solution of (1) exactly to zero in a time interval of length
\[ t_0 + t_1 = (2M)^4r^4 + \frac{C^*}{\log 1/r}. \]

Hence, the second inequality in (3) is proved.

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References