Local Null Controllability of the $N$-Dimensional Navier-Stokes System with Nolinear Navier-slip Boundary Condition and $N-1$ Scalar Controls

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Abstract

In this paper we deal with the local null controllability of the Navier-Stokes system with nonlinear Navier-slip boundary conditions and internal controls having one vanishing component.

Key Words: Navier-Stokes system, null controllability, Carleman estimates and Navier-slip boundary conditions.

1 Introduction

Let $\Omega$ be a nonempty bounded connected open subset of $\mathbb{R}^N$ ($N=2$ or $N=3$) of class $C^\infty$. Let $T > 0$ and let $\omega \subset \Omega$ be a (small) nonempty open subset which is the control domain. Here, we will use the notation $Q := \Omega \times (0, T)$, $\Sigma := \partial \Omega \times (0, T)$ and by $n(x)$ the outward unit normal vector to $\Omega$ at the point $x \in \partial \Omega$.

Let us consider the controlled Navier-Stokes system with nonlinear Navier-slip boundary conditions. Given a nonlinear regular function $f : \mathbb{R}^N \to \mathbb{R}^N$ and an initial state $y_0$, we consider the following system:

\[
\begin{align*}
    &\begin{cases}
        y_t - \nabla \cdot (Dy) + (y, \nabla)y + \nabla p = v\chi_{\omega} & \text{in} \ Q, \\
        \nabla \cdot y = 0 & \text{in} \ Q, \\
        y \cdot n = 0, \quad (\sigma(y, p) \cdot n)_{tg} + f(y)_{tg} = 0 & \text{on} \ \Sigma, \\
        y(\cdot, 0) = y_0(\cdot) & \text{in} \ \Omega,
    \end{cases}
\end{align*}
\]

where $v = v(x, t)$ stands for the control which acts in an arbitrary fixed domain $\omega \times (0, T)$ and $\text{supp} \ v \subset \omega \times (0, T)$, $\chi_{\omega}$ is a smooth positive function such that $\chi_{\omega} = 1$ in $\omega'$, where $\omega' \Subset \omega$. Respect to the boundary conditions, we mention that in 1823, C.L. Navier (see [17]) established a slip-with-friction boundary condition and claimed that the component of the fluid velocity tangential to the surface should be proportional to the rate of strain at the surface. The velocity’s component normal to the surface is naturally zero as mass is not able to penetrate an impermeable solid surface [17]. This can be expressed by

\[
y \cdot n = 0 \quad \text{and} \quad (\sigma(y, p) \cdot n)_{tg} + ky_{tg} = 0 \quad \text{on} \ \Sigma,
\]

where $\sigma(y, p) := -pId + Dy$ is the stress tensor, $D$ is the symmetrized gradient of $y$, $p$ is the pressure, $Id$ is the identity matrix, $(\sigma(y, p) \cdot n)_{tg}$ denotes the tangential component of $\sigma(y, p) \cdot n$, $y_{tg}$ is the tangential velocity along the solid surface and $k$ is a scalar friction function that

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measures the local viscous coupling between fluid and solid.

Physically a nonzero slip length arises from the unequal wall and fluid densities, the weak wall-fluid interaction, and the high temperature. These behavior types had been recently demonstrated and showed that the phenomenon of slip occurs with dependence on various factors, such as in aerodynamics processes when the high pressure is involved, in weather forecast where trees, buildings, water waves have to be taken into account, in turbulence, when \( k \) depends on pressure, etc. In consequence, the analysis is complicated as well as numerical solutions of the model and an alternative is then to reduce the no-slip condition on rough boundaries to \textit{ad hoc} boundary conditions, the so-called \textit{wall laws}, on a smooth domain.

Let us point out that our boundary conditions corresponds to a law of the wall that appear in turbulent flows, specifically when \( k \) may not depend on \( |y| \) linearly. We invite to interested reader to see \([5],[15]\) for a complete discussion on this subject.

In the context of controllability, the papers by Coron \([7]\) and Imanuvilov \([13]\) show results of the approximate controllability and local exact controllability for the Navier-Stokes system with Navier-slip boundary conditions in two dimensions, with some restrictions respectively. The system \((1.1)\) has been studied by Guerrero \([11]\), in this paper the author proved the local null controllability to the trajectories of \((1.1)\) in dimension \( N \) using Carleman estimates for the associated linear system and fixed point arguments. On the other hand, recent papers by Coron and Guerrero \([6]\), Carreño and Guerrero \([4]\) are evidence of the null controllability and local null controllability of the Navier-Stokes system with \( N - 1 \) scalar controls, even thought they use homogeneous Dirichlet boundary conditions. Then, the main objective of this paper is to obtain the local null controllability of system \((1.1)\) by means of \( N - 1 \) scalar controls, see Theorem 1.1.

Let us now introduce several spaces which are usual in the context of problems modelling incompressible fluids:

\[ V := \{ u \in H^1_0(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega \}, \]
\[ H := \{ u \in L^2(\Omega)^N : \nabla \cdot u = 0, \text{ in } \Omega \cdot u \cdot n = 0 \text{ on } \partial \Omega \}, \]
\[ W = \{ u \in H^1(\Omega)^N : \nabla \cdot u = 0 \text{ in } \Omega, \ u \cdot n = 0 \text{ on } \partial \Omega \}. \]

Our main result is given in the following theorem.

**Theorem 1.1.** Let us assume that \( i \in \{1, \ldots, N\} \) and \( f \in C^4(\mathbb{R}^N; \mathbb{R}^N) \) with \( f(0) = 0 \). Then, for every \( T > 0 \) and \( \omega \subset \Omega \), there exists \( \delta > 0 \) such that, for every \( y_0 \in H^3(\Omega)^N \cap W \) satisfying \( \|y_0\|_{H^3(\Omega)^N \cap W} \leq \delta \) and the compatibility condition

\[ (Dy_0 \cdot n)_{tg} + (f(y_0))_{tg} = 0 \text{ on } \partial \Omega, \]  

we can find a control

\[ v \in L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N), \]

with \( v_i \equiv 0 \) and an associated solution \( (y, p) \) to \((1.1)\) verifying \( y(\cdot, T) = 0 \) in \( \Omega \).

To prove Theorem 1.1, we first deduce a null controllability result for a linearized system around zero associated to \((1.1)\):

\[ \begin{cases}
  y_t - \nabla \cdot (Dy) + \nabla p = h + v \chi_\omega & \text{in } Q, \\
  \nabla \cdot y = 0 & \text{in } Q, \\
  y \cdot n = 0, (\sigma(y, p) \cdot n)_{tg} + (A(x, t)y)_{tg} = 0 & \text{on } \Sigma, \\
  y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega, 
\end{cases} \]  

where \( A \) is a \( N \times N \) matrix-valued function in a suitable space and \( h \) decreases exponentially to zero in \( T \). Finally, we apply Kakutani’s fixed point theorem and an inverse mapping theorem.
to conclude the local null controllability for the nonlinear system (1.1).

On the other hand, we highlight that some ideas as appears in [4] and [6] concerning to null controllability for the linear system (1.3) are not able to be considered. Indeed, this relevant detail arises from the different boundary conditions that we present here. The paper is organized as follows. In Section 2, we present a previous regularity result proved in [11] and other that we prove here for systems as (1.3). In section 3 we establish a Carleman inequality needed to deal with the controllability problems. In section 4 we prove the null controllability of the linear system (1.3). Finally, in Section 5 we give the proof of Theorem 1.1 using fixed point arguments.

Before starting with Section 2, we consider several Hilbert spaces for \( \varepsilon > 0 \) small enough:

\[
P_\varepsilon^0 := H^{1/2+\varepsilon}(0, T; H^{1+\varepsilon}(\partial\Omega)^{N\times N}), \quad P_\varepsilon^1 := H^{5/4+\varepsilon}(0, T; L^2(\partial\Omega)^{N\times N}), \quad P_\varepsilon^2 := L^2(0, T; H^{5/2}(\partial\Omega)^{N\times N}),
\]

\[
Z_\varepsilon := H^{5/4+\varepsilon}(0, T; H^1(\Omega)^N \cap W) \cap L^2(0, T; H^3(\Omega)^N \cap W)
\]

and

\[
Y_1 := L^2(0, T; H^2(\Omega)^N) \cap H^1(0, T; L^2(\Omega)^N), \quad Y_2 := L^2(0, T; H^4(\Omega)^N) \cap H^2(0, T; L^2(\Omega)^N).
\]

## 2 Preliminary results

In order to prove the main theorem of this paper, we introduce some preliminary results which will be used later on. More precisely, we present regularity results concerning the Stokes system with linear Navier-slip boundary conditions.

The proof of the following result can be found in [11].

**Lemma 2.1.** Let \( A \in P_\varepsilon^0 \), \( u_0 \in H \), \( f_0 \in L^2(0, T; W') \), \( f_2 \in L^2(0, T; H^{-1/2}(\partial\Omega)^N) \) and let \( u \) be the weak solution of the system

\[
\begin{cases}
    u_t - \nabla \cdot (Du) + \nabla \theta = f_0 & \text{ in } Q, \\
    \nabla \cdot u = 0 & \text{ in } Q, \\
    u \cdot n = 0, \ (\sigma(u, \theta) \cdot n)t_g + (A(x, t)u)t_g = f_2 & \text{ on } \Sigma, \\
    u(\cdot, 0) = u_0(\cdot) & \text{ in } \Omega,
\end{cases}
\]

namely, the function \( u \) satisfying

\[
\begin{align*}
\int_{\Omega} & u_t(t) \cdot vdx + \frac{1}{2} \int_{\Omega} Du(t) : Dvdx + \int_{\partial\Omega} Au(t) \cdot vds \\
= & \int_{\Omega} f_0(t) \cdot vdx + \int_{\partial\Omega} f_2(t) \cdot vds \quad \text{ a.e } t \in (0, T), \ \forall v \in W, \\
& u(\cdot, 0) = u_0(\cdot) \quad \text{ in } \Omega.
\end{align*}
\]

Then, if we further assume \( u_0 \in W \) and

\[
f_0 \in L^2(Q)^N, \quad f_2 \in L^2(0, T; H^{1/2}(\partial\Omega)^N), \quad f_2 \in H^{1/4+\varepsilon}(0, T; H^{-\varepsilon}(\partial\Omega)^N),
\]

\( u \) is actually, together with a pressure \( \theta \), the strong solution of (2.1), i.e., \((u, \theta) \in Y_1 \times L^2(0, T; H^1(\Omega)) \). Furthermore, there exists a positive constant \( C \) such that

\[
\begin{align*}
\|u\|_{Y_1} + \|\theta\|_{L^2(0, T; H^1(\Omega))} & \leq Ce^{CT\|A\|_{L^2}^2} (1 + \|A\|_{L^2}^2) \left( \|f_0\|_{L^2(Q)^N} \\
& + \|f_2\|_{L^2(0, T; H^{1/2}(\partial\Omega)^N)} + \|f_2\|_{H^{1/4+\varepsilon}(0, T; H^{-\varepsilon}(\partial\Omega)^N)} + \|u_0\|_{H^1(\Omega)^N} \right).
\end{align*}
\]
\[ A \in H^{1-\ell}(0, T; W^{\nu_1, \nu_1+1(\partial\Omega)})^{N \times N}, \]
where \( 0 < \ell < 1/2 \) is arbitrarily close to \( 1/2 \) and \( \nu_1 > 1 \) is arbitrarily close to 1. Observe that this hypothesis is satisfied if \( A \in P^0_\varepsilon \).

Using the above Lemma, we prove now a regularity result for the solution of (2.1). To this end, we will impose the following compatibility condition:
\[ (Du_0 \cdot n)_{tg} + (A(\cdot, 0)u_0)_{tg} = f_2(\cdot, 0) \quad \text{on} \quad \partial\Omega. \]  

Theorem 2.1. Let \( A \in P_0^1 \cap P^2, \ u_0 \in H^3(\Omega)^N \cap W \) satisfying (2.3), \( f_0 \in Y_1, \ f_2 \in L^2(0, T; H^{5/2}(\partial\Omega)^N) \cap H^1(0, T; H^{1/2}(\partial\Omega)^N) \), and let \( u \) be the strong solution of system
\[ \begin{cases} u_t - \nabla \cdot (Du) + \nabla \theta = f_0 & \text{in } Q, \\ \nabla \cdot u = 0 & \text{in } Q, \\ u \cdot n = 0, (\sigma(u, \theta) \cdot n)_{tg} + (A(x, t)u \cdot n)_{tg} = f_2 & \text{on } \Sigma, \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega. \end{cases} \]  

Then, \((u, \theta) \in Y_2 \times L^2(0, T; H^3(\Omega))\) and there exists a positive constant \( C \) such that
\[ \|u\|_{Y_2} + \|\theta\|_{L^2(0, T; H^3(\Omega))} \leq C(A) \left( \|f_0\|_{Y_1} + \|f_2\|_{L^2(0, T; H^{5/2}(\partial\Omega)^N)} + \|f_2\|_{H^1(0, T; H^{1/2}(\partial\Omega)^N)} + \|u_0\|_{H^3(\Omega)^N} \right), \]  

where
\[ C(A) = C e^{CT\|A\|_{L^2} \left( 1 + \|A\|_{L^2}^3 \right)} \left[ 1 + \|A\|_{L^2}^3 + \|A\|_{H^1}^3 \right]. \]

Proof of Theorem 2.1. We consider (2.4) like a stationary system, that is to say:
\[ \begin{cases} -\nabla \cdot (Du) + \nabla \theta = f_0 - u_t & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot n = 0, (\sigma(u, \theta) \cdot n)_{tg} + (A(x, t)u \cdot n)_{tg} = f_2 & \text{on } \partial\Omega, \end{cases} \]  

for almost every \( t \in (0, T) \).

The rest of the proof is divided in two steps.

Step 1. The goal will be to prove that the weak solution \((u, \theta)\) of the stationary system
\[ \begin{cases} -\nabla \cdot (Du) + \nabla \theta = g_0 & \text{in } \Omega, \\ \nabla \cdot u = g_1 & \text{in } \Omega, \\ u \cdot n = 0, (\sigma(u, \theta) \cdot n)_{tg} = g_2 & \text{on } \partial\Omega, \end{cases} \]  

actually belongs to \( H^3(\Omega)^N \times H^2(\Omega) \), whenever \( g_0 \in H^1(\Omega)^N, g_1 \in H^2(\Omega) \) and \( g_2 \in H^{3/2}(\partial\Omega)^N \).

In accordance with estimate (2.2) for the stationary case and for \( A = 0 \), we obtain that the weak solution of (2.8) satisfies
\[ \|u\|_{H^2(\Omega)}^N + \|\theta\|_{H^1(\Omega)} \leq C \left( \|g_0\|_{L^2(\Omega)}^N + \|g_1\|_{H^1(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)}^N \right), \]  

for a positive constant \( C \).

The interior regularity readily follows from the corresponding result with homogeneous Dirichlet boundary conditions, which can be found in [18], for instance. Then, for every \( \Omega' \subset \subset \Omega \), we have \( u \in H^3(\Omega')^N, \theta \in H^2(\Omega') \) and
\[ \|u\|_{H^3(\Omega')^N} + \|\theta\|_{H^2(\Omega')} \leq C \left( \|g_0\|_{H^1(\Omega)}^N + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{3/2}(\partial\Omega)^N} \right), \]  

for a positive constant \( C \).
Moreover, then, taking into account that for every $i \in H^3$ such that $\theta \in H^2(\Omega \cap \hat{U})$, for every $\hat{U} \subset \subset U_0$.

To this end, let $\psi$ be a $W^{3,\infty}$ diffeomorphism which sends the set

$$C_0 := \{(x', x_N) \in \mathbb{R}^N : |x_i| < a_0 \quad i = 1, \cdots, N - 1, |x_N| < \beta_0\}$$

into and onto $U_0$ and which verifies

$$\psi(C_0^+) = \Omega \cap U_0, \quad \psi(\partial_{\alpha_0}) = \partial \Omega \cap U_0,$$

where we have denoted $C_0^+ = C_0 \cap \mathbb{R}^N$ and $\partial_{\alpha_0} = \partial \mathbb{R}^N \cap C_0$. Let us now introduce a cut-off function $\zeta \in C^2(U_0)$ such that

$$\zeta \equiv 1 \text{ in } \hat{U} \quad \text{and} \quad \text{supp } \zeta \subset \subset U_1 \subset U_0,$$

where $U_1$ is a regular open set. Then, let us set $z = \zeta u$, $\tilde{h} = \zeta \theta$. They verify:

$$
\begin{cases}
-\nabla \cdot (Dz) + \nabla h = g_0^* & \text{in } \Omega \cap U_0, \\
\nabla \cdot z = g_1^* & \text{in } \Omega \cap U_0, \\
z \cdot n = 0, (\sigma(z, h) \cdot n)_{tg} = g_2^* & \text{on } \partial \Omega \cap U_0, \\
z = 0 & \text{on } \Omega \cap \partial U_0,
\end{cases}
$$

with

$$g_0^* = \zeta g_0 - 2\zeta \nabla \cdot \nabla u - \nabla \cdot \nabla \nabla u - \Delta \zeta u - \nabla \nabla \zeta \cdot u + \theta \nabla \zeta - g_1 \nabla \zeta \in H^1(\Omega \cap \Omega_0)^N,$$

$$g_1^* = \zeta g_1 + \nabla \zeta \cdot u \in H^2(\Omega \cap U_0) \quad \text{and} \quad g_2^* = \zeta g_2 + \frac{\partial \zeta}{\partial n} u \in H^{3/2}(\partial \Omega \cap \Omega_0)^N.$$ 

Let us now perform the change of variable $x = \psi(\xi)$. If we define $\tilde{z} = x \circ \psi$, $\tilde{h} = h \circ \psi$ and $\tilde{n} = n \circ \psi$, then

$$\frac{\partial}{\partial x_i} z_s = \sum_{k=1}^{N} \frac{\partial \tilde{z}_i}{\partial \xi_k} \frac{\partial \xi_k}{\partial x_i} = \nabla \tilde{z}_i \cdot \nabla \psi^{-1}, \quad \forall s = 1, \ldots, N,$$

where we have denoted $\nabla \psi^{-1}$ the $i$th-column of $\nabla \psi^{-1}$. Observe that

$$\frac{\partial}{\partial x_i} \left( \frac{\partial}{\partial x_i} z_s \right) = \sum_{k=1}^{N} \frac{\partial^2 \tilde{z}_i}{\partial \xi_k \partial \xi_j} \frac{\partial \xi_j}{\partial x_i} + \sum_{k=1}^{N} \frac{\partial \tilde{z}_i}{\partial \xi_k} \frac{\partial^2 \xi_k}{\partial x_i \partial x_i}.$$ 

Therefore

$$\Delta z_s = \sum_{i,j,k=1}^{N} \left( \frac{\partial^2 \tilde{z}_s}{\partial \xi_k \partial \xi_j} \frac{\partial \xi_j}{\partial x_i} \frac{\partial x_i}{\partial x_i} \right) + \sum_{i,k=1}^{N} \frac{\partial \tilde{z}_s}{\partial \xi_k} \frac{\partial^2 \xi_k}{\partial x_i \partial x_i} = Hess(\tilde{z}_s) : \left( \sum_{i=1}^{N} \frac{\partial \xi_i}{\partial x_i} \frac{\partial \xi_i}{\partial x_i} \right)_{i,j,k} + \sum_{k=1}^{N} \frac{\partial \tilde{z}_s}{\partial \xi_k} \Delta \xi_k$$

where $\text{Hess}(\tilde{z}_s)$ represents the Hessian matrix on $\tilde{z}_s$ and $\Delta \psi^{-1} := \Delta \xi := (\Delta \xi_1, \ldots, \Delta \xi_N)$. Moreover,

$$\text{div } z = \sum_{s,j=1}^{N} \frac{\partial \tilde{z}_s}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_s} = \nabla \tilde{z} \cdot \nabla \psi^{-1} \quad \text{and} \quad \frac{\partial}{\partial x_s} \tilde{h} = \sum_{j=1}^{N} \frac{\partial \tilde{h}}{\partial \xi_j} \frac{\partial \xi_j}{\partial x_s} = \nabla \tilde{h} \cdot \nabla \psi^{-1}.$$ 

Then, taking into account that for every $i = 1, \ldots, N$ we have

$$(\nabla \cdot Dz)_i = \Delta z_i + \partial_i \text{div } z,$$
we find from (2.12) that $\tilde{z}_i$ satisfies the following system for $i = 1, \ldots, N$:

$$
\begin{cases}
-Hess(\tilde{z}_i) : \nabla \psi^{-1} \nabla^t \psi^{-1} - \nabla \tilde{z}_i \cdot \Delta \psi^{-1} + \nabla \tilde{h} \cdot \nabla^t \psi^{-1} = (g_0^*)_i + \partial_i g_i^* & \text{in } C^+_0, \\
\nabla \tilde{z}_i : \nabla \psi^{-1} = \bar{g}_i^* & \text{in } C^+_0, \\
\tilde{z} \cdot \bar{n} = 0, & \text{on } \partial \mathbb{R}^N_+ \cap C_0, \\
(\bar{\sigma}(\tilde{z}) \cdot \bar{n})_{tg} = \bar{g}_2^* & \text{on } \partial C^+_0 \cap \mathbb{R}^N_+, \\
\tilde{z} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(2.14)

where we have denoted

$$
\bar{g}_0^* = g_0^* \circ \psi, \quad \bar{g}_1^* = g_1^* \circ \psi, \quad \bar{g}_2^* = g_2^* \circ \psi
$$

and

$$
(\bar{\sigma}(\tilde{z}))_{is} := \nabla \tilde{z}_i \cdot \nabla \psi^{-1} + \nabla \tilde{z}_s \cdot \nabla \psi^{-1}, \quad \forall i, s \leq N.
$$

On the other hand, note that for every function $F$ in $H^\ell(\Omega)$ $(\ell \in \mathbb{N}, \ell \leq 3)$, $\tilde{F} = F \circ \psi$ belongs to $H^\ell(C^+_0)$ and there exists a positive constant $C = C(\Omega)$ such that

$$
\|\tilde{F}\|_{H^\ell(C^+_0)} \leq C\|F\|_{H^\ell(\Omega)}.
$$

Now, observe that $\tilde{z} \in \tilde{X}_{0,2}$, with

$$
\tilde{X}_{0,2} := \{ \tilde{z} \in H^2(C^+_0)^N : \tilde{z} = 0 \text{ on } \partial C^+_0 \cap \mathbb{R}^N_+, \tilde{z} \cdot \bar{n} = 0 \text{ on } \partial \mathbb{R}^N_+ \cap C_0 \}.
$$

Let us introduce $C_1 = \psi(U_1)$ (recall that $U_1 \subset \subset U_0$) and $d = dist(\partial C^+_0, \partial C^+_1)$.) Then, we have $\delta_m \tilde{z} \in \tilde{X}_{0,2}$ for any $1 \leq k \leq N - 1$ and any $|m| < d/2$, where we have denoted

$$
\tilde{X}_{1,2} := \{ \tilde{z} \in H^2(C^+_1)^N : \tilde{z} = 0 \text{ on } \partial C^+_1 \cap \mathbb{R}^N_+, \tilde{z} \cdot \bar{n} = 0 \text{ on } \partial \mathbb{R}^N_+ \cap C_1 \},
$$

and

$$
\delta_m^k(f) := r_m(f) - f, \quad \tau_m^k(f) = (\xi \rightarrow f(\xi + me_k))
$$

(see (2.9) and (2.11)). We denote now $\tilde{w} = \delta_m^k \tilde{z}, \tilde{\bar{n}} = \delta_m^k \tilde{h}$. We have:

$$
\delta_m^k(Hess(\tilde{z}_i) : \nabla \psi^{-1} \nabla^t \psi^{-1}) = Hess(\tilde{w}_i) : \nabla \psi^{-1} \nabla^t \psi^{-1} + Hess(\tilde{z}_i(\xi + me_k)) : \delta_m^k(\nabla \psi^{-1} \nabla^t \psi^{-1}).
$$

$$
\delta_m^k(\nabla \tilde{z}_i \cdot \Delta \psi^{-1}) = \nabla \tilde{w}_i \cdot \Delta \psi^{-1} + \nabla \tilde{z}_i(\xi + me_k) \cdot \delta_m^k(\Delta \psi^{-1}).
$$

$$
\delta_m^k(\nabla \tilde{z} : \nabla \psi^{-1}) = \nabla \tilde{w} : \nabla \psi^{-1} + \nabla \tilde{z}(\xi + me_k) \cdot \delta_m^k(\nabla \psi^{-1}).
$$

Additionally,

$$
\delta_m^k(\tilde{z} \cdot \bar{n}) = \bar{w} \cdot \bar{n}
$$

and

$$
\delta_m^k((\sigma(\tilde{z}, \tilde{h}) \cdot \bar{n})_{tg}) = (\bar{\sigma}(\tilde{w}) \cdot \bar{n})_{tg} + \left[ \sum_{k=1}^{N} (\nabla \tilde{z}_k(\xi + me_k) \cdot \delta_m^k \nabla \psi^{-1} + \nabla \tilde{z}_i(\xi + me_k) \cdot \delta_m^k \nabla \psi^{-1}) \bar{n}_s \right]_{tg}
$$

on $\partial \mathbb{R}^N_+ \cap C_1$. The last two identities readily follow from (2.15) and the fact that $\bar{n}_j(\xi + me_k) = \bar{n}_j(\xi)$ on $C_1 \cap \partial \mathbb{R}^N_+$, for every $k = 1, \ldots, N - 1$ and for every $j = 1, \ldots, N$. Taking into account the above identities and (2.14), the pair $(\tilde{w}, \tilde{\bar{n}})$ satisfies:

$$
\begin{cases}
-Hess(\tilde{w}_i) : \nabla \psi^{-1} \nabla^t \psi^{-1} - \nabla \tilde{w}_i \cdot \Delta \psi^{-1} + \nabla \tilde{\bar{n}} \cdot \nabla \psi^{-1} = G_{0,i} + \partial_i G_1 & \text{in } C^+_1, \\
\nabla \tilde{w} : \nabla \psi^{-1} = G_1 & \text{in } C^+_1, \\
\tilde{w} \cdot \tilde{n} = 0, & \text{on } \partial \mathbb{R}^N_+ \cap C_1,
\end{cases}
$$

(2.16)
where
\[ G_{0,i} = \delta_m^{k}(\tilde{g}_0^*)_i + \text{Hess}(\tilde{z}_i(\xi + me_k)) : \delta_m^{k}(\nabla \psi^{-1} \nabla^i \psi^{-1}) + \nabla \tilde{z}_i(\xi + me_k) : \delta_m^{k} \Delta \psi^{-1} + \partial_i(\nabla \tilde{z}(\xi + me_k) : \delta_m^{k} \nabla^i \psi^{-1}) - \nabla \tilde{h}(\xi + me_k) : \delta_m^{k} \nabla^i \psi^{-1}, \]
\[ G_1 = \delta_m^{k}(\tilde{g}_1^*) - \nabla \tilde{z}(\xi + me_k) : \delta_m^{k} \nabla^i \psi^{-1}, \]
\[ G_2 = \delta_m^{k}(\tilde{g}_2^*) - \sum_{s=1}^{N} (\nabla \tilde{z}_s(\xi + me_k) : \delta_m^{k} \nabla^i \psi^{-1} + \nabla \tilde{z}_i(\xi + me_k) : \delta_m^{k} \nabla^i \psi^{-1}) \tilde{n}_s \] for \(i\).

Let us now estimate \(G_{0,i}\) in the \(L^2(C_1^+)\) norm. We have
\[ \left\| \delta_m^{k}(\tilde{g}_0^*)_i \right\|_{L^2(C_1^+)} \leq C|m|\|\nabla(\tilde{g}_0^*)_i\|_{L^2(C_1^+)} \leq C|m|\|\nabla(\tilde{g}_0^*)\|_{H^1(C_1^+)}, \]
\[ \left\| \text{Hess}(\tilde{z}_i(\xi + me_k)) : \delta_m^{k}(\nabla \psi^{-1} \nabla^i \psi^{-1}) \right\|_{L^2(C_1^+)} \leq C|m|\|\nabla \tilde{z}_i\|_{L^2(C_1^+)} \leq C(k, \Omega)|m|\|\nabla \tilde{z}_i\|_{L^2(C_1^+)} \leq C|m|\|\nabla \tilde{z}_i\|_{H^2(C_1^+)}. \]
\[ \left\| \nabla \tilde{h}(\xi + me_k) : \delta_m^{k} \nabla^i \psi^{-1} \right\|_{L^2(C_1^+)} \leq C|m|\|\nabla \tilde{h}\|_{L^2(C_1^+)} . \]

Therefore
\[ \left\| G_0 \right\|_{L^2(C_1^+)} \leq C|m|\left( \|g_0^*\|_{H^1(\Omega)} + |z|_{H^2(\Omega)} + \|\nabla h\|_{L^2(\Omega)} \right). \]

In the same way we can estimate \(G_1\) in \(H^1(C_1^+)\) from
\[ \left\| \delta_m^{k}(\tilde{g}_1^*) \right\|_{H^1(C_1^+)} \leq |m|\|\nabla \tilde{g}_1^*\|_{H^2(C_1^+)} \]
and we obtain
\[ \left\| G_1 \right\|_{H^1(C_1^+)} \leq C|m|\left( \|g_1^*\|_{H^2(\Omega)} + |z|_{H^2(\Omega)} \right). \]

Finally, for \(G_2\) we get
\[ \left\| G_2 \right\|_{H^{1/2}(\partial \Omega \cap C_1^+)} \leq C|m|\left( \|\tilde{g}_2^*\|_{H^{1/2}(\partial \Omega \cap C_1^+)} + \|\tilde{z}\|_{H^{1/2}(\partial \Omega \cap C_1^+)} \right). \]

Then, using the definition of \(g^*_i (i = 0, 1, 2)\) given in (2.13) and the estimate (2.9) for the solutions of the stationary problems (2.8) and (2.12), we obtain
\[ \left\| G_0 \right\|_{L^2(C_1^+)} \leq C|m|\left( \|g_0^*\|_{H^1(\Omega)} \right), \]
\[ \left\| G_1 \right\|_{H^1(C_1^+)} \leq C|m|\left( \|g_0^*\|_{L^2(\Omega)} + \|g_0^*\|_{H^2(\Omega)} + \|g_2^*\|_{H^1(\Omega)} \right) \]
and
\[ \left\| G_2 \right\|_{H^{1/2}(\partial \Omega \cap C_1^+)} \leq C|m|\left( \|g_2^*\|_{H^{1/2}(\partial \Omega)} + \|g_0^*\|_{L^2(\Omega)} + \|g_1^*\|_{H^1(\Omega)} \right). \]

In consequence, the solution of (2.16) belongs to \(\tilde{X}_{1,2} \times H^1(C_1^+)\) and satisfies
\[ \left\| \delta_m^{k}\tilde{z} \right\|_{H^2(C_1^+)} + \left\| \delta_m^{k}\tilde{h} \right\|_{H^1(C_1^+)} \leq C|m|\left( \|g_0^*\|_{H^1(\Omega)} + \|g_1^*\|_{H^2(\Omega)} + \|g_2^*\|_{H^{1/2}(\partial \Omega)} \right) \]
for \(k = 1, \ldots, N - 1\). Taking \(m \to 0\), this implies \((\partial_k \tilde{z}, \partial_k \tilde{h}) \in H^2(C_1^+) \times H^1(C_1^+)\) and
\[ \left\| \partial_k \tilde{z} \right\|_{H^2(C_1^+)} + \left\| \partial_k \tilde{h} \right\|_{H^1(C_1^+)} \leq C\left( \|g_0^*\|_{H^1(\Omega)} + \|g_1^*\|_{H^2(\Omega)} + \|g_2^*\|_{H^{1/2}(\partial \Omega)} \right) \]
for $1 \leq k \leq N - 1$. Now, we will prove that \( \frac{\partial z_i}{\partial x_k} \frac{\partial h}{\partial x_k} \in H^2(C_1^+) \times H^1(C_1^+) \) for every \( i = 1, \ldots, N \).

From (2.14) we have

\[
- \frac{\partial^2 z_i}{\partial x_k^2} \sum_{k=1}^{N} \left| \frac{\partial^2 z_i}{\partial x_k^2} \right|^2 + \frac{\partial^2 h}{\partial x_k \partial x_i} \in H^1(C_1^+), \quad \forall i = 1, \ldots, N. \tag{2.17}
\]

Then

\[
- \left( \sum_{k=1}^{N} \frac{\partial^2 z_i}{\partial x_k^2} \right)^2 \left( \sum_{i=1}^{N} \frac{\partial^2 h}{\partial x_i \partial x_i} \right) + \frac{\partial^2 h}{\partial x_k \partial x_i} \sum_{i=1}^{N} \left| \frac{\partial^2 z_i}{\partial x_k \partial x_i} \right|^2 \in L^2(C_1^+). \tag{2.18}
\]

On the other hand, from the divergence free condition (see (2.14)) we get

\[
\sum_{i=1}^{N} \frac{\partial z_i}{\partial x_i} = - \sum_{i=1}^{N} \left( \sum_{k=1}^{N-1} \frac{\partial z_i}{\partial x_k} \right) + 3 \in H^2(C_1^+),
\]

so that

\[
\sum_{i=1}^{N} \frac{\partial z_i}{\partial x_i} \frac{\partial h}{\partial x_i} \in L^2(C_1^+). \tag{2.19}
\]

From (2.18) and (2.19), we obtain that

\[
\frac{\partial^2 h}{\partial x_k^2} \sum_{i=1}^{N} \left| \frac{\partial^2 z_i}{\partial x_k \partial x_i} \right|^2 \in L^2(C_1^+)
\]

and therefore \( \tilde{h} \in H^2(C_1^+) \). Coming back to (2.17) we obtain that

\[
\frac{\partial^2 z_i}{\partial x_k^2} \sum_{k=1}^{N} \left| \frac{\partial^2 z_i}{\partial x_k^2} \right|^2 \in L^2(C_1^+), \quad \forall i = 1, \ldots, N.
\]

Therefore \( \tilde{h} \in H^2(C_1^+) \) and \( \tilde{z} \in H^3(C_1^+) \), so that \((\partial_z z, \partial_z h) \in H^2(\Omega \cap \tilde{U})^N \times H^1(\Omega \cap \tilde{U})\) for \( k = 1, \ldots, N \) and we can conclude that \((z, h) \in H^2(\Omega \cap \tilde{U})^N \times H^2(\Omega \cap \tilde{U})\) for every \( \tilde{U} \subset \subset U \) with the estimate

\[
\|z\|_{H^3(\Omega \cap \tilde{U})^N} + \|h\|_{H^2(\Omega \cap \tilde{U})} \leq C \left( \|g_0\|_{H^2(\Omega \cap \tilde{U})}^N + \|g_1\|_{H^2(\Omega \cap \tilde{U})} + \|g_2\|_{H^2(\Omega \cap \tilde{U})} \right). \tag{2.20}
\]

This, together with (2.10), gives the following estimate for the solution of the stationary system (2.7):

\[
\|u\|_{H^3(\Omega)^N} + \|\theta\|_{H^2(\Omega)} \leq C \left( \|f_0\|_{H^2(\Omega)^N} + \|u_t\|_{H^2(\Omega)^N} + \|f_2\|_{H^2(\Omega)^N} + \|A u\|_{H^2(\Omega)^N} \right). \tag{2.21}
\]

Now, to estimate the term \( \|u_t(t)\|_{H^1(\Omega)^N} \) we multiply (2.4) by

\[
\partial_t (B(u, \theta)) := - \nabla \cdot D u_t + \nabla \theta_t
\]

and integrate in \( \Omega \). We get

\[
- \int_{\Omega} u_t \nabla \cdot D u_t \, dx + \int_{\Omega} u_t \cdot \nabla \theta_t \, dx + \frac{1}{2} \int_{\Omega} |B(u, \theta)|^2 \, dx = \int_{\Omega} f_0 \cdot \nabla \theta_t \, dx - \int_{\Omega} f_0 \cdot D u_t \, dx.
\]

Integrating by parts and using that \( f_0 \) belongs to \( W \), we obtain

\[
\int_{\Omega} \nabla u_t \, dx + \frac{1}{2} \int_{\Omega} |B(u, \theta)|^2 \, dx - \int_{\Omega} u_t \cdot (D u_t \cdot n)_{tg} \, d\sigma = \int_{\Omega} \nabla f_0 \cdot \nabla u_t \, dx - \int_{\Omega} f_0 \cdot (D u_t \cdot n)_{tg} \, d\sigma.
\]

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where \( \theta \)

From this estimate and (2.22) we obtain

Then for almost every \( t \in (0, T) \). Coming back to (2.21), we get

In order to estimate the third term in (2.22) we use that

Hence we have

Then

This estimate and (2.22) we obtain

where \( \theta(0) \) is defined (up to a constant) by

Now, we estimate the boundary terms in (2.23). First, we find

for any \( \delta > 0 \). The second term can be estimated as follows:

Putting together these estimates and (2.23) we can deduce

\[
\begin{align*}
\| u_t \|^2_{L^2(H^1(\Omega)^N)} + \| u \|^2_{L^2(H^3(\Omega)^N)} + \| B(u, \theta) \|^2_{L^\infty(L^2(\Omega)^N)} + \| \theta \|^2_{L^2(H^2(\Omega))} \\
&\leq C\left( \| f_0 \|^2_{L^2(H^1(\Omega)^N)} + \| f_2 \|^2_{L^2(H^3(\Omega)^N)} + \| \partial_t f_2 \|^2_{L^2(H^1(\Omega)^N)} + \| B(u_0, \theta(0)) \|^2_{L^2(\Omega)^N} \\
&\quad + \left( 1 + \| A \|^2_{L^\infty(H^3/2(\Omega)^{N \times N})} + \| \partial_t A \|^2_{L^2(\Sigma)^{N \times N}} + \| A \|^2_{L^2(\Sigma)^{N \times N}} \right) \| u \|^2_{L^2(H^1(\Omega))} \right).
\end{align*}
\]
Using (2.2) in order to estimate \(\|u\| \|_{V^1}^2\) and elliptic estimates (2.24), we get

\[
\begin{align*}
\|u_u\|_{L^2_2(H^1_\Omega)^N}^2 + \|u_{\theta}\|_{L^2_2(H^2_\Omega)^N}^2 + \|B(u, \theta)\|_{L^2_\infty(\Omega)^N}^2 + \|\theta\|_{L^2_2(H^2_\Omega)}^2 \\
\leq C(A) \left(\|f_0\|_{L^2_{\partial\Omega}}^2 + \|f_2\|_{L^2_2(H^{1/2}(\partial\Omega))^N}^2 + \|\partial_t f_2\|_{L^2_2(H^{1/2}(\partial\Omega))^N}^2 + \|u_0\|_{H^3_\Omega}^2 \right),
\end{align*}
\]

(2.25)

where

\[
C(A) := C e^{CT\|A\|_{L^p}^2} \left(1 + \|A\|_{L^p}^4\right) \left(1 + \|A\|_{L^\infty(\Omega)}^2 + \|\partial_t A\|_{L^2(\Omega)}^2\right). \]

**Step 2.** Taking into account the previous step, we will prove that the weak solution \((u, \theta)\) of (2.8) belongs to \(H^3(\Omega)^N \times H^3_\Omega\) whenever

\[
g_0 \in H^2(\Omega \cap U_0)^N, \quad g_1^* \in H^3_\Omega(\Omega \cap U_0), \quad g_2^* \in H^{5/2}(\partial\Omega \cap U_0)^N,
\]

also, \(\psi\) is a \(W^{4,\infty}\) diffeomorphism. Here, we define

\[
\tilde{X}_{1,3} := \{z \in H^3(C_1^+) : \tilde{z} = 0 \text{ on } \partial C_1^+ \cap \mathbb{R}^N_+, \quad \tilde{z} \cdot \tilde{n} = 0 \text{ on } \partial \mathbb{R}^N_+ \cap C_1\}.
\]

Let us prove that \(\tilde{z}\) satisfies \(\delta^k_m \tilde{z} \in \tilde{X}_{1,3},\) for \(k = 1, \ldots, N - 1\) and \(|m| < d/2\) (recall that \(d = dist(\partial C_1^+, \partial C_1^+)\)), where \(\tilde{z}\) fulfills (2.14). We have the following estimates for \(G_0, G_1\) and \(G_2\) (which were defined right after (2.16)):

\[
\|G_0\|_{H^1(C_1^+)^N} \leq C|m| \left(\|g_0^*\|_{H^2(\Omega)^N} + \|\tilde{z}\|_{H^3(\Omega)^N} + \|\nabla h\|_{H^1(\Omega)}\right),
\]

\[
\|G_1\|_{H^2(C_1^+)} \leq C|m| \left(\|g_1^*\|_{H^3(\Omega)} + \|\tilde{z}\|_{H^3(\Omega)^N}\right)
\]

and

\[
\|G_2\|_{H^{5/2}(\partial\Omega \cap C_1)} \leq C|m| \left(\|g_2^*\|_{H^{5/2}(\partial\Omega \cap C_1)} + \|\tilde{z}\|_{H^{5/2}(\partial\Omega \cap C_1)}\right).
\]

Then, using (2.26) together with the definition of \(g_i^* (i = 0, 1, 2)\) given in (2.13) and the estimate (2.25) for the solutions of the stationary problems (2.8) and (2.12), we obtain

\[
\|G_0\|_{H^1(C_1^+)^N} \leq C|m| \left(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N}\right),
\]

\[
\|G_1\|_{H^2(C_1^+)} \leq C|m| \left(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N}\right)
\]

and

\[
\|G_2\|_{H^{5/2}(\partial\Omega \cap C_1)} \leq C|m| \left(\|g_0\|_{H^1(\Omega)} + \|g_1\|_{H^2(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N}\right).
\]

In consequence, \((\delta_m^k \tilde{z}, \delta_m^k \tilde{h}) \in \tilde{X}_{1,3} \times H^2(C_1^+)\) and

\[
\|\delta_m^k \tilde{z}\|_{H^3(C_1^+)^N} + \|\delta_m^k \tilde{h}\|_{H^3(C_1^+)} \leq C|m| \left(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N}\right)
\]

for \(k = 1, \ldots, N - 1\).

Arguing now as in **Step 1**, we find

\[
\|u\|_{H^4(\Omega)^N} + \|\theta\|_{H^3(\Omega)} \leq C \left(\|g_0\|_{H^2(\Omega)^N} + \|g_1\|_{H^3(\Omega)} + \|g_2\|_{H^{5/2}(\partial\Omega)^N}\right).
\]

(2.27)

From (2.27) we obtain the estimate for the solution of the stationary system (2.7):

\[
\|u\|_{H^4(\Omega)^N} + \|\theta\|_{H^3(\Omega)} \leq C \left(\|f_0\|_{H^2(\Omega)^N} + \|u_t\|_{H^2(\Omega)^N} + \|f_2\|_{H^{5/2}(\partial\Omega)^N} + \|Au\|_{H^{5/2}(\partial\Omega)^N}\right).
\]

(2.28)
for almost every \( t \in (0, T) \). Now, in order to estimate the second term of the right-hand side of (2.28), we consider the system satisfied by \((\partial_t u, \partial_t \theta)\) (see (2.4)):

\[
\begin{aligned}
\partial_t (u_t) - \nabla \cdot (D u_t) + \nabla \theta_t &= \partial_t f_0 \\
\nabla \cdot u_t &= 0 \\
u_t \cdot n &= 0, \quad (\sigma (u_t, \theta_t) \cdot n)_{tg} + (A u_t)_{tg} &= \partial_t f_2 - (A_t u)_t g \\
u_t (\cdot, 0) &= \nabla \cdot D u_0 (\cdot) - \nabla (\theta (\cdot, 0) + f_0 (\cdot, 0)) \\
\end{aligned}
\]

(2.29)

In virtue of Lemma 2.1 we have that \((u_t, \theta_t)\) is the strong solution of (2.29). Furthermore, we get \(u_t \in Y_1\) and

\[
\|u_t\|_{Y_1} \leq e^{CT\|A\|^2_{L^p}} (1 + \|A\|^2_{L^p}) \left( \|\partial_t f_0\|_{L^2(Q)^N} + \|f_0\|_{L^\infty(\Omega)^N} + \|\partial_t f_2\|_{L^2(\Omega)^N} + \|\partial_t f_2\|_{L^2(\Omega)^N} \right)
\]

\[
+ \|A u\|_{L^2(\Omega)^N} + \|A u\|_{L^2(\Omega)^N} + \|A u\|_{L^2(Q)^N} + \|u_0\|_{L^2(\Omega)^N \cap W}.
\]

(2.30)

Therefore, from (2.28) and (2.30) we obtain

\[
\|u_t\|_{Y_1} + \|u\|_{L^2(\Omega)^N} + \|\theta\|_{L^2(\Omega)}
\]

\[
\leq e^{CT\|A\|^2_{L^p}} (1 + \|A\|^2_{L^p}) \left( \|\partial_t f_0\|_{L^2(Q)^N} + \|f_0\|_{L^\infty(\Omega)^N} + \|\partial_t f_2\|_{L^2(\Omega)^N} + \|\partial_t f_2\|_{L^2(\Omega)^N} \right)
\]

\[
+ \|A u\|_{L^2(\Omega)^N} + \|A u\|_{L^2(\Omega)^N} + \|A u\|_{L^2(\Omega)^N} + \|u_0\|_{L^2(\Omega)^N \cap W}.
\]

(2.31)

Finally, we estimate \(\|A u\|_{L^2(\Omega)^N}, \|A u\|_{L^2(\Omega)^N}\) and \(\|A u\|_{L^2(\Omega)^N}\) by:

\[
\|A u\|_{L^2(\Omega)^N} \leq C \|A u\|_{L^2(\Omega)^N} \leq C \|A u\|_{L^2(\Omega)^N} \leq C \|A u\|_{L^2(\Omega)^N} \leq C \|A u\|_{L^2(\Omega)^N} \leq C \|A u\|_{L^2(\Omega)^N}
\]

and

\[
\|A u\|_{L^2(\Omega)^N} \leq C \left( \|A\|_{L^\infty(\Omega)^N} \|u\|_{L^2(\Omega)^N} + \|A\|_{L^2(\Omega)^N} \|u\|_{L^2(\Omega)^N} \right).
\]

Using (2.25), (2.31) and the previous estimates, we find the desired estimate (2.5). This concludes the proof of Lemma 2.1.

### 3 Carleman inequality for the adjoint system

In this section we will mainly prove a Carleman estimate for the adjoint system of (1.3). In order to do so, we are going to introduce some weight functions. Let \(\omega_0\) be a nonempty open subset of \(\mathbb{R}^N\) such that \(\omega_0 \Subset \omega_1 \Subset \omega' \Subset \omega\) and \(\eta \in C^2(\overline{\Omega})\) such that

\[
|\nabla \eta| > 0 \text{ in } \overline{\Omega} \setminus \omega_0, \quad \eta > 0 \text{ in } \Omega \text{ and } \eta \equiv 0 \text{ on } \partial \Omega.
\]

The existence of such a function \(\eta\) is proved in [10]. Then, for all \(\lambda \geq 1\) we consider the following weight functions:

\[
\alpha (x, t) = \frac{e^{2\lambda |\eta|_x} - e^{\lambda \eta (x)}}{(t(T - t))^{11}}, \quad \xi (x, t) = \frac{e^{\lambda \eta (x)}}{(t(T - t))^{11}},
\]

\[
\alpha^* (t) = \max_{x \in \Omega} \alpha (x, t), \quad \xi^* (t) = \min_{x \in \Omega} \xi (x, t),
\]

\[
\tilde{\alpha} (t) = \min_{x \in \Omega} \alpha (x, t), \quad \tilde{\xi} (t) = \max_{x \in \Omega} \xi (x, t).
\]

(3.1)
We consider now a backwards nonhomogeneous system associated to the Stokes equation:

\[
\begin{aligned}
&\begin{cases}
-\varphi_t - \nabla \cdot (D\varphi) + \nabla \pi = g \\
\nabla \cdot \varphi = 0 \\
\varphi \cdot n = 0, (\sigma(\varphi, \pi) \cdot n)_{tg} + (A^t(x, t)\varphi)_{tg} = 0 \\
\varphi(\cdot, T) = \varphi^T(\cdot)
\end{cases} & \text{ in } Q, \\
&\begin{cases}
\varphi = 0 \\
\varphi \cdot n = 0
\end{cases} & \text{ on } \Sigma, \\
&\begin{cases}
\varphi = 0
\end{cases} & \text{ in } \Omega,
\end{aligned}
\]

(3.2)

where \( g \in L^2(Q)^N \) and \( \varphi^T \in H \). Our Carleman estimate is given in the following proposition.

**Proposition 3.1.** Let \( A \in P^1_2 \cap P^2_2 \). There exists a constant \( \lambda_0 \), such that for any \( \lambda > \lambda_0 \) there exist two constants \( C(\lambda) > 0 \) increasing on \( \|A\|_{P^1_2 \cap P^2_2} \) and \( s_0(\lambda) > 0 \) such that for any \( i \in \{1, \ldots, N\} \), any \( g \in L^2(Q)^N \) and any \( \varphi^T \in H \), the solution of (3.2) satisfies

\[
s^3 \int_Q e^{-6s\alpha^*}(\xi^*)^3|\varphi|^2 dxdt \
\leq C \left( \int_Q e^{-4s\alpha^*}|\varphi|^2 dxdt + s^7 \sum_{j=1}^N \int_0^T \int_{\omega_j} e^{-4s\alpha^* - 2s\alpha^*}(\xi^*)^{12}|\varphi_j|^2 dxdt \right)
\]

(3.3)

for every \( s \geq s_0 \).

Before giving the proof of Proposition 3.1, we present some technical results. We first present a Carleman inequality proved in [9] for a general heat equation with Fourier boundary conditions. To this end, let us introduce the system

\[
\begin{aligned}
&\begin{cases}
-\psi_t - \Delta \psi = f_1 + \nabla \cdot f_2 \\
(\nabla \psi + f_2) \cdot n = f_3 \\
\psi(\cdot, T) = \psi^T(\cdot)
\end{cases} & \text{ in } Q, \\
&\begin{cases}
\nabla \psi = 0 \\
\psi \cdot n = 0
\end{cases} & \text{ on } \Sigma, \\
&\begin{cases}
\psi = 0
\end{cases} & \text{ in } \Omega,
\end{aligned}
\]

(3.4)

where \( f_1 \in L^2(Q), f_2 \in L^2(Q)^N \) and \( f_3 \in L^2(\Sigma) \). We present now this result:

**Lemma 3.1.** Under the previous assumptions on \( f_1, f_2 \) and \( f_3 \), there exist \( \overline{\lambda}, \sigma_1, \sigma_2 \) and \( C \), only depending on \( \Omega \) and \( \omega \), such that, for any \( \lambda \geq \overline{\lambda}, \) any \( s \geq \overline{s} = \sigma_1(e^{\sigma_2\lambda T} + T^2) \) and any \( \psi^T \in L^2(\Omega) \), the weak solution to (3.4) satisfies

\[
\int_Q e^{-2s\alpha}(s^2\xi^2|\nabla \psi|^2 + s^3\lambda^4\xi^3|\psi|^2)dxdt + s^2\lambda^3 \int_{\Omega} e^{-2s\alpha}\xi^2|\psi|^2 d\sigma dt 
\leq C \left( \int_Q e^{-2s\alpha}(|f_1|^2 + s^2\lambda^2\xi^2|f_2|^2)dxdt \\
+ s\lambda \int_{\Omega} e^{-2s\alpha}\xi|f_3|^2 d\sigma dt + s^3\lambda^4 \int_{\omega_1 \times (0, T)} e^{-2s\alpha}\xi^3|\psi|^2 d\sigma dt \right)
\]

(3.5)

The next lemma is a result for elliptic equations with non homogeneous boundary condition that can be found in [14] (see also [8]).

**Lemma 3.2.** Let \( y \in H^1(\Omega) \) satisfy

\[
\Delta y = f_0 + \sum_{j=1}^N \frac{\partial f_i}{\partial x_j}, \text{ in } \Omega; \quad y = g, \text{ on } \partial \Omega,
\]

with \( f_0, f_j \in L^2(\Omega) \) and \( g \in H^{1/2}(\partial \Omega) \). Then there exist three constants \( C > 0, \overline{\lambda} > 1 \) and \( \overline{\tau} > 1 \)
such that for any $\lambda \geq \tilde{\lambda}$ and any $\tau \geq \tilde{\tau}$, we have

$$\int_{\Omega} |\nabla y|^2 e^{2s_\lambda^\alpha} dx + \tau^2 \lambda^2 \int_{\Omega} e^{2\lambda\eta} |y|^2 e^{2s_\lambda^\alpha} dx \leq C \left( \tau^{1/2} e^{2\eta} \|y\|_{H^{1/2}(\partial\Omega)}^2 + \tau^{-1} \lambda^{-2} \int_{\Omega} e^{-\lambda\eta} |f_0|^2 e^{2s_\lambda^\alpha} dx \right) + \sum_{j=0}^N \int_{\Omega} e^{\lambda\eta} |f_j|^2 e^{2s_\lambda^\alpha} dx + \int_{\Omega} \left( |\nabla y|^2 + \tau^2 \lambda^2 e^{2\lambda\eta} |y|^2 \right) e^{2s_\lambda^\alpha} dx \right).$$

(3.6)

**Remark 3.1.** We can eliminate the local integral of $|\nabla y|^2$ in (3.6) at the price of having a local term of $|y|^2$ in a open set $\omega_2$ satisfying $\omega_1 \Subset \omega_2 \Subset \omega'$. For these details, we invite the interested reader to see [8].

The next technical result corresponds to the Lemma 3 in [6].

**Lemma 3.3.** Let $r \in \mathbb{R}$. There exists $C > 0$ depending only on $r, \Omega, \omega_0$ and $\eta$ such that, for every $T > 0$ and every $u \in L^2(0, T; H^1(\Omega))$,

$$s^2 \lambda^2 \iint_Q e^{-2s_\alpha^\tau} |x|^{r+2} |u|^2 dx dt \leq C \left( \iint_Q e^{-2s_\alpha^\tau} |\nabla u|^2 dx dt + s^2 \lambda^2 \iint_{\omega_0 \times (0, T)} e^{-2s_\alpha^\tau} |x|^{r+2} |u|^2 dx dt \right),$$

for every $\lambda \geq C$ and every $s \geq CT^{22}$.

**Remark 3.2.** In [6], [9] and [14] slightly different weight functions are used to prove the above results. However, this does not change the result since the important property is that $\alpha$ goes to $0$ polynomially when $t$ tends to $0$ and $T$.

We will now prove Proposition 3.1. Without any lack of generality, we treat the case $N = 2$ and $i = 2$. The arguments can be easily extended to the general case. Let us introduce $(w, q)$ and $(z, r)$, the solutions of the following systems:

$$\begin{cases} -w_t - \nabla \cdot (Dw) + \nabla q = \rho g & \text{in } Q, \\ \nabla \cdot w = 0 & \text{in } Q, \\ w \cdot n = 0, (\sigma(w, q) \cdot n)_{tg} + (A^t(x, t)w)_{tg} = 0 & \text{on } \Sigma, \\ w(\cdot, T) = 0 & \text{in } \Omega, \end{cases}$$

(3.8)

and

$$\begin{cases} -z_t - \nabla \cdot (Dz) + \nabla r = -\rho' \varphi & \text{in } Q, \\ \nabla \cdot z = 0 & \text{in } Q, \\ z \cdot n = 0, (\sigma(z, r) \cdot n)_{tg} + (A^t(x, t)z)_{tg} = 0 & \text{on } \Sigma, \\ z(\cdot, T) = 0 & \text{in } \Omega, \end{cases}$$

(3.9)

where $\rho(t) = e^{-2s_\alpha^\tau}$. Adding (3.8) and (3.9), we see that $(w + z, q + r)$ solves the same system as $(\rho \varphi, \rho \pi)$, where $(\varphi, \pi)$ is the solution of (3.2). By uniqueness of the Stokes system with Navier-slip boundary conditions, we have

$$\rho \varphi = w + z \quad \text{and} \quad \rho \pi = q + r.$$

(3.10)

For system (3.8) we will use Lemma 2.1 and the regularity estimate (2.2), namely

$$\|w\|_{L^2(0, T; H^2(\Omega)^2)}^2 + \|w\|_{H^1(0, T; L^2(\Omega)^3)}^2 \leq C\|\rho g\|_{L^2(Q)^2}^2,$$

(3.11)
and for the system (3.9) we will use the ideas of [4] and [6].

We apply the operator $\nabla$ to the equation satisfied by $z_1$ and we denote $\psi := \nabla z_1$. Then $\psi$ satisfies

$$-\psi_t - \Delta \psi = -\nabla (\rho' \phi_1) - \nabla \partial_t r \in Q.$$  

Using Lemma 3.1 with $f_1 = -\nabla (\rho' \phi_1) - \nabla \partial_t r$ and $f_2 = 0$, we obtain

$$s^3 \int_Q e^{-2s\alpha} \xi^3 |\psi|^2 dx dt \leq C \left( s^3 \int_0^T \int_{\omega_1} e^{-2s\alpha} \xi^3 |\psi|^2 dx dt \right.$$  

$$+ s \int_{\Omega} e^{-2s\alpha} \left| \frac{\partial \psi}{\partial n} \right|^2 ds dt + \int_Q e^{-2s\alpha} |\nabla (\rho' \phi_1) + \nabla \partial_t r|^2 dx dt \right)$$  

for every $\lambda \geq \bar{\lambda}$ and $s \geq \bar{s}_0$. Here and in the following, $C$ will denote a generic constant depending on $\Omega, \omega$ and $\lambda$.

The rest of the proof is divided into three steps.

a) In step 1, using Lemma 3.3 we estimate global integrals of $z_1$ and $z_2$. In addition, we partially estimate the pressure in the right-hand side of (3.12).

b) In step 2, we will estimate the normal derivative appearing in the right-hand side of (3.12) and the global term of the pressure obtained in step 1.

c) In step 3, we will estimate all the local terms by a local term of $\phi_1$.

**Step 1. Estimate of $z_1$.** We use Lemma 3.3 with $u = \nabla z_1$ and $r = 3$:

$$s^5 \int_Q e^{-2s\alpha} \xi^5 |z_1|^2 dx dt \leq C \left( s^3 \int_Q e^{-2s\alpha} \xi^3 |\psi|^2 dx dt + s^5 \int_0^T \int_{\omega_0} e^{-2s\alpha} \xi^5 |z_1|^2 dx dt \right)$$  

for every $s \geq C$.

**Estimate of $z_2$.** Let us first establish a general estimate: $\forall \varepsilon' > 0$, $\exists C \in \mathbb{R}$:

$$\|u\|_{H^{1/2 + \varepsilon'}(\Omega)}^2 \leq C(\|u_1\|_{L^2(\Omega)} + \|u_1 n_1\|_{L^2(\partial \Omega)} + \|\partial_t u_1\|_{H^{-1/2}(\Omega)}) \leq C\|u_1\|_{H^{1/2 + \varepsilon'}(\Omega)}, \forall u \in W.$$  

(3.14)

Indeed, for any $f \in H_{\varepsilon'} := H^{1/2 + \varepsilon'}(\Omega)^2 \cap H$, we have (after an integration by parts)

$$\int_\Omega u \cdot f dx = \int_\Omega u_1 f_1 dx - \int_{\partial \Omega} u_1 n_1 \tilde{f}_2 ds + \int_\Omega \partial_t u_1 \tilde{f}_2 dx,$$  

(3.15)

where $\tilde{f}_2 \in H^{1/2 + \varepsilon'}(\Omega)$ satisfies

$$\partial_t \tilde{f} = f_2 \quad \text{a.e. } \Omega \quad \text{and} \quad \|\tilde{f}_2\|_{H^{1/2 + \varepsilon'}(\Omega)} \leq C\|f\|_{H^{1/2 + \varepsilon'}(\Omega)}.$$  

Then, from (3.15), we readily obtain (3.14).

Let us now apply (3.14) for $u := z$. We deduce

$$\forall \varepsilon' > 0, \exists C \in \mathbb{R} : \|z\|_{H^{1/2 + \varepsilon'}(\Omega)}^2 \leq C\|z_1\|_{H^{1/2 + \varepsilon'}(\Omega)}^2,$$

so that, using that $H^{1/2 + \varepsilon'}(\Omega)$ is the interpolation space $(H^1(\Omega), L^2(\Omega))_{1/2 + \varepsilon', 2}$, we find

$$s^{4-2\varepsilon'} \int_0^T e^{-2s\alpha} \xi^{4-2\varepsilon'} |z|^2_{H^{1/2 + \varepsilon'}(\Omega)} dt \leq C s^3 \int_Q e^{-2s\alpha} \xi^3 \left( \frac{s^2 (\xi^*)^2}{2} |z_1|^2 + |\nabla z_1|^2 \right) dx dt.$$  

(3.16)
Putting together (3.12), (3.13) and (3.16) we have for the moment
\[
s_{5} \int_{Q} e^{-2s\alpha} \xi_{\ast}^{5} |z_{1}|^{2} dx dt + s^{4-2\epsilon} \int_{0}^{T} e^{-2s\alpha} (\xi_{\ast}^{5})^{4-2\epsilon} \| \xi \|^{2}_{(H^{\epsilon}_{\ast})} dt + s^{3} \int_{Q} e^{-2s\alpha} \xi_{\ast}^{3} |\psi|^{2} dx dt
\]
\[
\leq C \left( \int_{0}^{T} e^{-2s\alpha} (s^{5} \xi_{\ast}^{5} |z_{1}|^{2} + s^{3} \xi_{\ast}^{3} |\nabla z_{1}|^{2}) dx dt \right.
\]
\[
+ s \int_{\Sigma} e^{-2s\alpha} \xi_{\ast} \left| \frac{\partial \nabla z_{1}}{\partial \nu} \right|^{2} d\sigma dt + \int_{Q} e^{-2s\alpha} |\nabla (\rho \varphi_{1}) + \nabla \partial_{t} r|^{2} dx dt \right)
\]
for every \( s \geq C. \)

Taking into account that
\[
|\alpha_{z}^{\ast}| \leq C (\xi_{\ast}^{4})^{12/11}, \quad |\rho'| \leq C s \rho (\xi_{\ast}^{4})^{12/11}
\]
and (3.10), we obtain
\[
\int_{Q} e^{-2s\alpha} |\nabla (\rho \varphi_{1})|^{2} dx dt = \int_{Q} e^{-2s\alpha} |\rho'|^{2} |\rho|^{-2} |\nabla (\rho \varphi_{1})|^{2} dx dt
\]
\[
\leq C \left( s^{2} \int_{Q} e^{-2s\alpha} (\xi_{\ast})^{24/11} |\nabla w_{1}|^{2} + s^{2} \int_{Q} e^{-2s\alpha} (\xi_{\ast})^{24/11} |\nabla z_{1}|^{2} dx dt \right).
\]
(3.17)

The fact that \( s^{2} e^{-2s\alpha} (\xi_{\ast})^{24/11} \) is bounded allows us to estimate the first term in the right-hand side of (3.19) using (3.11). On the other hand, the second term in the right-hand side of (3.19) can be absorbed by the third term in the left-hand side of (3.17).

Additionally, using the divergence-free condition on the equation of (3.9), we see that
\[
\Delta r = 0 \quad \text{in } Q,
\]
then
\[
\Delta (\nabla \partial_{t} r) = 0 \quad \text{in } Q.
\]

Using Lemma 3.2 with \( y = \nabla \partial_{t} r \) and Remark 3.1 we obtain
\[
\tau^{2} \int_{\Omega} e^{2\lambda_{y}} |\nabla \partial_{t} r|^{2} e^{2\tau r \lambda_{y}} dx \leq C \left( \tau^{1/2} e^{\tau r} \| \nabla \partial_{t} r \|^{2}_{H^{1/2}(\partial \Omega)} + \tau^{2} \int_{w_{2}} e^{2\lambda_{y}} |\nabla \partial_{t} r|^{2} e^{2\tau r \lambda_{y}} dx \right)
\]
for every \( \tau \geq C. \) Now we take
\[
\tau = \frac{s}{(t(T-t))^{11}},
\]
multiply the last inequality by
\[
\exp \left( -2s \frac{e^{2\lambda_{y} \| y \|_{\infty}}}{(t(T-t))^{11}} \right),
\]
and integrate with respect to \( t \) in \((0, T)\) to obtain
\[
\int_{Q} e^{-2s\alpha} |\nabla \partial_{t} r|^{2} dx dt
\]
\[
\leq C \left( s^{-3/2} \int_{0}^{T} e^{-2s\alpha} (\xi_{\ast})^{-3/2} \| \nabla \partial_{t} r \|^{2}_{H^{1/2}(\partial \Omega)} dt + \int_{0}^{T} \int_{Q} e^{-2s\alpha} \xi_{\ast}^{2} |\nabla \partial_{t} r|^{2} dx dt \right),
\]
for all $s \geq C$.

Combining this with (3.17) and (3.19), we have for the moment

$$
s^5 \int_Q e^{-2s\alpha} |\xi|^2 z_1^2 dx \, dt + s^4 - 2\epsilon \int_0^T e^{-2s\alpha} (\xi^*)^{1-2\epsilon} \|z\|_{(H^\epsilon)'(\Omega)}^2 dt + s^3 \int_Q e^{-2s\alpha} |\nabla z_1|^2 dx \, dt
$$

\[ \leq C \left( \|\rho g\|_{L^2(Q)}^2 + s \int_\Sigma e^{-2s\alpha} \left| \frac{\partial z_1}{\partial n} \right|^2 \, d\sigma dt + s^{-3/2} \int_0^T e^{-2s\alpha} (\xi^*)^{-3/2} \|\nabla \partial_t r\|^2_{H^{1/2}(\partial\Omega)} dt \right. \\
+ \left. \int_0^T e^{-2s\alpha} (\xi^*)^{-3/2} \|\nabla \partial_t r\|^2_{H^{3/2}(\partial\Omega)} dt \right),
\]

for every $s \geq C$.

**Step 2.** In this step we deal with the boundary terms in (3.20), i.e.

$$
s \int_\Sigma e^{-2s\alpha} \left| \frac{\partial z_1}{\partial n} \right|^2 \, d\sigma dt \quad \text{and} \quad s^{-3/2} \int_0^T e^{-2s\alpha} (\xi^*)^{-3/2} \|\nabla \partial_t r\|^2_{H^{1/2}(\partial\Omega)} dt.
$$

Let us start by defining

$$
\tilde{z} := \tilde{\theta}(t) z, \quad \tilde{r} := \tilde{\theta}(t) r, \quad \tilde{\theta}(t) := s^{1-\epsilon'} e^{-\sigma\alpha} (\xi^*)^{10/11-\epsilon'}.
$$

From (3.9), we see that $(\tilde{z}, \tilde{r})$ is the solution of the Stokes system:

$$
\begin{align*}
-\tilde{z} - \nabla \cdot (D\tilde{z}) + \nabla \tilde{r} &= -\tilde{\theta}'(t) z - \tilde{\theta} r \varphi & \text{in} & \ Q, \\
\nabla \cdot \tilde{z} &= 0 & \text{in} & \ Q, \\
\tilde{z} \cdot n &= 0, \ (\sigma(\tilde{z}, \tilde{r}) \cdot n)_{tg} + (\mathcal{A}(x, t) \tilde{z})_{tg} &= 0 & \text{on} & \ \Sigma, \\
\tilde{z}(\cdot, T) &= 0 & \text{in} & \ \Omega.
\end{align*}
$$

For this system, we have

$$
\begin{align*}
\|\tilde{z}\|^2_{L^2(0, T; H^{3/2-\epsilon'}(\Omega)^2)} &\leq C \left( \|s^{2-\epsilon'} e^{-\sigma\alpha} (\xi^*)^{2-\epsilon'} \tilde{z}\|^2_{L^2(0, T; (H^\epsilon)'(\Omega)^2)} + \|s^2 e^{-\sigma\alpha} (\xi^*)^2 \rho \varphi\|^2_{L^2(Q)} \right) \\
&\leq C \left( \|s^{2-\epsilon'} e^{-\sigma\alpha} (\xi^*)^{2-\epsilon'} \tilde{z}\|^2_{L^2(0, T; (H^\epsilon)'(\Omega)^2)} + \|s^2 e^{-\sigma\alpha} (\xi^*)^2 w\|^2_{L^2(Q)} \right).
\end{align*}
$$

Observe that this inequality comes from Lemma 2.1 with a right-hand side in the interpolation space

$$
(L^2(0, T; W'), L^2(Q))_{1/2+\epsilon', 2} = L^2(0, T; (H^\epsilon)')).
$$

The fact that $s^{3/2} e^{-\sigma\alpha} (\xi^*)^{3/2}$ is bounded allows us to use (3.11) and conclude that $\|\tilde{z}\|^2_{L^2(0, T; H^{3/2-\epsilon'}(\Omega)^2)}$ is bounded by the left-hand side of (3.20) and $\|\rho g\|^2_{L^2(\Omega)^2}$. Using that

$$
L^2(\Omega)^2 = ((H^\epsilon)'), H^{3/2-\epsilon'}(\Omega)^2)_{3/4-\epsilon'/2, 2},
$$

we deduce that $s^{7/2-3\epsilon'} |e^{-\sigma\alpha} (\xi^*)^{7/4-3\epsilon'/2} z|^2_{L^2(Q)}$ is bounded by the left-hand side of (3.20) and $\|\rho g\|^2_{L^2(\Omega)^2}$. Taking $\epsilon' > 0$ small enough, we deduce in particular that

$$
s^3 \int_Q e^{-2s\alpha} (\xi^*)^3 |z|^2 dx \, dt
$$

is bounded by the left-hand side of (3.20) and $\|\rho g\|^2_{L^2(\Omega)^2}$.
Next, we define
\[ z^* := \theta^*(t)z, \quad r^* := \theta^*(t)r, \quad \theta^*(t) := s^{1/2}e^{-s\alpha^*}(\xi^*)^{9/22}. \]
From (3.9), we see that \((z^*, r^*)\) is the solution of (3.21) with \(\hat{\theta}\) replaced by \(\theta^*\). Using again (2.2) and taking into account (3.18), we deduce
\[
\begin{align*}
\|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)} & \leq C \left( s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2} \|z\|^2_{L^2(0,T;H^2(\Omega^2))} + s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}w\|\mathcal{L}_{\mathcal{Z}}(t)\|^2 \right).
\end{align*}
\] (3.23)
Arguing as before, we conclude that \(\|z^*\|^2_{L^2(0,T;H^2(\Omega)^2)}\) is bounded by the left-hand side of (3.20) and \(\|\rho g\|^2_{L^2(\Omega^2)}\).

Now, let \(\hat{z} := \hat{\theta}(t)z, \quad \hat{r} := \hat{\theta}(t)r, \quad \hat{\theta} := s^{-1/2}e^{-s\alpha^*}(\xi^*)^{-15/22}.\)
From (3.9), \((\hat{z}, \hat{r})\) is the solution of (3.21) with \(\hat{\theta}\) replaced by \(\hat{\theta}\). Observe that the right-hand side of this system can be considered in \(L^2(0, T; H^2(\Omega)^2) \cap H^1(0, T; L^2(\Omega)^2)\) and thus, using the regularity estimate (2.5) we have
\[
\begin{align*}
\|\hat{z}\|^2_{L^2(0,T;H^2(\Omega^2))} & \leq C \left( \|\hat{\theta}z\|^2_{L^2(0,T;H^2(\Omega^2))} + \|\hat{\rho}'\|_{L^2(0,T;H^2(\Omega^2))} \right) \\
& \leq C \left( \|\rho g\|^2_{L^2(\Omega)} + s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2} \|z\|^2_{L^2(0,T;H^2(\Omega^2))} + \|\theta^*z\|^2_{L^2(0,T;L^2(\Omega^2))} + s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}z\|\mathcal{L}_{\mathcal{Z}}(t)\|^2 \right).
\end{align*}
\] (3.24)
From (3.23), the right-hand side of (3.24) is bounded by
\[ \|s^{3/2}e^{-s\alpha^*}(\xi^*)^{3/2}z\|^2_{L^2(\Omega^2)} \quad \text{and} \quad \|\rho g\|^2_{L^2(\Omega^2)}. \]
Coming back to (3.20), we find in particular
\[
\begin{align*}
s^5 \int_Q e^{-2\alpha^*}\xi^5|z_1|^2dxdt + s^3 \int_Q e^{-2\alpha^*}(\xi^*)^3|z_2|^2dxdt + \|z^*\|^2_{L^2(0,T;H^2(\Omega^2))} & \\
+ \|\hat{z}\|^2_{L^2(0,T;H^1(\Omega^2))} + \|\hat{r}\|^2_{L^2(0,T;H^2(\Omega))} & \leq C \left( \|\rho g\|^2_{L^2(\Omega)} + \int_{\Omega} e^{-2\alpha^*}\xi^5 \left| \frac{\partial \nabla z_1}{\partial n} \right|^2 d\sigma + s^{3/2} \int_0^T e^{-2\alpha^*}(\xi^*)^{-3/2}\|\nabla \nabla r\|^2_{H^{1/2}(\partial \Omega)} dt \\
+ \int_0^T e^{-2\alpha^*}\xi^2 \left| \nabla \nabla r \right|^2 dxdt + \int_0^T e^{-2\alpha^*}(s^5\xi^5|z_1|^2 + s^3\xi^3|z_2|^2)dxdt \right).
\end{align*}
\] (3.25)
Observe that the boundary term
\[ s^{3/2} \int_0^T e^{-2\alpha^*}(\xi^*)^{-3/2}\|\nabla \nabla r\|^2_{H^{1/2}(\partial \Omega)} dt \]
can be absorbed by the fifth term of the left-hand side of (3.25).
In order to estimate the other boundary term, we notice that \(\alpha\) and \(\xi\) coincide with \(\alpha^*\) and \(\xi^*\) respectively on \(\Sigma\), so that
\[
\begin{align*}
s \int_{\Sigma} e^{-2\alpha^*}\xi^5 \left| \frac{\partial \nabla z_1}{\partial n} \right|^2 d\sigma dt & = s \int_{\Sigma} e^{-2\alpha^*}\xi^5 \left| \frac{\partial \nabla z_1}{\partial n} \right|^2 d\sigma dt \leq Cs \int_0^T e^{-2\alpha^*}\xi^* \left| z_1 \right|^2_{H^{5/2}+\epsilon(\Omega)} dt
\end{align*}
\] (3.26)
Thus, at this point we have
\[ s^{43/35} \int_0^T e^{-2sα^∗}\xi^* |z_1|^2 d\theta^{\alpha/s(\Omega)} dt \]
\[ \leq C \left( s^5 \int_0^T e^{-2sα^∗}(\xi^*)^5 \|z_1\|^2_{L^2(\Omega)} dt + s^{-1} \int_0^T e^{-2sα^∗}(\xi^*)^{-15/11} \|z_1\|^2_{H^\alpha(\Omega)} dt \right), \]
for every \( s \geq C \). Coming back to (3.26) and using the above inequality, the boundary term
\[ s \int_\Sigma e^{-2sα^∗} \xi \frac{\partial \nabla z_1}{\partial n} dx dt \]
can be absorbed by the left-hand side of (3.25). This ends Step 2.

Thus, at this point we have
\[ s^5 \int_Q e^{-2sα^∗\xi^5} |z_1|^2 dx dt + s^3 \int_Q e^{-2sα^∗(\xi^*)^3} |z_2|^2 dx dt \]
\[ + \|\theta z\|^2_{L^2(0,T;H^\alpha(\Omega))} + \|\theta\|_{L^2(0,T;H^\alpha(\Omega))} \]
\[ \leq C \left( \|ρg\|^2_{L^2(Q)^2} + \int_0^T e^{-2sα^∗\xi^2} |\nabla \theta_1 r|^2 dx dt + \int_0^T e^{-2sα^∗(s^5\xi^3) \|z_1\|^2 + s^3\xi^3 |\nabla z_1|^2} dx dt \right) \]
for every \( s \geq C \).

**Step 3.** In this step, we estimate the local term on \( \nabla \theta_1 r \) in the right-hand side of (3.27). The other two local terms can be estimated in an easier way.

Let \( \omega_j \) be an open subset satisfying \( \omega_2 \subset \omega_3 \subset \omega' \) and let \( \rho_1 \in C^2(\omega_3) \) with \( \rho_1 \equiv 1 \) in \( \omega_2 \) and \( \rho_1 \geq 0 \). Then, integrating by parts and using that \( \Delta r = 0 \) we get
\[ \int_0^T \int_{\omega_2} e^{-2sα^∗\xi^2} |\nabla \theta_1 r|^2 dx dt \leq C \int_0^T \int_{\omega_3} \Delta(\rho_1 e^{-2sα^∗\xi^2}) |\theta_1 r|^2 dx dt. \]

From (3.9) and the estimate
\[ |\Delta(\rho_1 e^{-2sα^∗\xi^2})| \leq C s^2 e^{-2sα^∗\xi^4} 1_{\omega_3}, \quad s \geq C, \]
we obtain
\[ \int_0^T \int_{\omega_2} e^{-2sα^∗\xi^2} |\nabla \theta_1 r|^2 dx dt \leq C s^2 \int_0^T \int_{\omega_3} e^{-2sα^∗\xi^4} (|z_{1,t}|^2 + |\Delta z_1|^2 + |\rho'|^2) dx dt \quad (3.28) \]
for every \( s \geq C \). We will now estimate the two first terms in the last integral of (3.28), the third one being estimated in an easier way.

i) Estimate of \( z_{1,t} \). We integrate by parts with respect to \( t \):
\[ s^2 \int_0^T \int_{\omega_3} e^{-2sα^∗\xi^4} |z_{1,t}|^2 dx dt = \frac{s^2}{2} \int_0^T \int_{\omega_3} \theta_t (e^{-2sα^∗\xi^4}) |z_1|^2 dx dt - s^2 \int_0^T \int_{\omega_3} e^{-2sα^∗\xi^4} z_{1,tt} z_1 dx dt \]
\[ \leq C \left( s^4 \int_0^T \int_{\omega_3} e^{-2sα^∗(\xi)^{68/11}} |z_1|^2 dx dt + s^2 \int_0^T \int_{\omega_3} \theta_{1,tt} e^{-2sα^∗\xi^4} |z_1| dx dt \right), \]
where we recall that $\hat{\theta} := s^{-1/2}e^{-s\alpha^*}(\xi^*)^{-15/22}$.

Using Young’s inequality for the second term we obtain for every $\varepsilon > 0$

$$s^2 \int_0^T \int_{\omega_3} e^{-2s\alpha \xi^4} \left| \Delta z_1 \right|^2 \, dx \, dt \leq C \left( s^4 \int_0^T \int_{\omega_3} e^{-2s\alpha \xi^7} \left| \nabla z_1 \right|^2 \, dx \, dt + \varepsilon \int_0^T \int_{\omega_3} \left| \xi^2 \right| \left| z_1 \right|^2 \, dx \, dt + C(\varepsilon) s^5 \int_0^T \int_{\omega_3} e^{-4s\alpha + 2s\alpha^* \xi^{10}} \left| z_1 \right|^2 \, dx \, dt \right).$$

(3.29)

The second term in the right-hand side of the above inequality can be absorbed by the left-hand side of (3.27).

ii) Estimate of $\Delta z_1$. Let $w_4$ be an open subset such that $w_3 \subset w_4$ and $\rho_2 \in C^2_\omega(w_4)$ with $\rho_2 \equiv 1$ in $\omega_3$ and $\rho_2 \geq 0$. Then, an integration by parts gives

$$s^2 \int_0^T \int_{\omega_3} e^{-2s\alpha \xi^4} \left| \Delta z_1 \right|^2 \, dx \, dt \leq s^2 \int_0^T \int_{\omega_4} \rho_2^2 e^{-2s\alpha \xi^4} \left| \Delta z_1 \right|^2 \, dx \, dt$$

$$= -s^2 \int_0^T \int_{\omega_4} \nabla(\rho_2^2 e^{-2s\alpha \xi^4}) \cdot \nabla z_1 \Delta z_1 \, dx \, dt - s^2 \int_0^T \int_{\omega_4} \rho_2^2 e^{-2s\alpha \xi^4} \nabla \Delta z_1 \cdot \nabla z_1 \, dx \, dt.$$ 

Using the estimate $|\nabla(\rho_2^2 e^{-2s\alpha \xi^4})| \leq C s e^{-2s\alpha \xi^5} \rho_2$, $s \geq C$, and again Young’s inequality for the first term, we obtain

$$s^2 \int_0^T \int_{\omega_3} e^{-2s\alpha \xi^4} \left| \Delta z_1 \right|^2 \, dx \, dt$$

$$\leq C \left( s^4 \int_0^T \int_{\omega_4} e^{-2s\alpha \xi^6} \left| \nabla z_1 \right|^2 \, dx \, dt - s^2 \int_0^T \int_{\omega_4} \rho_2^2 e^{-2s\alpha \xi^4} \nabla \Delta z_1 \cdot \nabla z_1 \, dx \, dt \right)$$

(3.30)

for every $s \geq C$.

Now, to estimate $I_1$ we consider $w_5$ an open subset such that $w_4 \subset w_5 \subset \omega'$ and $\rho_3 \in C^2_\omega(w_5)$ with $\rho_3 \equiv 1$ in $\omega_4$ and $\rho_3 \geq 0$. Then

$$I_1 \leq s^4 \int_0^T \int_{\omega_5} \rho_3 e^{-2s\alpha \xi^6} \left| \nabla z_1 \right|^2 \, dx \, dt$$

$$\leq C \left( s^6 \int_0^T \int_{\omega_5} e^{-2s\alpha \xi^8} \left| z_1 \right|^2 \, dx \, dt + s^4 \int_0^T \int_{\omega_5} \rho_3 e^{-2s\alpha \xi^6} \left| \Delta z_1 \right| \left| z_1 \right| \, dx \, dt \right)$$

$$= C \left( s^6 \int_0^T \int_{\omega_5} e^{-2s\alpha \xi^8} \left| z_1 \right|^2 \, dx \, dt + s^4 \int_0^T \int_{\omega_5} \rho_3 \theta^s \left| \Delta z_1 \right| e^{-2s\alpha \theta^s} (\xi^*)^{-1} \xi^6 \left| z_1 \right| \, dx \, dt \right),$$

for every $s \geq C$. We recall that $\theta^* := s^{1/2} e^{-s\alpha^*}(\xi^*)^{9/22}$. 

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Finally, using the definition of the weight functions and (3.11), we readily obtain
\[ I_1 \leq \left( s^6 \int_0^T e^{-2\alpha \xi} |z_1|^2 dxdt + \varepsilon \int_0^T \| \theta \Delta z_1 \|^2 dxdt + C(\varepsilon)s^7 \int_0^T e^{-4\alpha^* + 2\alpha^*\xi^2} |z_1|^2 dxdt \right), \]  
(3.31)
for every $s \geq C$. The second term in the right-hand side of the above inequality can be absorbed by the left-hand side of (3.27).

Now we estimate $I_2$. An integration by parts gives
\[ I_2 \leq C(s^3 \int_0^T e^{-2\alpha \xi} \xi_1 |\Delta z_1| |z_1| dxdt + s^3 \int_0^T e^{-2\alpha \xi} \Delta^2 z_1 |z_1| dxdt). \]

Using again the Young’s inequality, we obtain by an analogous argument the estimate:
\[ I_2 \leq C(\varepsilon \| \theta z_1 \|^2_{L^2(0,T;H^4(\omega))}) + C(\varepsilon)s^5 \int_0^T e^{-4\alpha^* + 2\alpha^*\xi^2} |z_1|^2 dxdt), \]  
(3.32)
for every $\varepsilon > 0$ and $s \geq C$. The first term in the right-hand side of (3.32) can be absorbed by the left-hand side of (3.27).

Finally, using the definition of the weight functions and (3.11), we readily obtain
\[ s^7 \int_0^T e^{-4\alpha^* + 2\alpha^* \xi^2} |z_1|^2 dxdt \]
\[ \leq 2s^7 \int_0^T e^{-4\alpha^* + 2\alpha^* \xi^2} |\hat{\xi}|^{12} |\rho|^2 |\varphi|^2 dxdt + 2s^7 \int_0^T e^{-4\alpha^* + 2\alpha^* \xi^2} |\hat{\xi}|^{12} |\varphi|^2 dxdt \]
\[ \leq 2s^7 \int_0^T e^{-4\alpha^* + 2\alpha^* \xi^2} |\hat{\xi}|^{12} |\rho|^2 |\varphi|^2 dxdt + C\| \rho g \|^2_{L^2(Q)}^2. \]

From (3.27) and (3.28)-(3.32), we conclude the proof of Proposition 3.1.

### 4 Null controllability of the linear system

Here we are concerned with the null controllability of the following system:
\[ \begin{cases} y_t - \nabla \cdot (Dy) + \nabla p = h + v \chi_\omega & \text{in } Q, \\
\nabla \cdot y = 0 & \text{in } Q, \\
y \cdot n = 0, (\sigma(y,p) \cdot n)_{tg} + (A(x,t)y)_{tg} = 0 & \text{on } \Sigma, \\
y(\cdot,0) = y_0(\cdot) & \text{in } \Omega, \end{cases} \]
(4.1)

where $y_0 \in W$, $h$ is in an appropriate weighted space. We look for a control $v \in L^2(0,T;H^2(\omega)^N) \cap H^1(0,T;L^2(\omega)^N)$ such that $v_i \equiv 0$ for some $i \in \{1, \ldots, N\}$.

To do this, let us first state a Carleman inequality with weight functions not vanishing in $t = 0$. Let $\ell \in C^2([0,T])$ be a positive function in $[0,T)$ such that $\ell(t) > t(T-t)$ for all $t \in [0,T/4]$ and $\ell(t) = t(T-t)$ for all $t \in [T/2,T]$.

Now, we introduce the following weight functions:
\[ \beta(x,t) = \frac{2\lambda \| \eta \|_\infty}{\ell^1(t)}, \quad \beta^*(t) = \max_{x \in \Omega} \beta(x,t), \]
\[ \beta(t) = \min_{x \in \Omega} \beta(x,t), \quad \beta(t) = \min_{x \in \Omega} \beta(x,t), \]
\[ \hat{\gamma}(t) = \max_{x \in \Omega} \gamma(x,t), \]  
(4.2)

\[ \gamma(x,t) = \frac{e^{\lambda_\eta(x)}}{\ell^1(t)}; \]
\[ \hat{\gamma}(t) = \max_{x \in \Omega} \gamma(x,t). \]
Lemma 4.1. Let \( i \in \{1, \ldots, N\} \) and let \( s \) and \( \lambda \) be like in Proposition 3.1. Then, there exists a constant \( C > 0 \) (depending on \( s \) and \( \lambda \) and increasing on \( \|A\|_{\rho_1} \)) such that every solution \( \varphi \) of (3.2) satisfies:

\[
\|\varphi(\cdot, 0)\|_{L^2(\Omega)^N} + \int_Q e^{-6s\beta^*} (\gamma^*)^3 |\varphi|^2 \, dx \, dt \\
\leq C \left( \int_Q e^{-4s\beta^*} |g|^2 \, dx \, dt + \sum_{j=1, j \neq i}^{N} \int_0^T \int_\omega e^{-4s\beta - 2s\beta^*} (\gamma)^{12} |\varphi_j|^2 \, dx \, dt \right). \tag{4.3}
\]

**Proof.** We start by an a priori estimate for the Stokes system (3.2). To do this, we introduce a function \( \nu \) in \( C^3([0, T]) \) such that

\[
\nu \equiv 1 \quad \text{in } [0, T/2], \quad \nu \equiv 0 \quad \text{in } [3T/4, T].
\]

We easily see that \( (\nu \varphi, \nu \pi) \) satisfies:

\[
\begin{cases}
-(\nu \varphi)_t - \nabla \cdot (D\nu \varphi) + \nabla (\nu \pi) = \nu g - \nu' \varphi & \text{in } Q, \\
\nabla \cdot (\nu \varphi) = 0 & \text{in } Q, \\
(\nu \varphi) \cdot n = 0, (\sigma(\nu \varphi, \nu \pi) \cdot n)_{tg} + (A^t(x, t)\nu \varphi)_{tg} = 0 & \text{on } \Sigma, \\
(\nu \varphi)(T) = 0 & \text{in } \Omega. \tag{4.4}
\end{cases}
\]

Using (2.2) we have in particular

\[
\|\varphi\|_{L^2(0,T/2;L^2(\Omega)^N)} + \|\varphi(\cdot,0)\|_{L^2(\Omega)^N} \\
\leq Ce^{CT\|A\|_{\rho_0}^2} \left( 1 + \|A\|_{\rho_0}^2 \right) \left( \|g\|_{L^2(0,3T/4;L^2(\Omega)^N)} + \|\varphi\|_{L^2(T/2,3T/4;L^2(\Omega)^N)} \right).
\]

Taking into account that

\[
e^{-4s\beta^*} \geq C > 0 \quad \forall t \in [0, 3T/4] \quad \text{and} \quad e^{-6s\beta^*} (\gamma^*)^3 \geq C > 0, \quad \forall t \in [T/2, 3T/4],
\]

we have

\[
\begin{align*}
\int_0^{T/2} \int_\Omega e^{-6s\beta^*} (\gamma^*)^3 |\varphi|^2 \, dx \, dt + \|\varphi(\cdot,0)\|^2_{L^2(\Omega)^N} \\
\leq Ce^{CT\|A\|_{\rho_0}^2} \left( 1 + \|A\|_{\rho_0}^2 \right) \left( \int_0^{3T/4} \int_\Omega e^{-4s\beta^*} |g|^2 \, dx \, dt + \int_{T/2}^{3T/4} \int_\Omega e^{-6s\beta^*} (\gamma^*)^3 |\varphi|^2 \, dx \, dt \right). \tag{4.5}
\end{align*}
\]

Note that, since \( \alpha = \beta \) in \( \Omega \times (T/2, T) \), we have:

\[
\int_{T/2}^{T} \int_{\Omega} e^{-6s\beta^*} (\gamma^*)^3 |\varphi|^2 \, dx \, dt = \int_{T/2}^{T} \int_{\Omega} e^{-6s\alpha^*} (\xi^*)^3 |\varphi|^2 \, dx \, dt
\]

and by virtue of Carleman inequality (3.3) (see Proposition 3.1), we obtain

\[
\int_{T/2}^{T} \int_{\Omega} e^{-6s\beta^*} (\gamma^*)^3 |\varphi|^2 \, dx \, dt \leq C \left( \int_Q e^{-4s\alpha^*} |g|^2 \, dx \, dt + \sum_{j=1, j \neq i}^{N} \int_0^T \int_\omega e^{-4s\alpha - 2s\beta^*} (\hat{\xi})^{12} |\varphi_j|^2 \, dx \, dt \right).
\]

Since \( \ell(t) = t(T-t) \) for any \( t \in [T/2, T] \) and

\[
e^{-4s\beta^*} \geq C \quad \text{and} \quad e^{-4s\beta^* - 2s\beta^*} (\hat{\gamma})^{12} \geq C \quad \text{in } [0, T/2],
\]
we readily get
\[
\int_{T/2}^{T} \int_{\Omega} e^{-6s\beta^* (\gamma^*)^3} |\varphi|^2 \, dx \, dt \leq C \left( \int_{Q} e^{-4s\beta^* |g|^2} \, dx \, dt + \sum_{j=1}^{N} \int_{0}^{T} \int_{\omega} e^{-4s\beta^* - 2s\beta^* (\tilde{\gamma})^2} |\chi_{\omega} \varphi|^2 \, dx \, dt \right).
\]
(4.6)

From (4.5) and (4.6) we obtain (4.3).

**Remark 4.1.** Observe that on the left-hand side of (4.3) it is possible to put the terms
\[
\|e^{-3s\beta^* (\gamma^*)^{11/22}} \varphi\|_{L^2(0,T;H^2(\Omega) \cap W)}^2 + \int_{Q} e^{-6s\beta^* (\gamma^*)^{9/11}} |\varphi|^2 \, dx \, dt.
\]
(4.7)

To this end, we consider \( \tilde{\gamma} := e^{-3s\beta^* (\gamma^*)^{9/22}} \) and \((\tilde{\theta}, \tilde{\varphi}, \tilde{\pi})\) the solution of (4.4) with \( \tilde{\theta} \) instead of \( \nu \). Next, taking into account that \( |\partial_t \beta^*| \leq C(\gamma^*)^{12/11} \), \( |\theta| \leq C e^{-3s\beta^* (\gamma^*)^{3/2}} \) and the regularity estimate (2.2), we obtain (4.7).

Now we are ready to prove the null controllability of system (4.1). The idea is to look for a solution in an appropriate weighted functional space. Let us set
\[
Ly = y_t - \nabla \cdot Dy
\]
and let us introduce the space, for \( N = 2 \) or \( N = 3 \) and \( i \in \{1, \ldots, N\} \),
\[
E^i_N := \{(y,p,v) : e^{2s\beta^*} y, e^{2s\beta^* + s\beta^* (\tilde{\gamma})}^{-6} v, \tilde{\rho} \partial_t v \in L^2(Q)^N, \tilde{p} v \in L^2(0,T;H^2(\Omega))^N, v_i \equiv 0, \text{ supp } v \subset \omega \times (0,T), e^{2s\beta^* (\gamma^*)^{-12/11}} y \in Y_1, e^{3s\beta^* (\gamma^*)^{-3/2}} (Ly + \nabla p - v \chi_{\omega}) \in L^2(Q)^N \},
\]
where
\[
\rho := e^{-4s\beta^* - 2s\beta^* (\tilde{\gamma})^2} \quad \text{and} \quad \tilde{\rho} := \rho^{-1} \tilde{\theta}.
\]
It is clear that \( E^i_N \) is a Banach space for the following norm:
\[
\|(y,p,v)\|_{E^i_N} = \left( \|e^{2s\beta^*} y\|_{L^2(Q)^N}^2 + \|e^{2s\beta^* + s\beta^* (\tilde{\gamma})}^{-6} v\|_{L^2(Q)^N}^2 + \|\tilde{\rho} \partial_t v\|_{L^2(Q)}^2 \right. \\
+ \|\tilde{p} v\|_{L^2(0,T;H^2(\Omega)^N)}^2 + \|e^{2s\beta^* (\gamma^*)^{-12/11}} y\|_{Y_1}^2 \\
+ \left. \|e^{3s\beta^* (\gamma^*)^{-3/2}} (Ly + \nabla p - v \chi_{\omega})\|_{L^2(Q)^N}^2 \right)^{1/2}.
\]

**Remark 4.2.** Observe in particular that \((y,p,v) \in E^i_N \) implies \( y(\cdot, T) = 0 \) in \( \Omega \).

**Proposition 4.1.** Assume the hypothesis of Lemma 4.1 and
\[
y_0 \in W \quad \text{and} \quad e^{3s\beta^* (\gamma^*)^{-3/2}} h \in L^2(Q)^N.
\]
(4.8)

Then, we can find a control \( v \) such that the associated solution \((y,p,v) \) to (4.1) satisfies \((y,p,v) \in E^i_N \). In particular, \( v_i \equiv 0 \) and \( y(\cdot, T) = 0 \) in \( \Omega \). Furthermore, there exists \( C > 0 \) increasing with respect to \( \|A\|_{P_1} \) such that
\[
\|v\|_{L^2(0,T;H^2(\omega)^N)} + \|v\|_{H^1(0,T;L^2(\omega)^N)} \leq C \left( \|y_0\|_{H^2(\omega)^N \cap W} + \|h\|_{L^2(Q)^N} \right).
\]
(4.9)

**Proof.** Following the arguments in [8], we introduce the space \( P_0 \) of functions \((\varphi, \pi) \in C^2(Q)^{N+1}\) such that
\( (i) \quad \nabla \cdot \varphi = 0 \quad \text{in} \ Q \),
\( (ii) \quad (\sigma(\varphi, \pi) \cdot n)_{tg} + (A^i(x,t)\varphi)_{tg} = 0 \quad \text{on} \ \Sigma \),
\( (iii) \quad \varphi \cdot n = 0 \quad \text{on} \ \Sigma \).
Also we define the bilinear form
\[
a((\hat{\varphi}, \hat{\pi}), (w, q)) := \iint_{Q} e^{-4s\beta} (L^* \hat{\varphi} + \nabla \hat{\pi})(L^* w + \nabla q) dxdt
+ \sum_{j=1,j\neq i}^{N} \int_{0}^{T} \int_{\omega} e^{-4s\beta - 2s\beta^*} (\hat{\gamma})^{12} \chi_{\omega} \hat{\varphi}_{j} \chi_{\omega} w_{j} dxdt,
\]
for every \((w, q) \in P_0\), and a linear form
\[
\langle G, (w, q) \rangle := \iint_{Q} h \cdot w dxdt + \int_{\Omega} y_0(\cdot) \cdot w(\cdot, 0) dx,
\]
(4.10)
where \(L^*\) is the adjoint operator of \(L\), i.e.,
\[
L^* w = -w_t - \nabla \cdot Dw.
\]
Observe that Carleman inequality (4.3) holds for all \((w, q) \in P_0\). Consequently,
\[
\iint_{Q} e^{-6s\beta} (\gamma^*)^{3} |w|^2 dxdt \leq Ca((w, q), (w, q)), \quad \forall (w, q) \in P_0.
\]
Therefore, \(a(\cdot, \cdot) : P_0 \times P_0 \to \mathbb{R}\) is a symmetric, definite positive bilinear form on \(P_0\). We denote by \(P\) the completion of \(P_0\) for the norm induced by \(a(\cdot, \cdot)\). Then, \(a(\cdot, \cdot)\) is well-defined, continuous and again definite positive on \(P\). Furthermore, in view of the Carleman inequality (4.3) and the assumption (4.8), the linear form \((w, q) \mapsto \langle G, (w, q) \rangle\) is well-defined and continuous on \(P\). Hence, from Lax-Milgram’s Lemma, there exists one and only one \((\hat{\varphi}, \hat{\pi}) \in P\) satisfying:
\[
a((\hat{\varphi}, \hat{\pi}), (w, q)) = \langle G, (w, q) \rangle, \quad \forall (w, q) \in P.
\]
(4.11)
Let us set
\[
\begin{align*}
\hat{y} &= e^{-4s\beta} (L^* \hat{\varphi} + \nabla \hat{\pi}) \quad \text{in} \ Q, \\
\hat{v}_j &= -e^{-4s\beta - 2s\beta^*} (\hat{\gamma})^{12} \hat{\varphi}_{j} \chi_{\omega}, \quad j \neq i, \quad \hat{v}_i \equiv 0 \quad \text{in} \ \omega \times (0, T).
\end{align*}
\]
(4.12)
Let us remark that \((\hat{y}, \hat{v})\) verifies
\[
a((\hat{\varphi}, \hat{\pi}), (\hat{\varphi}, \hat{\pi})) = \iint_{Q} e^{-4s\beta} (L^* \hat{\varphi} + \nabla \hat{\pi})^2 dxdt + \sum_{j=1,j\neq i}^{N} \int_{0}^{T} \int_{\omega} e^{-4s\beta - 2s\beta^*} (\hat{\gamma})^{12} |\chi_{\omega} \hat{\varphi}_{j}|^2 dxdt

= \iint_{Q} e^{4s\beta} |\hat{y}|^2 dxdt + \sum_{j=1,j\neq i}^{N} \int_{0}^{T} \int_{\omega} e^{4s\beta + 2s\beta^*} (\hat{\gamma})^{-12} |\hat{v}_j|^2 dxdt < +\infty.
\]
Let us prove that \(\hat{y}\) is, together with some pressure \(\hat{p}\), the weak solution of the Stokes system in (4.1) for \(v = \hat{v}\). In fact, we introduce the (weak) solution \((\hat{y}, \hat{p})\) to the Stokes system:
\[
\begin{align*}
L \hat{y} + \nabla \hat{p} &= h + \hat{v} \chi_{\omega} \quad \text{in} \ Q, \\
\nabla \cdot \hat{y} &= 0 \quad \text{in} \ Q, \\
\hat{y} \cdot n &= 0, \quad (\sigma(\hat{y}, \hat{p}) \cdot n)_{tg} + (A(x, t) \hat{y})_{tg} = 0 \quad \text{on} \ \Sigma, \\
\hat{y}(\cdot, 0) &= y_0(\cdot) \quad \text{in} \ \Omega.
\end{align*}
\]
(4.13)
Clearly, \(\hat{y}\) is the unique solution of (4.13) defined by transposition. This means that \(\hat{y}\) is the unique function in \(L^2(Q)^N\) satisfying
\[
\iint_{Q} \hat{y} \cdot g dxdt = \int_{\Omega} y_0(\cdot) \cdot w(\cdot, 0) dx + \iint_{Q} h \cdot w dxdt + \iint_{Q} \hat{v} \cdot w \chi_{\omega} dxdt, \quad \forall g \in L^2(Q)^N, \quad (4.14)
\]
Finally, from the continuity of $G$ (see (4.10)) and (4.11), we deduce (4.9). This ends the proof of Proposition 4.1.
5 Proof of the main result

In this section we give the proof of Theorem 1.1 using classical arguments. The first step is to apply Kakutani’s fixed point theorem on the boundary. Finally, we will deal with the nonlinear term in the Navier-Stokes equations through an inverse mapping theorem to conclude the proof of Theorem 1.1.

5.1 Nonlinearity on the boundary conditions.

In this section we present the local null controllability for the following system:

\begin{align*}
\begin{cases}
y_t - \nabla \cdot (Dy) + \nabla p = h + v x_\omega & \text{in } Q, \\
\nabla \cdot y = 0 & \text{in } Q, \\
y \cdot n = 0, (\sigma(y, p) \cdot n)_tg + (f(y))_{tg} = 0 & \text{on } \Sigma, \\
y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega.
\end{cases}
\end{align*}

(5.1)

**Theorem 5.1.** Let us assume that \( f \in C^4(\mathbb{R}^N; \mathbb{R}^N) \) and \( f(0) = 0 \). Then, for every \( T > 0 \), \( \omega \subset \Omega \) and \( i \in \{1, \ldots, N\} \), there exists \( \delta > 0 \) such that, for every \( y_0 \in H^3(\Omega)^N \cap W \), \( h \in Y_1 \) satisfying

\[
|\varepsilon|^3 \beta^2 (\gamma^*)^{-3/2} h \in L^2(Q)^N,
\]

and the compatibility condition (1.2), we can find a control

\[
v \in L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N)
\]

and an associated solution \((y, p)\) of (5.1) satisfying \( y \in Y_2 \) and such that \((y, p, v) \in E_N^i\).

**Proof.** For every \( z \in Z_c \) (recall that \( Z_c \) was defined in (1.4)) we consider the following system:

\begin{align*}
\begin{cases}
y_t - \nabla \cdot (Dy) + \nabla p = h + v x_\omega & \text{in } Q, \\
\nabla \cdot y = 0 & \text{in } Q, \\
y \cdot n = 0, (\sigma(y, p) \cdot n)_tg + (f(y))_{tg} = 0 & \text{on } \Sigma, \\
y(\cdot, 0) = y_0(\cdot) & \text{in } \Omega,
\end{cases}
\end{align*}

(5.3)

where

\[
g(z) := \frac{1}{N} \int_0^1 \nabla f(\tau z) d\tau.
\]

On the other hand, observe that since \( f \in C^4(\mathbb{R}^N; \mathbb{R}^N) \), each row and each column of \( g(z) \) belongs to \( Z_c \). Then, for every \( z \in Z_c \) we can use Proposition 4.1 with \( A = g(z) \) and deduce the existence of a control \( v_z \) belonging to \( L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N) \) such that the solution \((y_z, p_z, v_z)\) of (5.3) satisfies \((y_z, p_z, v_z) \in E_N^i\).

Moreover, from (4.9) we have

\[
\|v_z\|_{L^2(0, T; H^2(\omega)^N)} + \|v_z\|_{H^1(0, T; L^2(\omega)^N)} \leq C_1(\Omega, \omega, T, \|g(z)\|_{P_1^i \cap P_2^i}) \left(\|y_0\|_{H^3(\Omega)^N \cap W} + \|h\|_{L^2(Q)^N}\right),
\]

(5.4)

where \( C_1 \) is increasing with respect to \( \|g(z)\|_{P_1^i \cap P_2^i} \).

Next, taking into account that \( v_z, h \in Y_1 \) and the compatibility condition (2.3) with \( u_0 \) replaced by \( y_0 \), \( A(\cdot, 0) \) replaced by \( g(y_0(\cdot)) \) and \( f_2(\cdot, 0) \) replaced by \( 0 \) (see (1.2)), we can apply Theorem 2.1 to system (5.3). Combining this with (5.4), we can obtain that \( y_z \in Y_2 \) and

\[
\|y_z\|_{Y_2} \leq C_2(\Omega, \omega, T, \|g(z)\|_{P_1^i \cap P_2^i}) \left(\|y_0\|_{H^3(\Omega)^N \cap W} + \|h\|_{Y_1}\right),
\]

(5.5)

with \( C_2 \) increasing with respect to \( \|g(z)\|_{P_1^i \cap P_2^i} \) (see (2.6)).
Let \( C(z) \) be the set constituted by the controls \( v_z \in L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N) \) that satisfy (5.4) and drive the solution \( y_z \) of system (5.3) to zero at time \( T \). Then, let us introduce
\[
\Lambda(z) := \{ y_z \text{ solution of (5.3) : } v_z \in C(z) \}.
\]
Observe that, thanks to (5.5), \( \Lambda(z) \) is included in \( Y_2 \). Moreover, for any \( z \in Y_2 \) such that \( \|z\|_{Y_2} \leq 1 \), we have \( \|g(z)\|_{P_1 \cap P_2} \leq M \), where \( M > 0 \) is a constant only depending on \( \varepsilon, T \) and \( \Omega \). Consequently,
\[
\|y_z\|_{Y_2} \leq C_2(\Omega, \omega, T, M) \left( \|y_0\|_{H^1(\Omega)^N \cap W} + \|h\|_{Y_1} \right)
\]
(see (5.5)). Choosing now \( \delta := \frac{1}{C_2(\Omega, \omega, T, M)} \) in (5.2), we find \( \|y_z\|_{Y_2} \leq 1 \).

Now, we want to establish that the set-valued map \( \Lambda : K \rightarrow 2^K \) possesses a fixed-point, where
\[
K := \overline{B}_{Y_2}(0; 1) = \{ y \in Y_2 : \|y\|_{Y_2} \leq 1 \}.
\]
For this end, we will apply Kakutani’s fixed-point theorem (see for instance [2], Theorem 3.2.3, page 87):

i) \( \Lambda(z) \) is a nonempty closed convex set of \( L^2(Q)^N \), for every \( z \in K \).

ii) \( K \) is a nonempty convex compact set of \( L^2(Q)^N \).

iii) \( \Lambda \) is upper-hemicontinuous in \( L^2(Q)^N \), i.e., for any \( \lambda \in L^2(Q)^N \), the mapping
\[
z \mapsto \sup_{y \in \Lambda(z)} \langle \lambda, y \rangle_{L^2(Q)^N}
\]
is upper semicontinuous.

i) For every \( z \in K \), let \( \{y_z^k\} \subset C(z) \) such that \( y_z^k \rightarrow y_z \) in \( L^2(Q)^N \). From (5.4), we find (at least for a subsequence) that \( v_z^k \rightarrow v_z \) in \( L^2(Q)^N \). Let us denote \( w_z \) the solution of (5.3) associated to \( v := v_z \). Then, \( y_z^k - w_z \) satisfies (5.3) with \( h := 0 \), \( v := v_z^k - v_z \), and \( y_0 := 0 \). Thanks to (2.2), we have \( y_z^k \rightarrow w_z \) in \( L^2(Q)^N \) in particular and so \( y_z = w_z \). This shows that \( \Lambda(z) \) is closed. The convexity of \( \Lambda(z) \) is trivial.

ii) Since \( Y_2 \) is compactly embeded into \( L^2(Q)^N \), the second item holds true.

iii) Finally, let us prove the upper-hemicontinuity of \( \Lambda \). Assume \( z_k \rightarrow z \) in \( L^2(Q)^N \). In consequence from the compactness of \( \Lambda(z_k) \), we have
\[
\sup_{y \in \Lambda(z_k)} \langle \lambda, y \rangle_{L^2(Q)^N} = \langle \lambda, y_k \rangle_{L^2(Q)^N},
\]
for some \( y_k \in \Lambda(z_k) \). Then, we choose \( (z_k) \subset (z_k) \) such that
\[
\lim_{k' \rightarrow \infty} \sup_{y \in \Lambda(z_k')} \langle \lambda, y \rangle_{L^2(Q)^N} = \lim_{k' \rightarrow \infty} \langle \lambda, y_{k'} \rangle_{L^2(Q)^N}
\]
and denote \( v_{k'} \) the controls in \( C(z_{k'}) \) which are associated to \( y_{k'} \in \Lambda(z_{k'}) \). From (5.4), there exists \( v^* \in L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N) \) such that \( v_{k'} \rightarrow v^* \) in \( L^2(0, T; H^2(\omega)^N) \cap H^1(0, T; L^2(\omega)^N) \) and \( v^* \in \Lambda(z) \). In particular, \( v_{k'} \rightarrow v^* \) in \( L^2(Q)^N \) (for a subsequence).

Now, let \( (y^*, p^*) \) be the solution to (5.3) associated to \( v^* \). We set \( \tilde{y}_{k'} := y_{k'} - y^* \), \( \tilde{p}_{k'} := p_{k'} - p^* \) and \( \tilde{v}_{k'} := v_{k'} - v^* \). Then,
\[
\begin{cases}
(\tilde{y}_{k'})_t - \nabla \cdot (D\tilde{y}_{k'}) + \nabla \tilde{p}_{k'} = \tilde{v}_{k'} \chi, & \text{in } Q, \\
\nabla \cdot \tilde{y}_{k'} = 0, & \text{in } Q, \\
\tilde{y}_{k'} \cdot n = 0, & \text{on } \Sigma, \\
\tilde{y}_{k'}(\cdot, 0) = 0, & \text{in } \Omega.
\end{cases}
\]

Taking into account that \( g(z_{k'}) \to g(z) \) in \( Z_c \), one can prove that in particular
\[
\|g(z) - g(z_k')\|_{L^2(0,T;H^{1/2}(\partial \Omega)^N) \cap H^{1/4+\varepsilon}(0,T;H^{-\varepsilon}(\partial \Omega)^N)} \xrightarrow{k' \to \infty} 0.
\]
Then, from Lemma 2.1 we can deduce that \( y_{k'} \to y^* \) in \( Y_1 \). Additionally, \( y^* \in A(z) \) and therefore,
\[
\lim_{k' \to \infty} \sup_{y \in A(z_{k'})} \langle \lambda, y \rangle_{L^2(Q)^N} = \lim_{k' \to \infty} \langle \lambda, y_{k'} \rangle_{L^2(Q)^N} = \langle \lambda, y^* \rangle_{L^2(Q)^N} \leq \sup_{y \in A(z)} \langle \lambda, y \rangle.
\]
This concludes the proof of Theorem 5.1.

### 5.2 Nonlinearity in the main equation.

**Theorem 5.2.** Suppose that \( B_1, B_2 \) are Banach spaces and
\[
A : B_1 \to B_2
\]
is a continuously differentiable map. We assume that for \( b_1^0 \in B_1, b_2^0 \in B_2 \) the equality
\[
A(b_1^0) = b_2^0
\]
holds and \( A'(b_1^0) : B_1 \to B_2 \) is an epimorphism. Then there exists \( \delta > 0 \) such that for any \( b_2 \in B_2 \) which satisfies the condition
\[
\|b_2^0 - b_2\|_{B_2} < \delta
\]
there exists a solution \( b_1 \in B_1 \) of the equation
\[
A(b_1) = b_2.
\]

We apply this theorem for some given \( i \in \{1, \ldots, N\} \) and the spaces
\[
B_1 := \{(y, p, v) \in E^i_N : y \in Y_2\}
\]
and
\[
B_2 = [L^2(e^{3s\beta^*}(\gamma^*_{-3/2})(0,T);L^2(\Omega)^N) \cap Y_1] \times [H^3(\Omega)^N \cap W]
\]

We define the operator \( A \) by the formula
\[
A(y, p, v) = (Ly + (y \cdot \nabla)y + \nabla p - v\chi, y(\cdot, 0)).
\]

Let us see that \( A \) is of class \( C^1(B_1, B_2) \). Indeed, notice that all the terms in \( A \) are linear, except for \((y \cdot \nabla)y\). We prove now that the bilinear operator
\[
((y^1, p^1, v^1), (y^2, p^2, v^2)) \mapsto (y^1 \cdot \nabla)y^2
\]
is continuous from \( B_1 \times B_1 \) to \( L^2(e^{3s\beta^*}(\gamma^*_{-3/2})(0,T);L^2(\Omega)^N) \cap Y_1 \).

In fact, notice that (see the definition of the space \( E^i_N \))
\[
e^{2s\beta^*}(\gamma^*_{-12/11}y) \in L^2(0,T;L^\infty(\Omega)^N)
\]
and
\[
\nabla(e^{2s\beta^*}(\gamma^*_{-12/11}y) \in L^\infty(0,T;L^2(\Omega)^N \times L^\infty(\Omega)^N).
\]

Consequently, we obtain
\[
\|e^{3s\beta^*}(\gamma^*_{-3/2})(y^1 \cdot \nabla)y^2\|_{L^2(\Omega)^N}
\]
\[
\leq C\|e^{2s\beta^*}(\gamma^*_{-12/11}y^1 \cdot \nabla)e^{2s\beta^*}(\gamma^*_{-12/11}y^2\|_{L^2(\Omega)^N}
\]
\[
\leq C\|e^{2s\beta^*}(\gamma^*_{-12/11}y^1\|_{L^2(0,T;L^\infty(\Omega)^N)}\|e^{2s\beta^*}(\gamma^*_{-12/11}y^2\|_{L^\infty(0,T;W)}.
\]
On the other hand,
\[ \|(y^1 \cdot \nabla)y^2\|_{Y_1} \leq C\|y^1\|_{Y_2}\|y^2\|_{Y_2}. \]

Notice that \( A'(0, 0, 0) : B_1 \to B_2 \) is given by
\[ A'(0, 0, 0)(y, p, v) = (Ly + \nabla p - v \chi_\omega, y(\cdot, 0)), \quad \text{for all } (y, p, v) \in B_1. \]

In virtue of Theorem 5.1, this functional satisfies \( \text{Im}(A'(0, 0, 0)) = B_2. \)

Let \( b_1^0 = (0, 0, 0) \) and \( b_2^0 = (0, 0) \). Then equation (5.6) obviously holds. So all necessary conditions to apply Theorem 5.2 are fulfilled. Therefore there exists a positive number \( \delta \) such that, if \( \|y(\cdot, 0)\|_{H^3(\Omega) \times \Gamma_0^N} \leq \delta \), we can find a control \( v \) satisfying \( v_i \equiv 0 \), for some given \( i \in \{1, \ldots, N\} \) and an associated solution \((y, p)\) to (1.1) satisfying \( y(\cdot, T) = 0 \) in \( \Omega \). This finishes the proof of Theorem 1.1.
References


