From medical images to computational meshes

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Outline

1. Problem statement
   • Data acquisition: challenges, solutions
   • Mesh simplification

2. Surface reconstruction

3. Construction of a geometric mesh

4. Generation of a computational mesh

5. Mesh adaptation

6. More examples

Problem Statement: scanning devices

- **Data acquisition**
  - fast and accurate scanning and sensing devices available (CT, MRI, ...)
  - (very) large datasets of points of many physical objects or human organs collected in almost real time;
  - need for increasing accuracy related to the desire of capturing the geometry of organs as precisely as technology permit;
  - combination of several technologies (angiography, arteriography, X-ray);

- **Data conversion**
  - discrete data usually converted into polygonal surface meshes, using “brute-force” algorithms, often resulting in large meshes (Marching Cubes).
Problem Statement: data acquisition

- **Symbolic projects**

1. the Digital Michelangelo project (Stanford University)
   "virtual museum", scanning of italian statues
   accuracy 0.1\,mm, size 2.5\,m, 3\,Gb storage, \(10^9\) points

2. the Visual Human Project (National Library of Medicine)
   "anatomical atlas", MRI, \(512 \times 512\) slices
   \(10^7\) points

3. the Digitized Man (INRETS, Marseille)
   "crash tests", digitized data
   \(10^6\) points
Problem Statement: the challenges

- **Data processing**
  - challenge related to the size of the datasets,
  - range images or point clouds not suitable neither for archiving and processing data nor for representing 3D geometries,
  - need for *polygonal models* (triangulations) for computational purposes.

- **Requirements**
  - computer graphics
    - fast rendering of complex scene composed of polygonal objects,
    - frame rate (hardware technology) dictates the number of primitives.
  - numerical simulations
    - emphasis laid on geometric accuracy + element shape quality control,
    - compromise: number of DOF vs. quality of geometric approximation.
  - data archiving
    - polygonal meshes are more flexible than CAD files,
    - size of model adjusted to comply with network bandwidth.
Problem Statement: the solutions

- **Algorithms**
  - related to the fields of applications:
    - polygonal simplification algorithms tuned to generate LOD of objects,
    - mesh simplification + mesh adaptation,
    - hierarchical models,
  - need to tailor simplification algorithms to the needs of the application

The number of polygons we want always exceeds the budget polygon we can afford!

- **Mesh simplification** ($\neq$ geometry simplification)
  - **accurate** representation of the geometry with a **minimal** number of well-shaped elements.
Problem Statement: mesh simplification

- **Aim**: creation of an *optimal* mesh
  - surface geometry represented with a minimal number of regular elements,
  - element shape + size quality vs. vertex reduction: conflicting requirements,
  - equidistribution of the approximation error over the mesh elements,
  - vertices distributed evenly over the surface,

- **Suggested approach**
  - *a posteriori* geometric error estimate based on second derivatives,
  - a discrete anisotropic metric map,
  - mesh adaptation.
Problem Statement: context related

- **Decimation**
  - early papers of Schroeder and Turk (1992),
  - $>>1,000$ papers found on the Internet databases,
  - hot topic in computer graphics, VR, biomedical and numerical simulations, ... 
  - comprehensive surveys have been compelled (Heckbert, Krus, Luebcke),
  - emphasis laid on graphical issues.

- **Scientific computing**: toolboxes available
  - fast and accurate solvers,
  - efficient mesh generation packages available
  - complex biomedical simulations (hemodynamics) now possible.

- **Bottleneck**: gap between discrete (medical) images and FE models.
Surface Reconstruction: series of slices

- **Input**
  - series of $m$ equally spaced slices of size $n \times n$,
  - scalar field attached to each pixel (density of the material tissue).

- **Approach**
  - running along contours, triangles created between two adjacent slices,
  - surface triangulation obtained by merging altogether the sets of triangles,
  - various criteria suggested to control the resulting surface triangulation: volume maximization, surface minimization, edge length or angle minimization, . . .

- **Drawback**
  - branching problem: number of closed contours change from one slice to another,
  - solution: heuristic assumptions on vertical connections, global technique to generate a polyhedral representation (Delaunay-based reconstruction).
From slices to surface mesh

Figure 1: Delaunay-based surface reconstruction algorithm used for connecting series of parallel slices of an aneurysm (mesh: E. Saltel, INRIA).

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Surface Reconstruction: point clouds

- **Input**
  - (sparse) set of unorganized points possibly supplied with normal directions,
  - points assumed to lie on the surface: no image processing step,
  - common problem in reverse engineering, image processing, computer graphics.

- **Main issues**
  - need to deal with arbitrary topology, non-uniform sampling,
  - produce geometric meshes: accurate piecewise linear approximation.

- **Approach: 2 steps**
  - construction of a 3D Delaunay triangulation over the set of points,
  - extraction of a triangulation by selecting appropriate faces from the tetrahedra,
  - use of natural neighbour interpolation to refine the triangulation locally to match a sampling condition (Chew, Boissonnat).
Surface reconstruction from point clouds

Figure 2: Point cloud and reconstructed surface triangulation based on a Delaunay tetrahedralization (mesh: P.L. George, INRIA).
Surface Reconstruction: valued data

- **Input**
  - signed distance function and compute its zero set,
  - implicit surface seen as a levelset defined over the entire embedding space,
  - density function seen as a discrete implicit function known at sampled points.

- **Approach**
  - "Marching Cubes" algorithm and variants,
  - based on a cell by cell analysis to follow the implicit surface,
  - check based on the evaluation of the sign of the function at the cell corners,
  - $2^8 = 256$ configurations reduced to 16 patterns using rotation and symmetry.

- **Drawbacks**
  - unnecessary large number of triangles,
  - poor element quality.
Isosurface reconstruction

Figure 3: Volumetric MRI image and "Marching-Cubes" surface reconstruction (data: M. Seppa, Finland).
Surface Reconstruction: summary

● Algorithms
  numerous robust and efficient algorithms available to produce triangulations.

● Major drawbacks
  1. sampled data may be very noisy,
  2. the accuracy of devices leads to unnecessary dense datasets,
  3. the density is not related to the local geometric complexity,
  4. artefacts due to ”brute-force” reconstruction algorithms,
  5. no (or very few) attention is paid to the element shape quality.

● Remedy
  1. ”filtering” pass to remove the noise,
  2. mesh adaptation.
Surface Reconstruction: strategy

• **Description**
  \( \Omega \subset \mathbb{R}^3 \) bounded domain described by its boundary \( \Sigma \).
  Given \((M_{\text{ref}}(\Sigma), H_{\text{ref}}(\Sigma))\) construct an adapted mesh \( M(\Sigma) \).

• **Scheme**
  1. \( M_{\text{ref}}(\Sigma) \) is simplified and optimized based on the Hausdorff distance leading to a geometric reference mesh \( M_{\text{ref},g}(\Sigma) \),
  2. a piecewise \( C^1 \) continous geometric support is defined on the mesh \( M_{\text{ref},g}(\Sigma) \),
  3. the metric map \( H_{\text{ref},g}(\Sigma) \) is modified to account for the surface geometry and the desired mesh gradation,
  4. \( M_{\text{ref},g}(\Sigma) \) is adapted to \( H_{\text{ref},g}(\Sigma) \) yielding to the computational mesh \( M(\Sigma) \).
Table 1: General scheme of mesh adaptation.
Surface Mesh Adaptation: mesh smoothing

• **Aim**
  reduce apparent faceting, remove staircase artefacts.

• **Approach**
  ○ 2 steps of Gaussian smoothing,
  ○ update vertex position: weighted average of $P_i \in B(P)$

\[
\begin{align*}
P' &= P + \lambda \sum_i \omega_j (P_i - P) \\
P' &= P - \mu \sum_i \omega_j (P_i - P)
\end{align*}
\]

• **Analysis**
  ○ linear low-pass filter $0 < \lambda < -\mu$,
  ○ remove high curvature variations (curvature vs. frequency),
  ○ special care to ridges, borders.
Mesh smoothing

Figure 4: Original and “smoothed” triangulations (10 iterations, parameters: $\lambda = 0.33, \mu = 0.34$).
Surface Triangulation: basic definitions

**Definition 1** \((\Omega \subset \mathbb{R}^3, \Sigma)\). A *regular mesh* \(\mathcal{M}(\Omega)\) contains only regular elements.

- **Remark**: The existence of a regular mesh is not established in any dimension.

- **Surface meshes**
  2 types characterized by the element size:
  - **uniform mesh**: \(\mathcal{H} = \lambda \text{Id} \) on \(\Sigma\) (constant size),
  - **geometric mesh**: \(\mathcal{H} \sim (\rho_1, \rho_2)\) (size related to main curvatures).

- **Remark**
  1. A uniform mesh does not guarantee anything about the quality of the surface approximation, except being infinitely refined,
  2. A geometric mesh though, is a mesh adapted to the geometry of the surface: a geometric continuity of level 1 is sufficient.
Surface Triangulation: geometric mesh

Definition 2 Two main notions:
- **approximation**: a mesh is 0-order geometric iif each triangle is close to the surface,
- **smoothness**: a mesh is 1-order geometric if the plane supporting each triangle $K$ is close to $\Pi(P_i)$ at each point $P_i$ of $\Sigma$ close to $K$.

• **Remark**
  - approximation: relates the measure of the gap $G_1$ between the points of the triangle and the corresponding points on the portion of the surface discretized by the triangle
  - smoothness: related to the measure of the gap $G_2$ between the normal of the triangle and the normal at any point of the portion of the surface discretized by the triangle

• **Requirements**
  - minimal number of triangles,
  - element shape quality control.
Surface Mesh: curvatures

• Interest
  ○ handle a discretisation of a $C^k$ domain with free choice of the discretisation,
  ○ define second order estimates for discrete objects.

• Remark
  Approximations of $C^1$ surface required when surface is defined by a triangulation.

• Method
  ○ local surface geometry via augmented Darboux frame
    $$ \Delta(P) = (P, \tau_1, \tau_2, N, \kappa_1, \kappa_2) $$
  ○ but, estimating the Darboux frames of a unknown piecewise smooth surface is
difficult because of the inherently discrete nature of the data,
  ○ in spite of effectiveness, all techniques are very time consuming.
  ○ a discrete geometric metric $\mathcal{H}_{ref,g}$ defined in the tangent plane $\Pi(P)$.

Surface Mesh: metric definition

- **Analysis**
  - the analysis of the curvatures allows to minimize the deviation between the tangent planes of the piecewise linear interpolation and the true surface,
  - the two gaps $G_1$ and $G_2$ can be bounded by a given tolerance provided the triangle size at each vertex $h(P)$ is proportional to the two radii of curvatures.

- **Geometric metric**
  $H_{\text{ref},g}$ defined in the tangent plane of each vertex $P$ as:

  \[
  H_{\text{ref},g} = (h_1 \ h_2) \begin{pmatrix}
  1 & 0 \\
  \frac{1}{(\alpha \rho_1(P))^2} & 1 \\
  0 & \frac{1}{(\eta(\alpha, \rho_1(P), \rho_2(P))\rho_2(P))^2}
  \end{pmatrix}
  \begin{pmatrix}
  h_1 \\
  h_2
  \end{pmatrix} = 1.
  \]
Surface Decimation: Hausdorff measure

**Definition 3** the distance from a point $X \in \mathbb{R}^3$ to a bounded domain $F$ is given by:

$$d(X, F) = \inf_{Y \in F} d(X, Y),$$

If $\rho(F_1, F_2)$ measures the quantity $\sup_{X \in F_1} d(X, F_2)$, then Hausdorff distance defined as:

$$d_H(F_1, F_2) = \sup(\rho(F_1, F_2), \rho(F_2, F_1)). \forall F_1, F_2 \subset \mathbb{R}^3$$

- **Remark**
  - continuous measure is expensive to compute (as it must be checked for all points lying in the plane of the triangles)
  - use a discrete approximation of the measure.
Geometric Mesh: discrete Hausdorff distance

- **Proximity property**
  - a global tolerance envelope around the surface (at a given Hausdorff distance $\delta$) is introduced on both sides of the reference surface mesh $M_{ref}$,
  - it is sufficient to check that each triangle resulting from the mesh optimization procedure remains contained within this envelope.

- **Smoothness property**
  - another constraint added yielding to satisfy both requirements:
    
    $$d_H(K, M_{ref}) \leq \delta \quad \text{and} \quad \langle \nu_k(K), \nu(K) \rangle \geq \cos(\theta).$$

- **Discrete distance**
  - bound the distance between meshes: keep track of the deformations hierarchically,
  - associate majoration of the Hd with triangle $K^i$ of $M^i$ w/r $M_{ref}$:
    
    $$h(K^{i+1}(Q)) = d_H(K^{i+1}(Q), B^i(P)) + \max_i h(K^i(P)).$$
  - triangle $K^{i+1}_i(Q)$ does not degrade the piecewise linear approximation of the surface geometry if it is such that $h(K^{i+1}_i(Q)) \leq \delta$. 
Geometric Mesh: modifications

● **Principle**
  ○ remove iteratively the vertices from $\mathcal{M}_{ref}$,
  ○ three mesh modification operations are carried out: the edge flipping, the edge collapsing and the node relocation, enforcing the two previous requirements.

● **Operations**
  ○ edge flipping
    - replace the two triangles sharing an edge by the alternate configuration,
    - applied if the two triangles are strictly coplanar.
  ○ edge collapsing
    - remove a vertex $P$ and retriangulate the cavity, - operation possible if:
      \[d_H(K^{i+1}, \mathcal{M}_{ref}) \leq \delta \quad \text{and} \quad \langle \nu_k(K^{i+1}(Q)), \nu(K^{i+1}_i(Q)) \rangle \geq \cos(\theta),\]
  ○ node relocation: discrete Laplacian.

● **Control** of the shape quality
Figure 5: Geometric meshes obtained using the Hausdorff simplification procedure for tolerance values of $\delta = 0.1\%$ and $\delta = 0.5\%$ of the bounding box size.
Computational Mesh: volume mesh

- Generation
  - constrained Delaunay triangulation,
  - create an empty mesh, boundary recovery,
  - saturate edges, internal point creation based on \( \mathcal{H} \),
  - insert vertices via the Delaunay kernel.

- Delaunay kernel
  - Delaunay measure is defined as:
    \[
    \alpha(K, P) = \frac{\text{dist}(P, O)}{r_K}.
    \]
  - \( K \in C_P \) if \( \alpha(K, P) \leq 1 \).
  - can be extended to anisotropic case.
Volume Mesh

Figure 6: Two cutting planes through a volumic computational mesh corresponding to the head model.
Mesh Adaptation

• Context
  general PDE problem defined on a bounded domain $\Omega$, $u$ the exact solution, $u_h$ the FE solution obtained on a mesh $M_h$.

• Error estimate
  in mesh adaptation, the problem consists in computing at stage $i$:
  \[ e_h^i = \|u - u_h^i\|. \]

**Definition 4** Cea’s lemma for elliptic problems provides:

\[ \|u - u_h\| \leq c\|u - \Pi_h u\|. \]

• Aim
  - problem is to define a metric tensor $\mathcal{H}$
  - to equidistribute the interpolation error over the mesh.
Error Estimate: in one dimension

- Let consider the segment $AB = [x_0, x_n]$,

\[ \ell_i = \int_{x_i}^{x_{i+1}} \sqrt{|u''|} \]

find the subdivision of $AB$ into $N$ segments such that $\ell_i = \frac{\ell}{N}$ is not dependent of $i$.

- if we fix $\int_{x_0}^{x_n} \sqrt{|u''|} \leq \varepsilon$ then we are looking for $c$ (constant) and $e$ (exponent) such that:

\[ \|u - \Pi_1 u\| \leq c \varepsilon^e. \]

giving the majorations in various norms:

\[ \|u - \Pi_1 u\|_{L^\infty} \leq \frac{(b-a)^2}{8} |\sup_{[AB]} u''| \]
\[ \|u - \Pi_1 u\|_{L^1} \leq \frac{(b-a)^3}{12} |\sup_{[AB]} u''| \]
\[ \|u - \Pi_1 u\|_{L^2} \leq \frac{(b-a)^{5/2}}{2\sqrt{30}} |\sup_{[AB]} u''| \]
Error estimate: in three dimensions

**Definition 5** the $P_1$ interpolation error on $K = (a, b, c, d)$ is given by a Taylor expansion with integral rest:

\[
(u - \Pi_h u)(a) = (u - \Pi_h u)(x) + \langle \vec{x} a, \nabla (u - \Pi_h u)(x) \rangle \\
+ \int_0^1 (1 - t) \langle \vec{a} \vec{x}, H_u(x + t\vec{x} a) \vec{a} \vec{x} \rangle \, dt,
\]

and ... finally:

\[
\|u - \Pi_h u\|_{L^\infty,K} \leq \frac{9}{32} \max_{y \in K} \max_{\vec{v} \subset K} \langle \vec{a} a', H_u(y)\vec{a} a' \rangle,
\]

where $a'$ is the point corresponding to the intersection of line $ax$ with the face opposite to $a$: $\vec{a} x = \lambda a a'$, $\lambda \leq 3/4$.

**Definition 6** The bound on the interpolation error can be written as:

\[
\|u - \Pi_h u\|_{\infty,K} \leq \frac{9}{32} \max_{y \in K} \max_{\vec{v} \subset K} \langle \vec{v}, |H_u(y)| \vec{v} \rangle.
\]
Error estimate: numerical computation

**Definition 7** the previous relation can be written as:

\[ \|u - \Pi_h u\|_{\infty,K} \leq c_d \max_{x \in K} \max_{e \in E_K} \langle \bar{e}', |H_u(x)| \bar{e}' \rangle. \]

- **Remark**
  - the right hand side term is hard to compute,
  - assume that \( \exists \mathcal{M}(K) \) such that:
    \[ \max_{x \in K} \langle \bar{e}', |H_u(x)| \bar{e}' \rangle \leq \langle \bar{e}', \mathcal{M}(K) \bar{e}' \rangle, \quad \forall e \in E_K, \]
    such that the region defined by: \{ \langle \bar{v}, \mathcal{M}(K) \bar{v} \rangle | \forall \bar{v} \subset K \} is minimal. Hence,
    \[ \|u - \Pi_h u\| = c \max_{e \in E_K} \langle \bar{e}', \mathcal{M}(K) \bar{e}' \rangle. \]
  - interpolation error related to \( \bar{h}, \) diameter of \( K, \)
  - controlling the length of the mesh edges allows to control the interpolation error.
Error estimate: metric tensor

**Definition 8** $\varepsilon$ max interp. error, $h_{\text{min}}, h_{\text{max}}$ min and max edge size. Then:

$$M = R \tilde{\Lambda} R^{-1}, \quad \text{with} \quad \tilde{\Lambda} = \begin{pmatrix} \tilde{\lambda}_1 & 0 & 0 \\ 0 & \tilde{\lambda}_2 & 0 \\ 0 & 0 & \tilde{\lambda}_3 \end{pmatrix}$$

$$\tilde{\lambda}_i = \min \left( \max \left( \frac{c |\lambda_i|}{\varepsilon}, \frac{1}{h_{\text{max}}^2} \right), \frac{1}{h_{\text{min}}^2} \right),$$

where $R$ is the eigenvector matrix and the coefficients

**Definition 9** it is possible to define a relative error as:

$$\left\| \frac{u - \Pi_h u}{\alpha |u|_\varepsilon + \bar{h} \|\nabla u\|_2} \right\|_{\infty,K} \leq c \max_{x \in K} \max_{\bar{e} \in E_K} \frac{|H_u(x)|}{\alpha |u(x)|_\varepsilon + \bar{h} \|\nabla u(x)\|_2} \langle \bar{e}, \frac{1}{\alpha |u(x)|_\varepsilon + \bar{h} \|\nabla u(x)\|_2} \rangle,$$

$0 < \alpha < 1.$
Mesh optimisation example

Figure 7: From geometric to computational mesh (data: Pr. Viano, St Jacques de Compostelle).
Figure 8: Computational surface mesh of the thorax for solving problems in electrocardiology (data A. Pages, M. Sermesant, PF).
Numerical simulations

Figure 5. Resulting computational surface mesh of the thorax: lungs, heart and bone structures are visible (left-hand side) and visualisation of thorax potentials during a heart depolarisation (right-hand side).

Figure 9: Frontal cut through the thorax and thorax potentials during a heart depolarisation (data A. Pages, M. Sermesant, PF).
Numerical simulations

Figure 10: Fluid and structure meshes of cerebral aneurism (data J.F. Gerbeau, M. Vidrascu, PF).
Numerical simulations

Figure 11: Propagation of a pressure wave in cerebral aneurism (data J.F. Gerbeau, M. Vidrascu, PF).
Mesh Adaptation

Figure 12: Surface mesh adaptation: meshes at iterations 0, 1, 3 and 5.
Conclusions

- **Conclusions**
  - global approach for generating (computational, geometric) meshes,
  - element size and shape quality controlled by metric tensor,
  - suitable to handle large deformations or rigid body motion problems,
  - needs a watertight mesh to start with.

- **Perspectives**
  - investigate alternate approaches for surface reconstruction (levelsets),
  - need to deal with topology modification,
  - use imaging as well as functional information.
Differential Geometry: curvatures

- surface curvature on differentiable manifolds: important invariants in differential geometry - describes the local shape of a surface

- concept rooted in differential geometry, origins in the XVIII\textsuperscript{th} century:

- registration, smoothing, simplification, reverse engineering, visualisation, . . . .


L. Euler (1707-1783)  
C.F. Gauss (1777-1895)  
A.M. Legendre (1752-1833)
Differential Geometry: a primer

The curvature of a surface describes the local shape of that surface. Let consider a $C^\infty$ immersion of a surface $\Sigma$:

$$U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \quad (u, v) \mapsto M = M(u, v)$$

$U$ is an open set of $\mathbb{R}^2$ and $\Sigma$ is oriented by the normal $M_u \wedge M_v$

The metric induced by that of $\mathbb{R}^3$ (Euclidean) is represented in the map $(u, v)$ by the quadratic (1st fundamental) form of the Riemannian surface $\Sigma$:

$$ds^2 = Edu^2 + 2F dudv + Gdv^2$$

$$E = |M_u|^2; \quad F = \langle M_u, M_v \rangle; \quad G = |M_v|^2$$

The length of a curve $M(u(t), v(t))$ is given by:

$$\int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} \, dt$$

and the area of $\Sigma$ by:

$$\int_U |M_u \wedge M_v| dudv \quad \text{with} \quad |M_u \wedge M_v| = \sqrt{EG - F^2}$$
Differential Geometry: curvatures

With the immersion is associated a $C^\infty$ application such that:

\[
\Sigma \rightarrow S^2 \quad M \rightarrow N(M)
\]

where $N(M)$ is the outward unit normal.

The tangent plane to $\Sigma$ at $M$ is naturally identified to the tangent plane of $S^2$ at $N(M)$, hence the tangent application $dN : T\Sigma \rightarrow TS^2$ is a symmetric linear application. $dN(M)$ can be diagonalized into an orthonormal basis.

The application $II := -dN$ is the 2nd fundamental form. The eigenvalues $\kappa_1, \kappa_2$ at $M_0$ are the principal curvatures:

- mean: $H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2} tr dN(M_0)$
- Gaussian: $K = \kappa_1 \kappa_2 = \text{det} dN(M_0)$
**Differential Geometry: curvatures computation**

**Theorema Egregium** (Gauss): all curvatures can be computed in function of $E, F, G$ and partial derivatives. In the 2nd fundamental form:

\[
e = -\langle dN.M_u, M_u \rangle = \langle N, M_{uu} \rangle
\]
\[
f = -\langle dN.M_u, M_v \rangle = \langle N, M_{uv} \rangle
\]
\[
g = -\langle dN.M_v, M_v \rangle = \langle N, M_{vv} \rangle
\]

let $(a_{ij})$ be the matrix of $-dN$ in the basis $(M_u, M_v)$:

\[-dN.M_u = a_{11}M_u + a_{21}M_v \quad -dN.M_v = a_{12}M_u + a_{22}M_v\]

then we obtain:

\[
\begin{pmatrix}
e & f \\
f & g
\end{pmatrix}
= \begin{pmatrix}
a_{11} & a_{21} \\
a_{12} & a_{22}
\end{pmatrix}
\begin{pmatrix}
E & F \\
F & G
\end{pmatrix}.
\]

Finally (inverting a $2 \times 2$ matrix):

\[
K = \frac{eg - f^2}{EG - F^2} \quad H = \frac{eG + Eg - 2fF}{EG - F^2}
\]
Differential Geometry: metric and distance

The metric provides a natural distance on $\Sigma$.

Let $M_0 \in \Sigma$ and let

$$D(\varepsilon) = \{ M \in \Sigma, d(M, M_0) \leq \varepsilon \}$$

$$C(\varepsilon) = \{ M \in \Sigma, d(M, M_0) = \varepsilon \}$$

be the disk (resp. circle) of radius $\varepsilon$.

if $\varepsilon \to 0$, the limited expansions

$$\text{Length } C(\varepsilon) = 2\pi \varepsilon \left( 1 - \frac{K(M_0)}{6} \varepsilon^2 + o(\varepsilon^2) \right)$$

$$\text{Area } D(\varepsilon) = \pi \varepsilon^2 \left( 1 - \frac{K(M_0)}{12} \varepsilon^2 + o(\varepsilon^2) \right)$$

correspond to the so-called Puiseux and Diquet formula, respectively.
A surface is locally a graph above its tangent plane. In the vicinity of $O \in \Sigma$, assuming $T_O \Sigma = \{z = 0\}$, there is a parametrisation $z = f(x, y)$. If the surface is oriented with the normal outward then:

$$II = D^2 f(O).$$

The osculating paraboloid to $\Sigma$ at $O$ is defined by

$$z = II(x, y) = \kappa_1 \xi_1^2 + \kappa_2 \xi_2^2$$

where $\xi_i$ are the main coordinates at $O$.

We have the following properties:

- if $K > 0$, $\Sigma$ is locally strictly convex,
- if $K < 0$, each tangent plane crosses the surface.
Differential Geometry: practical computation

1. find a parametrisation \((u, v)\) of \(\Sigma\),
2. compute \(E, F, G\),
3. form \(M_u \wedge M_v\),
4. compute \(|M_u \wedge M_v|^2 = |M_u|^2 |M_v|^2 \sin^2 \theta = EG - F^2\).
5. compute \(N = M_u \wedge M_v / \sqrt{EG - F^2}\),
6. compute \(M_{uu}, M_{uv}, M_{vv}\), then \(e, f, g\),
7. finally, compute \(K, H, \kappa_1, \kappa_2\).
Curvature estimation on triangular mesh

- old topic but first recent work in XXth century by Alexandrov,
- polyhedral surface described by a set of points and a structure,
- Requirements
  - *intrinsic behavior*: result must be invariant by isometries and be modified in a reasonable way by affine transformations,
  - *convergence*: to the continuous curvatures when number of points $\to \infty$,
  - *local behavior*: not reasonable to use 'long-distance' information to find properties analogous to derivatives,
  - *independence*: of the discretisation.

- Remark: information of curvature, primarily of 2nd order, is closely related to the estimation of normal vector (1st order) as well as to length, angle and area.

Discrete Surfaces: problème statement

- **Zoology**
  - formulas using angles and lengths (angular defects),
  - local approximation by an elementary surface,
  - convergence: asymptotic analysis.

- **Results**
  - dependent of the data structure,
  - neighborhood,
  - distances and angles between adjacent vertices
    (e.g., matrices of voxels provide equidistance properties, in quadrilateral mesh regular cutting result in 6 adjacent neighbors per vertex).
1. **Gauss-Alexandrov** angular defect: discrete equivalent of Gauss’s definition. considers the normal vectors and defines a spherical indicatrix (joining points on the unit sphere), the area of the polygon is then $A_C = 2\pi - \sum \alpha_i$ and Gaussian curv. is the limit of ratio $A_{I(C)}/A_C$ when areas tend toward 0.

2. **Borrelli-Boix**: connect angles, lengths, areas, Gaussian curvature.
   - geodesic (intrinsic) triangles:
     \[
     \alpha - \alpha' = area(T) \frac{2K(A) + K(B) + K(C)}{12} + o(a^3 + b^3 + c^3)
     \]
     Legendre’s formula for all surrounding triangles gives:
     \[
     K = \frac{2\pi - \sum \alpha_i}{\frac{1}{3} \sum area(T_i)}
     \]
   - Euclidean triangles: geodesics of $\mathbb{R}^3$ in which surface embedded
     \[
     K = \frac{2\pi - \sum \alpha_i}{\frac{1}{2} \sum area(T_i) - \frac{1}{8} \sum cotg \alpha_i l_i^2}
     \]
     \[
     l = s - \frac{\kappa^2}{24} s^3 + O(s^3)
     \]
Discrete surfaces: computing curvatures

3. **Circular fitting**: use Meusnier and Euler results involving the normal curvature:

\[ \kappa_n = A - B \cos 2\alpha + C \sin 2\alpha \]

\(\alpha\) angle between tangent direction corresponding to \(\kappa_n\) and reference direction \(T_0\). Then:

\[ \kappa_i = A \pm \sqrt{B^2 + C^2} \]

4. **Paraboloid fitting**: approximates a small neighborhood of \(P\) by an osculating paraboloid. Assume an arbitrary direction \(x\) and \(y = z \times x\) then the canonical form is:

\[ z = ax^2 + bxy + cy^2 \]

and using a least square fit to \(P\) and \(P_i\)

\[ K = 4ac - b^2 \quad H = a + c \]
5. Taubin’s approach: defines the symmetric matrix $M$ by the integral formula:

$$M = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \kappa^P_n(T_\theta)T_\theta T^t_\theta \, d\theta$$

where $\kappa^P_n(T_\theta)$ is the normal curvature of $\Sigma$ at $P$ in direction $T_\theta = \cos(\theta)\tau_1 + \sin(\theta)\tau_2$.

Since $N(P)$ is an eigenvector of $M$ associated with the eigenvalue 0 it comes:

$$M = T^t_{12} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} T_{12}$$

where $T_{12} = [\tau_1, \tau_2]$ is the $3 \times 2$ matrix constructed by concatenating $\tau_1$ and $\tau_2$.

Principal curvatures are obtained as functions of the nonzero eigenvalues of $M$:

$$\kappa_1 = 3m_{11} + m_{22} \quad \kappa_2 = 3m_{11} - m_{22}.$$
Metric: basic definitions

**Definition 10** a metric $\mathcal{M}$ is a field of symmetric positive definite matrices $\mathcal{M} = (a_{ij}(x)) \in \mathcal{M}_3(\mathbb{R})$, $a_{ii}(x) > 0$ and $\det(\mathcal{M}) > 0$

**Proposition 1** such a matrix $\mathcal{M}$ can be decomposed as:

$$A = R \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} R^{-1}$$

$R = (v1 \ v2 \ v3)$ rotation matrix, $v_i$ and $\lambda_i$ unit eigenvectors (values) of $\mathcal{M}$.

**Definition 11** the scalar product is defined as:

$$\forall u, v \in \mathbb{R}^d, \quad \langle u , v \rangle_\mathcal{M} = \langle u , \mathcal{M}v \rangle = t^u \mathcal{M} v \in \mathbb{R}.$$ and the Euclidean norm of $u$ is:

$$\|u\|_\mathcal{M} = \sqrt{\langle u , \mathcal{M}u \rangle}.$$
Metric: geometric interpretation

If $\mathcal{M}(x) = \mathcal{M}$, the Euclidean geometry can be used and

**Definition 12** a metric $\mathcal{M}$ can be represented by its unit ball

$$\mathcal{E}_\mathcal{M} = \{x \in \mathbb{R}^d, \langle x, \mathcal{M}x \rangle = 1\}$$

The geometric locus of all points $M$ equidistant to $P$ is an ellipsoid.
Metric: unit length

Let $\gamma$ be a curve in $\mathbb{R}^d$ with a normal parametrization $\gamma(t)$, $t \in [0, 1]$.

**Definition 13** the length $|\gamma|$ of $\gamma$ is defined as:

$$|\gamma| = \int_0^1 \sqrt{\langle \gamma'(t), \mathcal{M}(\gamma(t))\gamma'(t) \rangle} \, dt$$

and for a mesh edge $AB = A + tAB$:

$$\ell(AB) = \int_0^1 \sqrt{\langle AB, \mathcal{M}(\gamma(t))AB \rangle} \, dt$$

**Definition 14** the mesh size $h_i$ in direction $v_i$ is defined as:

$$h_i = \frac{1}{\sqrt{\lambda_i}}, \quad \lambda_i = \frac{1}{h_i^2}.$$
And then ... problem arises!

- If $M(x) \neq M$, Riemannian geometry is involved
- straight edges are ‘replaced’ by geodesics
- simplifications are needed for sake of efficiency: linear edges

Euclid of Alexandria (325-265)  B. Riemann (1826-1866)
Definition 15  the metric $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$ is represented by the ellipsoid:

$$\mathcal{E}_\mathcal{M} = \sup_{\mathcal{M}_i} \mathcal{E}_{\mathcal{M}_i} = \sup_{\mathcal{M}_i} \left\{ M | \sqrt{\langle \overrightarrow{PM}, \overrightarrow{PM} \rangle_{\mathcal{M}_i}} = 1 \right\} \subset \mathcal{E}_{\mathcal{M}_1} \cap \mathcal{E}_{\mathcal{M}_2}.$$
Metric interpolation

Let $\gamma(t) = AB$ be a parametrized mesh edge and $M_A$ and $M_B \in \mathcal{M}_3(\mathbb{R})$.

**Definition 16** The metric $M(t)$ at $t$ on $\gamma(t)$ is:

$$M(t) = \left( (1 - t)M_A^{-\frac{1}{2}} + tM_B^{-\frac{1}{2}} \right)^{-2}, \quad 0 \leq t \leq 1.$$
Metric: size variation

- the quality of the elements is related to the variation of $\mathcal{M}$
- the size variations must be bounded (i.e. truncated)

**Definition 17** isotropic case: $\mathcal{M}(x) = \frac{1}{h(x)} I_3$

$$\ell(AB) = |AB| \int_0^1 \frac{1}{h(t)} dt, \quad h(t) = (1 - t)h(A) + th(B),$$

$$\ell(AB) = \frac{|AB|}{h(B) - h(A)} \ln \frac{h(B)}{h(A)},$$

size variation defined as:

$$\nu(AB) = \frac{h(B) - h(A)}{|AB|}$$
Metric: mesh gradation

**Definition 18** $\nu(AB)$ is bounded by a threshold value $\alpha$:

- **isotropic case:**

  \[
  \begin{align*}
  h(A) &= \min(h(A), h(B) + \alpha |AB|) \\
  h(B) &= \min(h(B), h(A) + \alpha |AB|)
  \end{align*}
  \]

- **anisotropic case:**

  \[
  \begin{align*}
  \mathcal{M}_A &= \mathcal{M}_A \cap \mathcal{M}_B (1 + \alpha \ell_A(AB))^{-2} \\
  \mathcal{M}_B &= \mathcal{M}_B \cap \mathcal{M}_A (1 + \alpha \ell_B(AB))^{-2}
  \end{align*}
  \]

If $\nu_{\text{max}} = \max \nu(AB)$ and if $r_{\text{max}}$ is the maximal ratio of edge lengths,

$$\nu_{\text{max}} \approx \ln(r_{\text{max}})$$