Perspectives of local anisotropic Delaunay mesh adaptation

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Perspectives . . . ?

• various meanings:
  ◦ a particular attitude toward or a way of regarding something; a point of view;
  ◦ true understanding of the relative importance of things.

• showing the relationships between objects:
  ◦ curvature estimation for surface meshing,
  ◦ Delaunay kernel for simplicial mesh generation,
  ◦ metric tensor for mesh adaptation,

• common discussion thread: anisotropy
General framework

- numerical simulations: PDE’s models, FEM schemes
- mesh adaptation: accuracy and reliability
- anisotropy: why and where...?
Metric: basic definitions

Definition 0.1 a metric $\mathcal{M}$ is a field of symmetric positive definite matrices
$\mathcal{M} = (a_{ij}(x)) \in \mathcal{M}_3(\mathbb{R})$, $a_{ii}(x) > 0$ and $\det(\mathcal{M}) > 0$

Proposition 0.1 such a matrix $\mathcal{M}$ can be decomposed as:

$$A = \mathcal{R} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathcal{R}^{-1}$$

$\mathcal{R} = (v_1 \ v_2 \ v_3)$ rotation matrix, $v_i$ and $\lambda_i$ unit eigenvectors (values) of $\mathcal{M}$.

Definition 0.2 the scalar product is defined as:

$$\forall u, v \in \mathbb{R}^d, \quad \langle u, v \rangle_{\mathcal{M}} = \langle u, \mathcal{M}v \rangle = t_u \mathcal{M} v \in \mathbb{R}.$$ 

and the Euclidean norm of $u$ is:

$$\|u\|_{\mathcal{M}} = \sqrt{\langle u, \mathcal{M}u \rangle}.$$
Metric: geometric interpretation

If $\mathcal{M}(x) = \mathcal{M}$, the Euclidean geometry can be used and

**Definition 0.3** a metric $\mathcal{M}$ can be represented by its unit ball

$\mathcal{E}_\mathcal{M} = \{x \in \mathbb{R}^d, \langle x, \mathcal{M} x \rangle = 1\}$

The geometric locus of all points $M$ equidistant to $P$ is an ellipsoid.
Metric: unit length

Let $\gamma$ be a curve in $\mathbb{R}^d$ with a normal parametrization $\gamma(t), t \in [0, 1]$.

**Definition 0.4** the length $|\gamma|$ of $\gamma$ is defined as:

$$|\gamma| = \int_0^1 \sqrt{\langle \gamma'(t), M(\gamma(t))\gamma'(t) \rangle} dt$$

and for a mesh edge $AB = A + tAB$:

$$\ell(AB) = \int_0^1 \sqrt{\langle AB, M(\gamma(t))AB \rangle} dt$$

**Definition 0.5** the mesh size $h_i$ in direction $v_i$ is defined as:

$$h_i = \frac{1}{\sqrt{\lambda_i}}, \quad \lambda_i = \frac{1}{h_i^2}.$$
And then ... problem arises!

- If $M(x) \neq M$, Riemannian geometry is involved
- straight edges are ‘replaced’ by geodesics
- simplifications are needed for sake of efficiency: linear edges

Euclid of Alexandria (325-265)  B. Riemann (1826-1866)
Metric intersection

**Definition 0.6** The metric \( \mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2 \) is represented by the ellipsoid:

\[
\mathcal{E}_\mathcal{M} = \sup_{\mathcal{M}_i} \mathcal{E}_{\mathcal{M}_i} = \sup_{\mathcal{M}_i} \left\{ M \mid \sqrt{\langle \vec{P}M, \vec{P}M \rangle_{\mathcal{M}_i}} = 1 \right\} \subset \mathcal{E}_{\mathcal{M}_1} \cap \mathcal{E}_{\mathcal{M}_2}.
\]

Intersection of metrics using the simultaneous reduction of quadratic forms.
Metric interpolation

Let $\gamma(t) = AB$ be a parametrized mesh edge and $M_A$ and $M_B \in M_3(\mathbb{R})$.

Definition 0.7 The metric $M(t)$ at $t$ on $\gamma(t)$ is:

$$M(t) = \left((1 - t)M_A^{-\frac{1}{2}} + tM_B^{-\frac{1}{2}}\right)^{-2}, \quad 0 \leq t \leq 1.$$
Metric: size variation

- the quality of the elements is related to the variation of $\mathcal{M}$
- the size variations must be bounded (i.e. truncated)

**Definition 0.8** isotropic case: $\mathcal{M}(x) = \frac{1}{h(x)}I_3$

\[ \ell(AB) = |AB| \int_0^1 \frac{1}{h(t)}dt, \quad h(t) = (1 - t)h(A) + th(B), \]

\[ \ell(AB) = \frac{|AB|}{h(B) - h(A)} \ln \frac{h(B)}{h(A)}, \]

size variation defined as:

\[ \nu(AB) = \frac{h(B) - h(A)}{|AB|} \]
Metric: mesh gradation

**Definition 0.9** \( \nu(AB) \) is bounded by a threshold value \( \alpha \):

- **isotropic case:**
  \[
  \begin{align*}
  h(A) &= \min(h(A), h(B) + \alpha |\overrightarrow{AB}|) \\
  h(B) &= \min(h(B), h(A) + \alpha |\overrightarrow{AB}|)
  \end{align*}
  \]

- **anisotropic case:**
  \[
  \begin{align*}
  M_A &= M_A \cap M_B \left(1 + \alpha \ell_A(AB)\right)^{-2} \\
  M_B &= M_B \cap M_A \left(1 + \alpha \ell_B(AB)\right)^{-2}
  \end{align*}
  \]

If \( \nu_{max} = \max \nu(AB) \) and if \( r_{max} \) is the maximal ratio of edge lengths,

\[
\nu_{max} \approx \ln(r_{max})
\]
Surface meshes ...

Control of the geometric approximation...
Discrete surfaces

- Surfaces
  - various surface definitions: explicit, parameterized, implicit, discrete, ...
  - several meshing approaches

- Discrete surfaces defined by:
  - discrete (sampled) data
  - piecewise linear approximation: a triangulation.

- Advantages of discrete data:
  - standard (self-contained) data format,
  - often the only available datum (CAD-free),
  - suitable representation for large deformations,
  - support of mesh optimisation algorithms.
Discrete surfaces

from point cloud to surface triangulation
Discrete surfaces

from MRI images to surface triangulation
Discrete surfaces

Delaunay triangulations of the Crater Lake.
Triangulation corresponding to a local deformation error of $\delta = 0.2$ m and
triangulation corresponding to 1% of the initial vertices
Surface meshes: the challenges

- Requirements differ from one application to another:
  - in numerical simulations:
    - emphasis laid on geometric accuracy + element shape quality control
    - compromise sought between number of DOF and quality of the geometric approximation of the surface
  - in computer graphics:
    - fast rendering of complex scene composed of polygonal objects
    - frame rate (hardware technology) dictates the number of primitives (budget)
  - in data archiving:
    - polygonal meshes more flexible than CAD files
    - size of model adjusted to comply with network bandwidth
Surface meshes: the solutions

- Algorithms tailored to the needs of applications:
  - polygonal simplification algorithms tuned to generate LOD of objects
  - mesh simplification + mesh adaptation
  - hierarchical models
  - Delaunay triangulations
  - etc.

- Warning:
  'the number of polygons we want always exceeds the budget polygon we can afford'
Surface meshes: motivations

- generation of an *optimal mesh*:
  - accurate approximation of the surface geometry
  - minimal number of regular (well-shaped) elements

- element shape and size quality and vertex reduction: two conflicting requirements

- preserve the surface geometry:
  - the approximation error must be equi-distributed over the mesh elements

- vertices are distributed evenly over the surface

- use an *a posteriori* geometric error estimate based on the Hessian of the surface to define a discrete anisotropic metric map

- metric used to control the mesh generation
Surface mesh: curvatures estimation

- surface curvature on differentiable manifolds: important invariants in differential geometry describes the local shape of a surface
- concept rooted in differential geometry, origins in the XVIIIth century:

  L. Euler (1707-1783)  
  C.F. Gauss (1777-1895)  
  A.M. Legendre (1752-1833)

- key role: registration, smoothing, simplification, reverse engineering, visualisation, etc.
- many estimation methods for estimating curvatures proposed: > 200 citations since 1992
Differential geometry: a primer

The curvature of a surface describes the local shape of that surface. Let consider a $C^\infty$ immersion of a surface $\Sigma$:

$$U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \quad (u, v) \longrightarrow M = M(u, v)$$

$U$ is an open set of $\mathbb{R}^2$ and $\Sigma$ is oriented by the normal $M_u \wedge M_v$

The metric induced by that of $\mathbb{R}^3$ (Euclidean) is represented in the map $(u, v)$ by the quadratic (1st fundamental) form of the Riemannian surface $\Sigma$:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

$$E = |M_u|^2; \quad F = \langle M_u, M_v \rangle; \quad G = |M_v|^2$$

The length of a curve $M(u(t), v(t))$ is given by:

$$\int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} \, dt$$

and the area of $\Sigma$ by:

$$\int_U |M_u \wedge M_v| \, dudv \quad \text{with} \quad |M_u \wedge M_v| = \sqrt{EG - F^2}$$
Differential geometry: curvatures

With the immersion is associated a $C^\infty$ application such that:

$$\Sigma \longrightarrow S^2 \quad M \longrightarrow N(M)$$

where $N(M)$ is the outward unit normal.

The tangent plane to $\Sigma$ at $M$ is naturally identified to the tangent plane of $S^2$ at $N(M)$, hence the tangent application $dN : T\Sigma \longrightarrow TS^2$ is a symmetric linear application. $dN(M)$ can be diagonalized into an orthonormal basis.

The application $II \equiv -dN$ is the 2nd fundamental form. The eigenvalues $\kappa_1, \kappa_2$ at $M_0$ are the principal curvatures:

mean: $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} tr\ dN(M_0)$

Gaussian: $K = \kappa_1 \kappa_2 = det\ dN(M_0)$
Differential geometry: curvatures computation

*Theorema Egregium* (Gauss):
all curvatures can be computed in function of $E, F, G$ and partial derivatives.

In the 2nd fundamental form:

$$
e = -\langle dN . M_u, M_u \rangle = \langle N, M_{uu} \rangle$$

$$f = -\langle dN . M_u, M_v \rangle = \langle N, M_{uv} \rangle$$

$$g = -\langle dN . M_v, M_v \rangle = \langle N, M_{vv} \rangle$$

let $(a_{ij})$ be the matrix of $-dN$ in the basis $(M_u, M_v)$:

$$-dN . M_u = a_{11} M_u + a_{21} M_v \quad -dN . M_v = a_{12} M_u + a_{22} M_v$$

then we obtain:

$$
\begin{pmatrix}
e & f \\
f & g
\end{pmatrix} =
\begin{pmatrix}
a_{11} & a_{21} \\
a_{12} & a_{22}
\end{pmatrix}
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix}.
$$

Finally (inverting a $2 \times 2$ matrix):

$$K = \frac{eg - f^2}{EG - F^2} \quad H = \frac{eG + Eg - 2fF}{EG - F^2}$$
The metric provides a natural distance on $\Sigma$.
Let $M_0 \in \Sigma$ and let
$$D(\varepsilon) = \{M \in \Sigma, d(M, M_0) \leq \varepsilon\}$$
$$C(\varepsilon) = \{M \in \Sigma, d(M, M_0) = \varepsilon\}$$
be the disk (resp. circle) of radius $\varepsilon$.

If $\varepsilon \to 0$, the limited expansions
$$\text{Length } C(\varepsilon) = 2\pi\varepsilon \left(1 - \frac{K(M_0)}{6}\varepsilon^2 + o(\varepsilon^2)\right)$$
$$\text{Area } D(\varepsilon) = \pi\varepsilon^2 \left(1 - \frac{K(M_0)}{12}\varepsilon^2 + o(\varepsilon^2)\right)$$
correspond to the so-called Puiseux and Diquet formula, respectively.
Differential geometry: curvature and convexity

A surface is locally a graph above its tangent plane. In the vicinity of $O \in \Sigma$, assuming $T_O \Sigma = \{z = 0\}$, there is a parametrisation $z = f(x, y)$. If the surface is oriented with the normal outward then:

$$II = D^2 f(O).$$

The osculating paraboloid to $\Sigma$ at $O$ is defined by

$$z = II(x, y) = \kappa_1 \xi_1^2 + \kappa_2 \xi_2^2$$

where $\xi_i$ are the main coordinates at $O$.

We have the following properties:

- if $K > 0$, $\Sigma$ is locally strictly convex,
- if $K < 0$, each tangent plane crosses the surface.
Differential geometry: practical computation

1. find a parametrisation \((u, v)\) of \(\Sigma\),
2. compute \(E, F, G\),
3. form \(M_u \wedge M_v\),
4. compute \(|M_u \wedge M_v|^2 = |M_u|^2 |M_v|^2 \sin^2 \theta = EG - F^2\).
5. compute \(N = M_u \wedge M_v / \sqrt{EG - F^2}\),
6. compute \(M_{uu}, M_{uv}, M_{vv}\), then \(e, f, g\),
7. finally, compute \(K, H, \kappa_1, \kappa_2\).
Surface mesh: discrete curvatures

Discrete curvatures are of interest for geometric modelling for (at least) 2 reasons:

- opportunity to handle a discretisation of a continuous object with a free choice of the discretisation
- to define second order estimates for discrete objects

Approximations of smooth surface required when surface is defined by a set of discrete points or by a triangulation.

estimating the local surface geometry: augmented Darboux frame (Sander, Zucker, 1990):

\[ \Delta(P) = (P, \tau_1, \tau_2, N, \kappa_1, \kappa_2) \]

- but, estimating the Darboux frames of an unknown piecewise smooth surface is difficult because of the inherently discrete nature of the data.
- in spite of effectiveness, all techniques are very time consuming.

Surface mesh: curvature estimation on triangular mesh

- old topic but first recent work in XXth century by Alexandrov.
- polyhedral surface described by a set of points and a structure
- several requirements on method and result (MD):
  - intrinsic behavior: result must be invariant by isometries and be modified in a reasonable way by affine transformations,
  - convergence: to the continuous curvatures when number of points $\to \infty$,
  - local behavior: not reasonable to use 'long-distance' information to find properties analogous to derivatives,
  - independence: of the discretisation (change of triangulation).

- remark: information of curvature, primarily of 2nd order, is closely related to the estimation of normal vector (1st order) as well as to length, angle and area.

Discrete surfaces: problematics

3 categories:

- formulas using angles and lengths (angular defects),
- local approximation by an elementary surface,
- convergence: asymptotic analysis.

results are partially dependent on the data structure:

- neighborhood
- distances and angles between adjacent vertices
  (e.g., matrices of voxels provide equidistance properties, in quadrilateral mesh regular cutting result in 6 adjacent neighbors per vertex).
Discrete surfaces: computing curvatures

1. (Gauss-Alexandrov) angular defect: discrete equivalent of Gauss’s definition. considers the normal vectors and defines a spherical indicatrix (joining points on the unit sphere), the area of the polygon is then \( A_C = 2\pi - \sum \alpha_i \) and Gaussian curv. is the limit of ratio \( A_{I(C)}/A_C \) when areas tend toward 0.

2. (Borrelli-Boix) connect angles, lengths, areas, Gaussian curvature.
   - geodesic (intrinsic) triangles:
     \[
     \alpha - \alpha' = area(T) \frac{2K(A) + K(B) + K(C)}{12} + o(a^3 + b^3 + c^3)
     \]
     Legendre’s formula for all surrounding triangles gives:
     \[
     K = \frac{2\pi - \sum \alpha_i}{\frac{1}{3} \sum area(T_i)}
     \]
     (equally sharing the area between 3 vertices)
   - Euclidean triangles: geodesics of \( \mathbb{R}^3 \) in which surface embedded (chords joining vertices)
     \[
     K = \frac{2\pi - \sum \alpha_i}{\frac{1}{2} \sum area(T_i) - \frac{1}{8} \sum \cotg \alpha_i l_i^2}
     \]
     \[
     l = s - \frac{\kappa^2}{24} s^3 + O(s^3)
     \]
Discrete surfaces: computing curvatures

3. circular fitting: use Meusnier and Euler results involving the normal curvature:

\[ \kappa_n = A - B \cos 2\alpha + C \sin 2\alpha \]

\(\alpha\) angle between tangent direction corresponding to \(\kappa_n\) and reference direction \(T_0\). Then:

\[ \kappa_i = A \pm \sqrt{B^2 + C^2} \]

4. paraboloid fitting: approximates a small neighborhood of \(P\) by an osculating paraboloid. Assume an arbitrary direction \(x\) and \(y = z \times x\) then the canonical form is:

\[ z = ax^2 + bxy + cy^2 \]

and using a least square fit to \(P\) and \(P_i\)

\[ K = 4ac - b^2 \quad H = a + c \]
Discrete surfaces: computing curvatures

5. Taubin’s approach: defines the symmetric matrix $M$ by the integral formula:

$$M = \frac{1}{2\pi} \pi \int_{-\pi}^{\pi} \kappa_{\nu}^{P}(T_\theta)T_\theta T_\theta^t \, d\theta$$

where $\kappa_{\nu}^{P}(T_\theta)$ is the normal curvature of $\Sigma$ at $P$ in direction $T_\theta = \cos(\theta)\tau_1 + \sin(\theta)\tau_2$.

Since $N(P)$ is an eigenvector of $M$ associated with the eigenvalue 0 it comes:

$$M = T_{12}^t \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} T_{12}$$

where $T_{12} = [\tau_1, \tau_2]$ is the $3 \times 2$ matrix constructed by concatenating $\tau_1$ and $\tau_2$.

Principal curvatures are obtained as functions of the nonzero eigenvalues of $M$:

$$\kappa_1 = 3m_{11} + m_{22} \quad \kappa_2 = 3m_{11} - m_{22}.$$
Lemma 0.1 \( \mathcal{T} \) is a Delaunay mesh iff

\[ \forall (K, K') \in \mathcal{T}, K = \text{adj}(K') \; ; \; (\text{open circumsphere of } K) \cap (\text{vertices of } K') = \emptyset \]
Delaunay kernel

- Incremental algorithm (Delaunay kernel, [Bowyer, Watson 1981]):

\[ \mathcal{T}_{n+1} = \mathcal{T}_n - \mathcal{C}_n \cup \mathcal{B}_{n+1} \]
Delaunay kernel

- Incremental algorithm (Delaunay kernel, [Bowyer, Watson 1981]):

\[ T_{n+1} = T_n - C_n \cup B_{n+1} \]

Triangulation \( T_n - C_n \)
Delaunay kernel

- Incremental algorithm (Delaunay kernel, [Bowyer, Watson 1981]):

\[ T_{n+1} = T_n - \mathcal{C}_n \cup \mathcal{B}_{n+1} \]
Delaunay kernel

To find $O$ center of the ball circumscribed to tetra $K$, solve:

$$\text{dist}(P_i, O) = \text{dist}(P_j, 0), \quad \forall P_i \neq P_j \in K,$$

it yields:

$$\begin{pmatrix}
  x_{12} & y_{12} & z_{12} \\
  x_{13} & y_{13} & z_{13} \\
  x_{14} & y_{14} & z_{14}
\end{pmatrix}
\cdot
\begin{pmatrix}
  O_x \\
  O_y \\
  O_z
\end{pmatrix}
= \begin{pmatrix}
  d_{12} \\
  d_{13} \\
  d_{14}
\end{pmatrix},$$

with $d_{1i} = (x_{i}^2 + y_{i}^2 + z_{i}^2) - (x_{1}^2 + y_{1}^2 + z_{1}^2)$ and $x_{1i} = 2(x_i - x_1)$. The solution of the system $A \cdot O = d$ is then obtained using Cramer’s formula, the in-radius is simply $r_K = \text{dist}(P_1, O)$.

**Definition 0.10** The Delaunay measure is defined as:

$$\alpha(K, P) = \frac{\text{dist}(P, O)}{r_K}.$$

and $K \in \mathcal{C}_P$ iif $\alpha(K, P) \leq 1$. 
Delaunay kernel: anisotropic case

The Delaunay kernel remains valid.

Given a metric tensor \( \mathcal{M}(x) \), the previous system becomes:

\[
\ell_{\mathcal{M}}(P_i, O) = \ell_{\mathcal{M}}(P_j, 0), \quad \forall P_i \neq P_j \in K.
\]

There are several ways of solving this equations, i.e., to define the Delaunay measure:

1. \( \alpha(K, P)_{\mathcal{M}(P)} \leq 1 \),

2. \( \alpha(K, P)_{\mathcal{M}(P)} + \sum_{i=1}^{4} \alpha(K, P)_{\mathcal{M}(P_i)} \leq 5 \),

3. \( \alpha(K, P)_{\mathcal{M}(P)} + \sum_{i=1}^{4} \omega_i \alpha(K, P)_{\mathcal{M}(P_i)} \leq 1 + \sum_{i=1}^{4} \omega_i \).
Delaunay triangulation: point insertion

General scheme:
- localisation of the tetrahedron containing \( P \)
- construction of the cavity \( C_P \)
- correction of the cavity (to avoid slivers)
- remeshing of \( C_P \).

use a bucket to avoid creating point too close to already existing vertices. This requires finding the bounding box of the ellipsoid associated with \( M \):

\[
F(u) = \langle u, Mu \rangle - 1 = 0 \quad \nabla F(u) = 2Mu
\]

finally it comes:

\[
\delta_x = \sqrt{\frac{\text{min}_x}{\Delta}} \quad \delta_y = \sqrt{\frac{\text{min}_y}{\Delta}} \quad \delta_z = \sqrt{\frac{\text{min}_z}{\Delta}}
\]

where \( \Delta \) is the determinant of \( M \).
Delaunay triangulation: local modifications

the construction of a unit volume mesh if then straightforward: given an (empty) volume mesh it simply consists in analysing the lengths of edges and:

- split large edges and insert points using the Delaunay kernel
- collapse small edges
- improve mesh quality using edge/face swaps
- use a node relocation procedure to improve the overall histogramm.

advantages:

1. only one mesh in memory: the current mesh is modified iteratively,
2. well suited for mesh adaptation,
3. amount of modifications decreases when number of adaptations increase (in steady state problem),
4. natural framework for moving mesh strategies.
Error estimate

consider a general PDE problem defined on a bounded domain \( \Omega \),
\( u \) the exact solution, \( u_h \) the FE solution obtained on a mesh \( T_h \).

in mesh adaptation, the problem consists in computing at stage \( i \):

\[
d_{h}^{i} = \|u - u_{h}^{i}\|.
\]

Definition 0.11 Using Cea’s lemma (elliptic problems) we know:

\[
\|u - u_{h}\| \leq c \|u - \Pi_{h}u\|.
\]

The problem turns to define a metric tensor \( M \) in order to equidistribute the interpolation error over the mesh.
Error estimate: in one dimension

- Let consider the segment $AB = [x_0, x_n]$, as seen before,
  \[
  \ell_i = \int_{x_i}^{x_{i+1}} \sqrt{|u''|}
  \]
  find the subdivision of $AB$ into $N$ segments such that $\ell_i = \frac{\ell}{N}$ is not dependent of $i$.

- if we fix $\int_{x_0}^{x_n} \sqrt{|u''|} \leq \varepsilon$ then we are looking for $c$ (constant) and $e$ (exponent) such that:
  \[
  \|u - \Pi_1 u\| \leq c \varepsilon^e.
  \]
  we find (F. Hecht):
  \[
  \|u - \Pi_1 u\|_{L^\infty} \leq \frac{(b-a)^2}{8} \sup_{[AB]} u''
  \]
  \[
  \|u - \Pi_1 u\|_{L^1} \leq \frac{(b-a)^3}{12} \sup_{[AB]} u''
  \]
  \[
  \|u - \Pi_1 u\|_{L^2} \leq \frac{(b-a)^{5/2}}{2\sqrt{30}} \sup_{[AB]} u''
  \]
Error estimate: in three dimensions

**Definition 0.12** With a $P_1$ finite element, the interpolation error on tetrahedron $K = (a, b, c, d)$ is given by a Taylor expansion with integral rest:

\[
(u - \Pi_h u)(a) = (u - \Pi_h u)(x) + \langle \vec{x} \hat{a}, \nabla (u - \Pi_h u)(x) \rangle \\
+ \int_0^1 (1 - t) \langle \vec{a} \hat{x}, H_u(x + t \vec{x} \hat{a}) \vec{a} \hat{x} \rangle \, dt,
\]

and ... finally:

\[
\left\| u - \Pi_h u \right\|_{L^\infty, K} \leq \frac{9}{32} \max_{y \in K} \max_{\vec{v} \subset K} \langle \vec{a} \hat{a}', H_u(y) \vec{a} \hat{a}' \rangle,
\]

where $a'$ is the point corresponding to the intersection of line $ax$ with the face opposite to $a$: $\vec{a} \hat{x} = \lambda \vec{a} \hat{a}'$, $\lambda \leq 3/4$.

**Definition 0.13** The bound on the interpolation error can be written as:

\[
\left\| u - \Pi_h u \right\|_{\infty, K} \leq \frac{9}{32} \max_{y \in K} \max_{\vec{v} \subset K} \langle \vec{v}, |H_u(y)| \vec{v} \rangle.
\]
Error estimate: numerical computation

**Definition 0.14** the previous relation can be written as:

\[ \| u - \Pi_h u \|_{\infty,K} \leq c_d \max_{x \in K} \max_{e \in E_K} \langle \vec{e}, |H_u(x)| \vec{e} \rangle. \]

- the right hand side term is hard to compute
- assume that it exists a metric tensor \( \overline{M}(K) \) such that:

\[ \max_{x \in K} \langle \vec{e}, |H_u(x)| \vec{e} \rangle \leq \langle \vec{e}, \overline{M}(K) \vec{e} \rangle, \quad \forall e \in E_K, \]

and such that the region defined by: \{ \langle \vec{v}, \overline{M}(K) \vec{v} \rangle | \forall \vec{v} \subset K \} is minimal. Consequently:

\[ \| u - \Pi_h u \| = c \max_{e \in E_K} \langle \vec{e}, \overline{M}(K) \vec{e} \rangle. \]

this relation relates the interpolation error to the diameter of \( K \), w/r \( \overline{M}(K) \).

hence, controlling the length of the mesh edges will allow to control the interpolation error.
from the estimation of the interpolation error, we can state that if \( \|u - \Pi_h u\| \) is fixed, the only variable left is the edge length.

hence, mesh adaptation will consists in adjusting mesh edges so that the interpolation error will be equidistributed over the adapted mesh.

the aim is to construct an \textit{optimal} mesh.

- mesh edges must solve:

\[
\varepsilon = c \langle \vec{e}, \overline{M}(K) \vec{e} \rangle, \quad \forall e \in E_K.
\]

let \( M(K) = \frac{c}{\varepsilon} \overline{M}(K) \) be the desired metric tensor, we have:

\[
\langle \vec{e}, M(K) \vec{e} \rangle = 1, \quad \forall e \in E_K.
\]

notion of \textit{unit mesh} as the desired optimal mesh.
Error estimate: metric tensor

**Definition 0.15** Let \( \varepsilon \) be the maximum interpolation error and let \( h_{\min} \) (resp. \( h_{\max} \) be the minimal (resp. maximal) edge size. We define the metric tensor at the mesh vertices as follows:

\[
\mathcal{M} = \mathcal{R} \tilde{\Lambda} \mathcal{R}^{-1}, \quad \text{with} \quad \tilde{\Lambda} = \begin{pmatrix}
\tilde{\lambda}_1 & 0 & 0 \\
0 & \tilde{\lambda}_2 & 0 \\
0 & 0 & \tilde{\lambda}_3
\end{pmatrix}
\]

\[
\tilde{\lambda}_i = \min\left(\max\left(\frac{c|\lambda_i|}{\varepsilon}, \frac{1}{h_{\max}^2}\right), \frac{1}{h_{\min}^2}\right)
\]

where \( \mathcal{R} \) is the eigenvector matrix and the coefficients

**Definition 0.16** it is possible to define a relative error as:

\[
\frac{\|u - \Pi_h u\|}{\alpha|u|_\varepsilon + \bar{h}\|\nabla u\|_2} \leq c \max_{x \in K} \max_{\bar{e} \in E_K} \langle \bar{e}, \frac{|H_u(x)|}{\alpha|u(x)|_\varepsilon + \bar{h}\|\nabla u(x)\|_2} \rangle,
\]

where \( \bar{h} \) is the diameter and \( 0 < \alpha < 1 \).
CFD simulations

Transonic viscous flow: adapted meshes (iter. 0, 1, 3, 9).
CFD simulations

Adapted meshes and density fields (iter. 0, 9).
Analytical surface

Initial and adapted surface meshes.
Analytical surface

Initial and adapted surface meshes (zooms).
Analytical surface

Cut through initial and adapted volume meshes.
Onera M6 airfoil

- Context: Euler simulation
- Mach 0.8395, angle of attack: 3.06°

- Parameters:
  - adaptation based on Mach number
  - 9 adaptations every 250 solver iterations
  - $\varepsilon = 0.001$, $h_{min} = 0.4 \text{ m}$, $h_{max} = 60 \text{ m}$, $h_{grad} = 1.5$ et $\gamma = 0.5$

- Simulation:
  - HP workstation 650 MHz, 1 Gb RAM
  - 10h comp.: $\approx 95\%$ in solver, $\approx 2.5\%$ mesh
  - initial mesh: $ns = 7,815$, $nt = 37,922$
  - final iso mesh: $ns = 231,113$, $nt = 1,316,631$
  - final aniso mesh: $ns = 23,516$, $nt = 132,676$. 
Onera M6 airfoil: isotropic adaptation

Isotropic surface and volume mesh adaptations (iterations 1,9).
Onera M6 airfoil: isotropic adaptation

Isotropic surface and volume mesh adaptations (iterations 1,9).
Onera M6 airfoil: anisotropic adaptation

Isotropic surface and volume mesh adaptations (iterations 1,9).
Onera M6 airfoil: anisotropic adaptation

Anisotropic surface and volume mesh adaptations (iterations 1,9).
Accurate simulation of air-cooled structures

Anisotropic mesh adaptation using local modifications (courtesy: CEA Cadarache, mesh: C. Dobrzynski).

Evaluation of the cooling properties for the design of a hot nuclear spent fuel.
Accurate simulation of air-cooled structures

Simulation of air-conditioning system in furnished house (C. Dobrzynski).
original geometry and domain decomposition for parallel computing.
Accurate simulation of air-cooled structures

Simulation of air-conditioning system in furnished house (C. Dobrzynski).

Streamlines of vector velocity in third floor.
Toward moving mesh applications

mesh movement based on local Delaunay mesh modifications.
Anisotropy

Flow features:

- Phenomena concentrated in small regions, mesh size is important
- Anisotropic phenomena: shock waves, boundary layers,...
  the mesh is not optimal w/r directions
- Regions move when unsteady phenomena
  uniformly fine mesh everywhere

(→ general framework)