Z. Angew. Math. Phys. 64 (2013), 1177–1185

0044-2275/13/041177-9
published online November 23, 2012
DOI 10.1007/s00033-012-0277-x

Zeitschrift für angewandte Mathematik und Physik ZAMP

The optimal shape of a pipe

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Abstract. In shape optimization, recently the question arose, whether or not the cylindrical pipe has the optimal shape for the transport of an incompressible fluid. In this short note, a proof will be presented that a cylindrical pipe with Poiseuille's flow inside indeed is optimal for the transportation of an incompressible fluid under the criterion "energy dissipated by the fluid." The proof reduces the problem to the minimization of a two-dimensional Dirichlet's integral. This simpler problem can be solved with a symmetrization argument.

Mathematics Subject Classification (2010). 76D55.

Keywords. Shape optimization · Laminar fluid flow · Poiseuille's flow.

1. Introduction

It is well known that the drag of swimming bodies can be reduced by the construction of riblets on the surface [4]. With that in mind, it is a natural idea to transfer this construction to optimize the shape of a pipe for fluid transport. Recently, Henrot and Privat [2] did some work on that topic and presented a proof for the non-optimality of the cylindrical pipe. Here, it will be shown that this result relies only on the special choice of boundary values on the in- and outflow of the pipe. In a second step, these boundary values will be replaced by a physically motivated inflow condition. This allows then to show optimality of the cylindrical pipe with Poiseuille's flow inside.

This note is organized as follows: In Sects. 2 and 3, the result of Henrot and Privat will be discussed and a new expression for the first variation will be shown. An independent approach in Sect. 4, replacing the boundary conditions by an inflow condition, shows optimality of Poiseuille's flow through a cylindrical pipe.

2. A first model for the problem

Consider a pipe $\Omega \subset \mathbb{R}^3$ of length L, volume V, with in- and outflow E and S given as two discs of radius R and lateral boundary Γ, with $\partial\Omega = \overline{E} \cup \overline{\Gamma} \cup \overline{S}$, cf. Fig. 1.

Through the pipe runs a fluid of velocity $\mathbf{u} : \Omega \to \mathbb{R}^3$, determined by the stationary incompressible Navier–Stokes system

$$\left.\begin{aligned} -\mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{0} \qquad & \text{in } \Omega, \\ \nabla\cdot\mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_E \qquad & \text{on } E, \\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \Gamma, \\ T(\mathbf{u},p)\cdot\mathbf{n} = \mathbf{h} \qquad & \text{on } S. \end{aligned}\right\} \tag{2.1}$$

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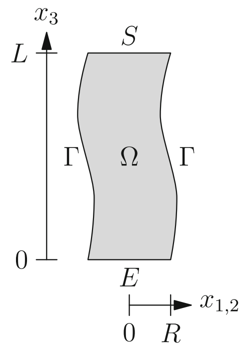

Fig. 1. Example of a pipe

Here, p denotes the pressure, μ the viscosity, $\mathbf{n}$ the outer unit normal, and $T(\mathbf{u}, p)$ the stress tensor

$$T(\mathbf{u}, p) = -p\mathbb{1} + 2\mu\varepsilon(\mathbf{u}), \quad \varepsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^t\right),$$

where ε is the stretching tensor. The task is to find the optimal shape of the pipe, that is, the shape of the lateral boundary Γ, in a certain class of domains (see [2] for details), which minimizes the energy dissipated by the fluid

$$J(\Omega) = 2\mu \int_\Omega |\varepsilon(\mathbf{u})|^2 \, \mathrm{d}\mathbf{x}$$

under a volume constraint, where $\mathbf{u} = \mathbf{u}(\Omega)$ is the solution of system (2.1). To assure a well-posed problem, the viscosity μ has to be large enough. Therefore, only laminar flows are considered.

Finally, the boundary conditions have to be specified. The lateral boundary Γ is the actual surface of the pipe, where a no-slip condition $\mathbf{u} = \mathbf{0}$ is used. In- and outflow E and S are non-physical boundaries, where the boundary conditions are not prescribed by physical laws, so that their choice is a question of modeling. In [2], non-homogeneous Dirichlet and Neumann boundary values

$$\begin{aligned} \mathbf{u}_E &= \left(0, 0, c(x_1^2 + x_2^2 - R^2)\right), \\ \mathbf{h} &= (2\mu c x_1, 2\mu c x_2, -p_1) \end{aligned} \tag{2.2}$$

are prescribed. These are the parabolic inflow and the normal component of the stress tensor taken from Poiseuille's flow through a cylinder of radius R, i.e.,

$$\begin{aligned} \mathbf{u}_P &= \left(0, 0, c(x_1^2 + x_2^2 - R^2)\right), \\ p &= 4\mu c(x_3 - L) + p_1, \end{aligned} \tag{2.3}$$

where $c < 0$ is a negative constant and p_1 is a fixed pressure value.

In [2, Th. 4], the main result is that for this special choice of boundary values (2.2), the first variation of $J(\Omega)$ under perturbation of the cylinder Ω is not zero.

But to choose boundary values from Poiseuille's flow for every pipe Ω, is only one possibility to define the mathematical model. Here, in Sect. 3, it will be shown that the first variation $\mathrm{d}J(\Omega, \mathbf{V})$ at the cylinder Ω only depends on the boundary behavior of the fluid's velocity at the outflow S (as long as the parabolical inflow is fixed). The calculated decrease in the energy dissipated by the fluid is a boundary effect and not related to the shape of the pipe.

3. The first variation of $J(\Omega)$

The cylinder Ω is considered under a perturbation of the form $\Omega_t = (\mathrm{Id} + t\mathbf{V})(\Omega)$, with a smooth, compactly supported vector field $\mathbf{V} \in C_c^\infty\left(\mathbb{R}^3 \setminus (\overline{E} \cup \overline{S})\right)$, so that in- and outflow E and S are not perturbed. In particular, $\mathbf{V} = \mathbf{0}$ on E and S. The first variation of $J(\Omega)$ is now defined as

$$\mathrm{d}J(\Omega, \mathbf{V}) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} J(\Omega_t).$$

The volume constraint is taken into account by a restriction to vector fields $\mathbf{V}$ that fulfill

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} |\Omega_t| = \int\limits_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, \mathrm{d}\sigma = 0. \tag{3.1}$$

Now, the following expression for the first variation

$$\mathrm{d}J(\Omega, \mathbf{V}) = 4\mu \int\limits_{S} \mathbf{u}' \cdot \varepsilon(\mathbf{u}) \, \mathbf{n} \, \mathrm{d}\sigma \tag{3.2}$$

will be shown. Here, Ω is the cylinder, $\mathbf{u} = \mathbf{u}_P$ is Poiseuille's flow (2.3), $\mathbf{n}$ is the outer unit normal on S, and $\mathbf{u}'$ is the shape derivative of $\mathbf{u}$ given by $\mathbf{u}' = \dot{\mathbf{u}} - \mathbf{V} \cdot \nabla \mathbf{u}$, where

$$\dot{\mathbf{u}} = \lim_{t \searrow 0} \frac{\mathbf{u}(\Omega_t) \circ (\mathrm{Id} + t\mathbf{V}) - \mathbf{u}(\Omega)}{t} \tag{3.3}$$

is the material derivative (for details, see [6, Chap. 3], [7]). The following properties of $\mathbf{u}'$ can be found in [2, eq. (10)]

$$\nabla \cdot \mathbf{u}' = 0, \quad \mathbf{u}'|_E = \mathbf{0}, \quad \mathbf{u}'|_\Gamma = -\frac{\partial \mathbf{u}}{\partial \mathbf{n}} (\mathbf{V} \cdot \mathbf{n}) \tag{3.4}$$

and will be used later on.

The proof of (3.2) starts with

$$\mathrm{d}J(\Omega, \mathbf{V}) = 4\mu \int\limits_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}') \, \mathrm{d}\mathbf{x} + 2\mu \int\limits_{\partial\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u})(\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}\sigma$$

(see [2, eq. (11)]), where $A : B = \sum_{ij} A_{ij} B_{ij}$ denotes the usual inner product for matrices. Integration by parts leads to

$$\begin{aligned}\mathrm{d}J(\Omega, \mathbf{V}) = &-2\mu \int\limits_{\Omega} \mathbf{u}' \cdot \Delta \mathbf{u} \, \mathrm{d}\mathbf{x} + 4\mu \int\limits_{\partial\Omega} \mathbf{u}' \cdot \varepsilon(\mathbf{u}) \, \mathbf{n} \, \mathrm{d}\sigma \\ &+ 2\mu \int\limits_{\partial\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u})(\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}\sigma.\end{aligned}$$

Since $\mathbf{u} = \mathbf{u}_P$ (2.3), the derivatives are

$$\Delta \mathbf{u} = (0, 0, 4c), \quad \nabla \mathbf{u} = \begin{pmatrix} 0 & 0 & 2cx_1 \\ 0 & 0 & 2cx_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \varepsilon(\mathbf{u}) = \begin{pmatrix} 0 & 0 & cx_1 \\ 0 & 0 & cx_2 \\ cx_1 & cx_2 & 0 \end{pmatrix}. \tag{3.5}$$

Gauss' theorem together with (3.4) yields (note that $\mathbf{u}' \cdot \mathbf{n}|_\Gamma = 0$)

$$\int\limits_{\Omega \cap \{x_3 = \mathrm{const.}\}} u_3' d(x_1, x_2) = 0$$

and with $\Delta \mathbf{u}$ given explicitly in (3.5) this leads to

$$\int_\Omega \mathbf{u}' \cdot \Delta \mathbf{u} \, \mathrm{d}\mathbf{x} = 0.$$

With the use of $\mathbf{n}|_\Gamma = R^{-1}(x_1, x_2, 0)$, it can be seen that

$$\int_\Gamma \mathbf{u}' \cdot \varepsilon(\mathbf{u}) \, \mathbf{n} \, \mathrm{d}\sigma = -\int_\Gamma \mathbf{n} \cdot \nabla \mathbf{u} \, \varepsilon(\mathbf{u}) \, \mathbf{n} (\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}\sigma = -\int_\Gamma \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) (\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}\sigma$$

and thus

$$\mathrm{d}J(\Omega, \mathbf{V}) = 4\mu \int_S \mathbf{u}' \cdot \varepsilon(\mathbf{u}) \, \mathbf{n} \, \mathrm{d}\sigma - 2\mu \int_\Gamma \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) (\mathbf{V} \cdot \mathbf{n}) \, \mathrm{d}\sigma.$$

Finally, the last term on the right-hand side vanishes, due to the volume constraint (3.1) and

$$\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u})|_\Gamma = 2c^2 R^2 = \mathrm{const}.$$

This concludes the proof of expression (3.2) for the first variation.

In (3.2), the term

$$\varepsilon(\mathbf{u}) \, \mathbf{n}|_S = (cx_1, cx_2, 0)$$

is fixed, so the first variation $\mathrm{d}J(\Omega, \mathbf{V})$ only depends on $\mathbf{u}'|_S$, that is, the derivative of the trace of $\mathbf{u}$ on S. When that trace $\mathbf{u}|_S$ is perturbed in a direction away from the center of S, the energy dissipated by the fluid $J(\Omega)$ will decrease. One can do the same calculations for a fixed cylindrical pipe, where the boundary values on the outflow S are perturbed. This leads to the same decrease in the energy dissipated by the fluid.

In the mathematical model of Henrot and Privat [2], the perturbation of the cylindrical pipe Ω via the vector field $\mathbf{V}$ induces a change in the outflow behavior of the fluid, which leads to a decrease in the energy dissipated by the fluid. But this connection is non-physical and just a result of the choice of boundary values (2.2).

4. The optimal shape of a pipe

In this section, a different approach to the optimization problem replaces the boundary values on E and S by a physical inflow condition $\int_E u_3 \, \mathrm{d}\sigma = f$, that is, the amount of liquid running through the pipe is prescribed. With a symmetrization argument, the optimality of Poiseuille's flow (2.3) with regard to the energy dissipated by the fluid will be shown under a volume constraint on the pipe and for regular divergence-free velocity fields $\mathbf{u}$ with a no-slip condition on the lateral boundary. First, it will be shown that Poiseuille's flow minimizes Dirichlet's integral. The connection to the energy dissipated by the fluid will be obtained with an integration by parts, where the boundary term will be discussed separately. For the proof, it is not necessary to restrict the velocity fields $\mathbf{u}$ to the solutions of Navier–Stokes' equations.

Now, the classes of admissible pipes and velocity fields will be introduced.

Definition 4.1. (*The class of admissible pipes*) Let $\Omega \subset \mathbb{R}^3$ be a bounded domain contained in the strip $\{\mathbf{x} \in \mathbb{R}^3 | \ 0 < x_3 < L\}$, $L > 0$, with Lipschitz boundary and finite volume $|\Omega| = V < \infty$. Let the inflow $E \subset \overline{\Omega}$ be a two-dimensional non-empty open set with Lipschitz boundary contained in the plane $\{\mathbf{x} \in \mathbb{R}^3 | \ x_3 = 0\}$. The same is assumed for the outflow S contained in the plane $\{\mathbf{x} \in \mathbb{R}^3 | \ x_3 = L\}$. Let additionally $\Omega^a = \{\mathbf{x} \in \Omega | \ 0 < x_3 < a\}$ have a Lipschitz boundary for almost every $0 < a < L$ and let almost every cross-section $\Omega_a = \{\mathbf{x} \in \Omega | \ x_3 = a\}$ have a Lipschitz boundary (in two dimensions). The lateral boundary will be denoted $\Gamma = \partial\Omega \setminus (E \cup S)$.

For given $0 < V < \infty$, the class of all such Ω will be denoted O_V.

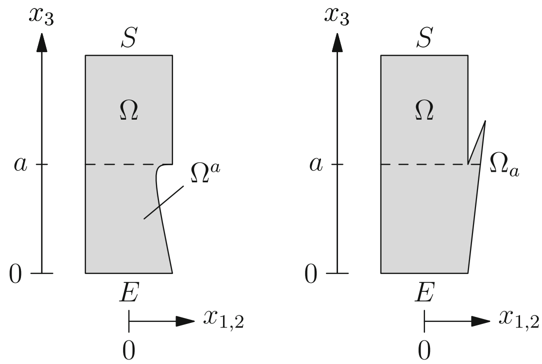

FIG. 2. This figure shows two examples, where the Lipschitz boundary of Ω does not carry on to the sets Ω^a and Ω_a.

Remark 4.2. One may think that the assumptions on the Lipschitz boundary of Ω^a and Ω_a are redundant and already implied by the Lipschitz boundary of Ω. Obviously, these implications are false if the "almost" is omitted, cf. Fig. 2.

The construction in [1] shows that for the set Ω^a, this implication is also false, even when the "almost" is kept.

Definition 4.3. (*The class of admissible velocity fields*) Let $\Omega \in O_V$ be an admissible pipe and let $\mathbf{u} \in C^1(\overline{\Omega})^3$ be divergence free, that is, $\nabla \cdot \mathbf{u} = 0$, with zero boundary conditions on the lateral boundary, that is, $\mathbf{u} = \mathbf{0}$ on Γ and with the inflow condition $\int_E u_3 \,\mathrm{d}\sigma = f$.

For given $f \in \mathbb{R}$, the class of all such $\mathbf{u}$ will be denoted $U_f(\Omega)$.

Note that, due to the divergence-free condition, the inflow condition also holds for almost every cross-sectional area Ω_a and the outflow S

$$\int_E u_3 \,\mathrm{d}\sigma = \int_S u_3 \,\mathrm{d}\sigma = \int_{\Omega_a} u_3 \,\mathrm{d}\sigma = f. \tag{4.1}$$

The main theorem of this work can now be stated.

Theorem 4.4. (Main Theorem) *Let $\Omega_P \subset \{\mathbf{x} \in \mathbb{R}^3 |\ 0 < x_3 < L\}$ be the cylinder centered at the x_3-axis of length L, radius R, and with volume $V = \pi R^2 L$ and let $\mathbf{u}_P$ be Poiseuille's parabolical flow (2.3) on Ω_P, with the inflow condition $\int_{B_R(\mathbf{0})} (\mathbf{u}_P)_3 \,\mathrm{d}(x_1, x_2) = f$ that means $c = -\frac{2f}{\pi R^4}$. Then, for all $\Omega \in O_V$ and $\mathbf{u} \in U_f(\Omega)$, it holds*

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{u} \,\mathrm{d}\mathbf{x} \geq \int_{\Omega_P} \nabla \mathbf{u}_P : \nabla \mathbf{u}_P \,\mathrm{d}\mathbf{x}.$$

Here, in contrast to the last section, c does not have to be negative any more. Also note that, due to the inflow condition, $\mathbf{u}_P$ and $\mathbf{u}$ transport the same amount of liquid through the corresponding pipes and that Ω_P and Ω have the same volume.

Remark 4.5. In Theorem 4.4, it is not necessary for the velocity field $\mathbf{u}$ to solve Navier–Stokes' equations. Of course, when the velocity fields $U_f(\Omega)$ are restricted to solutions of Navier–Stokes' equations, $\mathbf{u}_P$ stays the minimizer of Dirichlet's integral, since $\mathbf{u}_P$ itself is a solution to Navier–Stokes' equations.

To prove this theorem, an estimate concerning Dirichlet's integral in two dimensions is needed.

Lemma 4.6. *Let $\omega \subset \mathbb{R}^2$ be a non-empty open bounded set with Lipschitz boundary, $v \in H_0^1(\omega)$ with $\int_\omega v \, \mathrm{d}\mathbf{x} = f$. Then, a lower bound for Dirichlet's integral is given by*

$$\int\limits_\omega \nabla v \cdot \nabla v \, \mathrm{d}\mathbf{x} \geq \frac{8\mathrm{f}^2}{\pi \mathrm{r}^4} = \frac{8\mathrm{f}^2\pi}{|\omega|^2}, \tag{4.2}$$

with $r > 0$ defined by $|\omega| = \pi r^2$.

Proof. Let $v^* \in H_0^1(B_r(\mathbf{0}))$ denote the symmetric-decreasing rearrangement of v (see [3, Sect. 3.3] for details). Dirichlet's integral is non-increasing under symmetrization [3, Lem. 7.17] and therefore

$$\int\limits_\omega \nabla v \cdot \nabla v \, \mathrm{d}\mathbf{x} \geq \int\limits_{B_r(\mathbf{0})} \nabla v^* \cdot \nabla v^* \, \mathrm{d}\mathbf{x}.$$

Note that $\int_{B_r(\mathbf{0})} v^* \, \mathrm{d}\mathbf{x} = \int_\omega |v| \, \mathrm{d}\mathbf{x} \geq |f|$. On a ball, the minimizer $\tilde{w}$ of Dirichlet's integral in $H_f = \{w \in H_0^1(B_r(\mathbf{0})) | \int_{B_r(\mathbf{0})} w \, \mathrm{d}\mathbf{x} = f\}$ is well known, namely $\tilde{w}(x_1, x_2) = -\frac{2f}{\pi r^4}(x_1^2 + x_2^2 - r^2)$ (a proof can be found in [6, Lem 5.1]). Hence,

$$\int\limits_{B_r(\mathbf{0})} \nabla w \cdot \nabla w \, \mathrm{d}\mathbf{x} \geq \int\limits_{B_r(\mathbf{0})} \nabla \tilde{w} \cdot \nabla \tilde{w} \, \mathrm{d}\mathbf{x} = \frac{8f^2}{\pi r^4} \quad \forall w \in H_f.$$

This concludes the proof of inequation (4.2). □

In the following Lemma, the function $r : (0, L) \to (0, \infty)$ can be considered a x_3-dependent radius of a pipe that is symmetric with respect to the x_3-axis.

Lemma 4.7. *Let $r : (0, L) \to (0, \infty)$ be measurable and $0 < R < \infty$, with*

$$\int\limits_0^L \pi r^2(z) \, \mathrm{d}z = \pi R^2 L = V.$$

Then, it holds

$$\int\limits_0^L \frac{1}{r^4(z)} \, \mathrm{d}z \geq \int\limits_0^L \frac{1}{R^4} \, \mathrm{d}z = \frac{L}{R^4}. \tag{4.3}$$

Proof. The definition $r_1(z) = r(z) - R$ leads to

$$\int\limits_0^L \pi \left(R^2 + 2Rr_1(z) + r_1^2(z)\right) \, \mathrm{d}z = \pi R^2 L$$

and therefore to

$$\int\limits_0^L r_1^2(z) \, \mathrm{d}z = -\int\limits_0^L 2Rr_1(z) \, \mathrm{d}z.$$

With the convexity of $r \mapsto r^{-4}$ for $r > 0$ and a first-order Taylor expansion, the desired result is obtained:

$$\frac{1}{r^4(z)} \geq \frac{1}{R^4} - 4\frac{1}{R^5}r_1(z) \Rightarrow \int_0^L \frac{1}{r^4(z)}\,\mathrm{d}z \geq \int_0^L \frac{1}{R^4} - 4\frac{1}{R^5}r_1(z)\,\mathrm{d}z$$
$$= \int_0^L \frac{1}{R^4}\,\mathrm{d}z + \frac{2}{R^6}\int_0^L r_1^2(z)\,\mathrm{d}z \geq \frac{L}{R^4}.$$

□

Now, the tools are ready to prove the main theorem 4.4.

Proof (Theorem 4.4). With the definition of $r_a > 0$ by $|\Omega_a| = \pi r_a^2$ and (4.1), the following estimates hold:

$$\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} \geq \int_\Omega (\partial_{x_1}u_3)^2 + (\partial_{x_2}u_3)^2\,\mathrm{d}\mathbf{x}$$
$$= \int_0^L \int_{\Omega_a} (\partial_{x_1}u_3)^2 + (\partial_{x_2}u_3)^2\,\mathrm{d}(x_1,x_2)\,\mathrm{d}a$$
$$\overset{(4.2)}{\geq} \int_0^L \frac{8f^2}{\pi r_a^4}\,\mathrm{d}a \overset{(4.3)}{\geq} \frac{8f^2 L}{\pi R^4} = \int_{\Omega_P} \nabla\mathbf{u}_P : \nabla\mathbf{u}_P\,\mathrm{d}\mathbf{x}.$$

□

It has been shown that Poiseuille's flow minimizes Dirichlet's integral. The term that has to be minimized, that is, the energy dissipated by the fluid, is connected to Dirichlet's integral via integration by parts (in case of sufficient regularity of $\mathbf{u}$):

Lemma 4.8. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded with Lipschitz boundary and let $\mathbf{u} \in H^2(\Omega)^N$ fulfill $\nabla \cdot \mathbf{u} = 0$. Then, it holds*

$$\int_\Omega \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u})\,\mathrm{d}\mathbf{x} = \frac{1}{2}\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{u}\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_{\partial\Omega} \mathbf{u}\cdot\nabla\mathbf{u}\,\mathbf{n}\,\mathrm{d}\sigma. \tag{4.4}$$

Proof. Integration by parts. □

The boundary integral in (4.4) vanishes in case of periodic boundary conditions

$$\mathbf{u}|_E = \mathbf{u}|_S, \quad \partial_{x_3}\mathbf{u}|_E = \partial_{x_3}\mathbf{u}|_S \tag{4.5}$$

(here, E and S have to have the same shape) and also in the case

$$\mathbf{u} \times \mathbf{n} = \mathbf{0} \quad \text{on } E \text{ and } S. \tag{4.6}$$

Periodic boundary conditions model a periodic pipe of infinite length. They are a good way to avoid artificial boundary effects. In the latter case of boundary conditions, the fluid enters and leaves the pipe parallel to the x_3-axis.

For both cases of boundary conditions (4.5) and (4.6), Poiseuille's flow minimizes the energy dissipated by the fluid exactly. In the case of a very long pipe and arbitrary boundary conditions on E and S, the boundary integral in (4.4), which extends just over the in- and outflow (note that $\mathbf{u}|_\Gamma = \mathbf{0}$), can be neglected against the volume integral, where the domain of integration tends to infinite measure. So Poiseuille's flow is also optimal in that approximation for very long pipes.

But even if the energy dissipation in the pipe Ω is decreased compared to Poiseuille's flow, it is not really an improvement. To see this, consider a longer pipe divided into two pipes Ω and Ω', cf. Fig. 3,

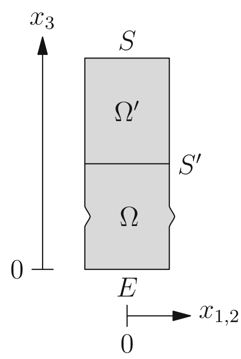

FIG. 3. A pipe divided into two parts

with an energy dissipation (4.4) for both parts. Let S' be the connection between the pipes. When the energy dissipation in Ω is decreased compared to Poiseuille's flow, ([2] states that this is possible), this is due to a decrease in the value of the boundary integral in (4.4), since Dirichlet's integral is optimal for Poiseuille's flow. Let $\mathbf{u}$ be a flow through both pipes Ω and Ω' with, for example, a negative integral

$$\int_{S'} \mathbf{u} \cdot \nabla \mathbf{u}\, \mathbf{n}\, \mathrm{d}\sigma < 0,$$

where $\mathbf{n}$ is the outer unit normal on Ω. Then, this value counts positive to the energy dissipation in Ω', since the normal vector changes its sign. So the energy dissipation that seems to be saved in Ω is just shifted to Ω'. Meanwhile, Dirichlet's integral is increased (or at least not decreased) compared to Poiseuille's flow. That is why there is no better flow than Poiseuille's flow.

Acknowledgments

For pipes with an axis of symmetry, the presented approach is contained in [6]. I am very grateful to Tobias Baier (TU Darmstadt) and Wolfgang Reichel (KIT), for their advice to the references which allow to include non-symmetric pipes. I would also like to thank Christian Wieners (KIT) for his support in the production of this note.

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(Received: August 7, 2012; revised: November 6, 2012)