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SHAPE OPTIMIZATION FOR A FLUID-ELASTICITY SYSTEM

JEAN-FRANÇOIS SCHEID AND JAN SOKOLOWSKI

Abstract. In this paper, we are interested in a shape optimization problem for a fluid-structure interaction system composed by an elastic structure immersed in a viscous incompressible fluid. The cost functional to minimize is an energy functional involving together the fluid and the elastic parts of the structure. The shape optimization problem is introduced in the 2-dimensional case. However the results in this paper are obtained for a simplified free-boundary 1-dimensional problem. We prove that the shape optimization problem is wellposed. We study the shape differentiability of the free-boundary 1-dimensional model. The full characterization of the associated material derivatives is given together with the shape derivative of the energy functional. A special case is explicitly solved, showing the relevancy of this shape optimization approach for a simplified free boundary 1-dimensional problem. The full model in two spatial dimensions is under studies now.

1. Introduction

Free boundary problems are classical models e.g., for phase transitions or contact problems in structural mechanics. The optimal control or shape optimization of free boundary problems are challenging fields of research in the calculus of variations and in the theory of nonlinear partial differential equations. The obtained results can be verified by using numerical methods specific for the models. The questions to be adressed within the shape optimization framework are the existence and uniqueness of optimal shapes as well as the necessary and sufficient optimality conditions. The velocity method of shape sensitivity analysis can be applied to shape optimization problems. The existence of topological derivatives for the energy type shape functionals in multiphysics can be considered.

An important class of free boundary problems [2] are variational inequalities [3]. The optimal control [1] and the shape optimization [10] of variational inequalities are well understood for unilateral constraints. In such a case the polyhedricity property of the solution with respect to the shape can be exploited. The concurrent approach is the penalization technique as it is described e.g., in [1]. The multiphysics models are new and important

Date: March 18, 2017.

2010 *Mathematics Subject Classification.* 35Q35, 35Q74, 35Q93, 49Q10, 74F10, 74B05.

Key words and phrases. Shape optimization, Fluid-structure interaction, Stokes equations coupled with linear elasticity.

branch of applied shape optimization. In this paper a simple model of this type is rigorously analyzed from the point of view of sensitivity analysis. We present an approach of shape optimization to fluid structure interaction which can be generalized to more complex structures.

We consider an elastic structure immersed in a viscous incompressible fluid. Let $\omega \subset\subset \Omega'_S \subset\subset \Omega \subset \mathbb{R}^2$ be three bounded domains where Ω'_S and Ω are simply-connected domains. The deformed elastic body occupies the domain $\Omega_S = \Omega'_S \setminus \overline{\omega} \subset \mathbb{R}^2$ and the elastic structure is attached to the inner fixed boundary $\partial\omega$. The fluid fills up a bounded domain $\Omega_F = \Omega \setminus \overline{\Omega'_S} = \Omega \setminus (\overline{\Omega_S} \cup \overline{\omega})$ surrounding the elastic body Ω_S. We denote by

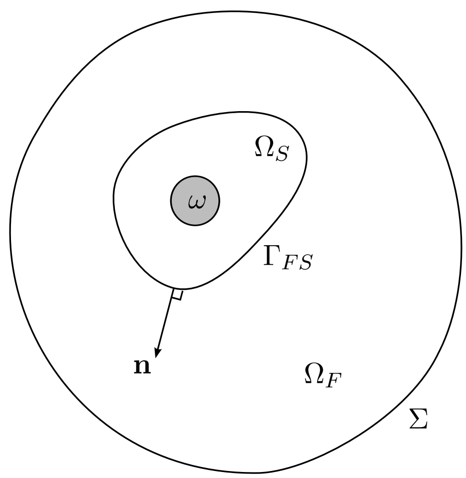

FIGURE 1. The geometry of the fluid-elasticity system

$\Gamma_{FS} = \partial\Omega_F \cap \partial\Omega_S$ the boundary between the fluid and the elastic structure and we have $\partial\Omega_F = \Gamma_{FS} \cup \Sigma$ where $\Sigma = \partial\Omega$. The boundary Σ corresponds also to the outer boundary of the fluid domain Ω_F (see Figure 1).

The fluid flow is governed by the Stokes equations for the velocity $\mathbf{u}$ and the pressure p of the fluid:

$$-\mathrm{div}\, \sigma(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega_F \tag{1.1}$$

$$\mathrm{div}\, \mathbf{u} = 0 \quad \text{in } \Omega_F \tag{1.2}$$

where $\sigma(\mathbf{u}, p) = 2\nu D(\mathbf{u}) - pI_d$ is the Cauchy stress tensor with the symetric strain tensor $D(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^\top\right)$. The fluid is subjected to a given force $\mathbf{f}$ and ν is the viscosity of the fluid. At the boundary of the fluid domain, we impose

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega_F = \Gamma_{FS} \cup \Sigma. \tag{1.3}$$

The elastic structure Ω_S is a deformation of a given reference bounded domain $\Omega_0 \subset \mathbb{R}^2$ by a mapping $\mathbf{X}$ i.e. $\Omega_S = \mathbf{X}(\Omega_0)$ (see Figure 2).

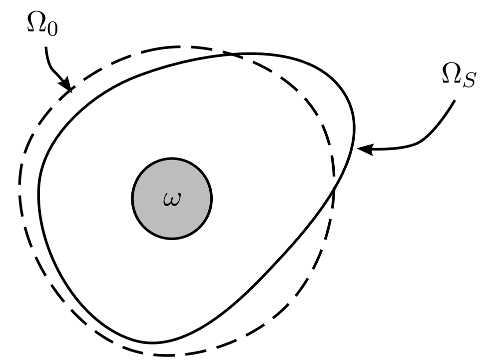

FIGURE 2. The elastic structure Ω_S is a deformation of a reference domain Ω_0

The deformation mapping is given by $\mathbf{X} = I_d + \mathbf{w}$ where $\mathbf{w}$ is the elastic displacement of the structure which satisfies the linearized elasticity equation

$$-\operatorname{div}\Pi(\mathbf{w}) = \mathbf{g} \quad \text{in } \Omega_0 \tag{1.4}$$

where Π is the second Piola-Kirchhoff stress tensor of the elastic structure given by

$$\Pi(\mathbf{w}) = \lambda \mathrm{tr}(D(\mathbf{w}))I_d + 2\mu D(\mathbf{w}) \tag{1.5}$$

with the Lamé coefficients $\lambda > 0, \mu > 0$. The elastic body is subjected to a given external force $\mathbf{g}$. Since the elastic structure is clamped to the inner boundary $\partial\omega$, we have $\mathbf{X}(\partial\omega) = \partial\omega$ and

$$\mathbf{w} = 0 \quad \text{on } \partial\omega. \tag{1.6}$$

We also denote by Γ_0 the outer boundary of Ω_0 and we have $\Gamma_{FS} = \mathbf{X}(\Gamma_0)$.

According to the action-reaction principle, we have

$$\int_{\Gamma_0} \Pi(\mathbf{w})\mathbf{n}_0 \cdot \mathbf{v} \circ \mathbf{X} \, d\Gamma = \int_{\Gamma_{FS}} \sigma(\mathbf{u}, p)\mathbf{n} \cdot \mathbf{v} \, d\Gamma$$

for all function $\mathbf{v}$ defined on Ω_F. We denote by $\mathbf{n}_0$ the normal unit vector directed outwards to the domain Ω_0 and $\mathbf{n}$ is the unit normal vector to Γ_{FS} directed from Ω_S to Ω_F. This leads to the local relation

$$\Pi(\mathbf{w})\mathbf{n}_0 = (\sigma(\mathbf{u}, p) \circ \mathbf{X}) \operatorname{cof}(\nabla\mathbf{X}) \, \mathbf{n}_0 \quad \text{on } \Gamma_0, \tag{1.7}$$

where $\operatorname{cof}(\nabla\mathbf{X})$ denotes the cofactor matrix of the jacobian matrix (for an invertible matrix A, we have $A^{-1} = \frac{1}{\det(A)}\mathrm{cof}(A)^\top$). The relation (1.7) can also be written on the boundary Γ_{FS} with

$$\sigma(\mathbf{u}, p)\mathbf{n} = \left(\Pi(\mathbf{w}) \circ \mathbf{X}^{-1}\right) \operatorname{cof}\left(\nabla\mathbf{X}^{-1}\right) \mathbf{n} \quad \text{on } \Gamma_{FS}. \tag{1.8}$$

In summary, the fluid-elasticity system for $(\mathbf{u}, p, \mathbf{w})$ reads as

$$
\begin{aligned}
-\operatorname{div} \sigma(\mathbf{u}, p) &= \mathbf{f} \quad \text{in } \Omega_F && (1.9)\\
\operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega_F && (1.10)\\
\mathbf{u} &= 0 \quad \text{on } \partial\Omega_F = \Gamma_{FS} \cup \Sigma && (1.11)\\
-\operatorname{div} \Pi(\mathbf{w}) &= \mathbf{g} \quad \text{in } \Omega_0 && (1.12)\\
\mathbf{w} &= 0 \quad \text{on } \partial\omega && (1.13)\\
\Pi(\mathbf{w})\mathbf{n}_0 &= (\sigma(\mathbf{u}, p) \circ \mathbf{X}) \operatorname{cof}(\nabla \mathbf{X})\, \mathbf{n}_0 \quad \text{on } \Gamma_0. && (1.14)
\end{aligned}
$$

In [7], the authors prove the existence of a solution to (1.9)–(1.14) using a fictitious domain approach and a fixed point procedure involving convergence of domains. This article contains in particular some interesting ideas that should be helpful for the shape optimization study associated to (1.9)–(1.14). We also mention the results in [6] where the existence of a solution to a coupled fluid-elasticity system for Stokes equation with a nonlinear elastic structure is established. A similar system to (1.9)–(1.14) has also been studied in [4] with the stationary Navier-Stokes equations and where the elastic structure is assumed to be a $\mathrm{S^t}$Venant–Kirchhoff material involving the *first* nonlinear Piola–Kirchhoff stress tensor (see also [11]).

Remark. Due to the incompressibility property of the fluid, the volume of the elastic structure is conserved during the deformation. Hence, we must have $|\Omega_S| = |\Omega_0|$ and the elastic displacement $\mathbf{w}$ satisfies

$$\int_{\Omega_0} \det(\nabla \mathbf{X})\, d\mathbf{y} = |\Omega_0|. \qquad (1.15)$$

We shall consider the shape optimization for a free boundary problem originated from the fluid-structure interaction. There is the following structure of coupled fields. Given a reference domain Ω_0 for the elasticity part of the system and a vector field $\mathbf{V}$ defined on Γ_0, we solve the elasticity subproblem and find the displacement field $\mathbf{w} = \mathbf{w}(\mathbf{V})$ on Γ_0 from the following boundary value problem with nonhomogeneous Neumann boundary condition

$$
\begin{aligned}
-\operatorname{div} \Pi(\mathbf{w}) &= \mathbf{g} \quad \text{in } \Omega_0 && (1.16)\\
\mathbf{w} &= 0 \quad \text{on } \partial\omega && (1.17)\\
\Pi(\mathbf{w})\mathbf{n}_0 &= \mathbf{V} \quad \text{on } \Gamma_0. && (1.18)
\end{aligned}
$$

In other words, we consider the Neumann-to-Dirichlet mapping associated with the elastic body. As a result, the deformation field $\mathbf{X} = \mathbf{X}(\mathbf{V})$ is determined for the boundary of the fluid subdomain

$$\mathbf{X} = I_d + \mathbf{w}.$$

The Stokes problem for $(\mathbf{u}, p) = (\mathbf{u}(\mathbf{V}), p(\mathbf{V}))$ is solved in the new subdomain Ω_F:

$$-\operatorname{div}\sigma(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega_F \tag{1.19}$$

$$\operatorname{div}\mathbf{u} = 0 \quad \text{in } \Omega_F \tag{1.20}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega_F = \Gamma_{FS} \cup \Sigma \tag{1.21}$$

and the fixed point condition for $\mathbf{V}$ on Γ_0 reads

$$\mathbf{V} = (\sigma(\mathbf{u}(\mathbf{V}), p(\mathbf{V})) \circ \mathbf{X}(\mathbf{V})) \operatorname{cof}(\nabla \mathbf{X}(\mathbf{V})) \, \mathbf{n}_0 \quad \text{on } \Gamma_0$$

The existence of solutions for the free boundary problem is already shown in [7] and in [6] for a nonlinear elastic structure. We are interested in the question of shape sensitivity analysis for the free boundary problem. The first problem to solve is the stability of the free boundary with respect to the sequence of domains Ω_0^k. Such sequence is produced by shape optimization techniques applied to a given shape functional. In such a case, $\Omega_0^k \to \Omega_0^\infty$ is the minimizing sequence and we want to assure that the corresponding fixed point conditions on Γ_0^k the outer boundary of Ω_0^k:

$$\mathbf{V}_k = (\sigma(\mathbf{u}_k(\mathbf{V}_k), p_k(\mathbf{V}_k)) \circ \mathbf{X}_k(\mathbf{V}_k)) \operatorname{cof}(\nabla \mathbf{X}_k(\mathbf{V}_k)) \, \mathbf{n}_0 \quad \text{on } \Gamma_0^k,$$

also converges to the fixed point condition in the limiting domain Ω_0^∞. To our best knowledge such results are not known in the literature.

Shape optimization formulation. We describe the shape optimization problem associated to (1.9)–(1.14). We aim to determine the optimal reference domain for which an energy type functional is minimum. More precisely, we want to determine a bounded domain $\Omega_0^\star \in \mathcal{U}_{ad}$ which minimizes

$$\min_{\Omega_0 \in \mathcal{U}_{ad}} J(\Omega_0) \tag{1.22}$$

where $\mathcal{U}_{ad}$ is the set of admissible domains :

$$\mathcal{U}_{ad} = \{\Omega_0 \subset \mathbb{R}^2,\ \Omega_0 = D_0 \setminus \overline{\omega} \ \text{ where } D_0 \text{ is a simply-connected,} \\ \text{bounded and regular domain containing } \overline{\omega}\}.$$

The energy functional $J(\Omega_0)$ is defined by

$$J(\Omega_0) = \int_{\Omega_F} |D(\mathbf{u})|^2 \, d\mathbf{x} + \eta \int_{\Omega_0} |D(\mathbf{w})|^2 \, d\mathbf{y} \tag{1.23}$$

with a given parameter $\eta > 0$ and where $\mathbf{u}$ and $\mathbf{X} = I_d + \mathbf{w}$ satisfy (1.9)–(1.14). In (1.23), we use the notation $|D(\mathbf{u})|^2 = D(\mathbf{u}) : D(\mathbf{u})$ where the double product « : » is defined by $A : B = \sum_{i,j} A_{ij} B_{ij}$ for two matrices A and B. The energy functional $J(\Omega_0)$ is composed by a fluid energy term and the elastic energy of deformation weighted by the parameter η.

2. A one-dimensional free-boundary model

In order to appreciate the relevance of the shape optimization problem presented in the introduction, we study a simplified one-dimensional free-boundary model. This system reads as follows. Let $y_0 \in (0,1)$ be given. We are seeking for two scalar functions u and w satisfying

$$\begin{aligned} -\partial_{xx}u(x) &= f(x),\ x \in (0,x^*) \\ u(0) &= u(x^*) = 0 \end{aligned} \tag{2.1}$$

$$\begin{aligned} -\partial_{yy}w(y) &= g(y),\ y \in (y_0,1) \\ w(1) &= 0 \end{aligned} \tag{2.2}$$

The (free) boundary point x^* is obtained by the deformation of the reference point y_0 with

$$x^* = x^*(y_0) = y_0 + w(y_0). \tag{2.3}$$

We also impose

$$\partial_x u(x^*) = \partial_y w(y_0) \tag{2.4}$$

which is the 1d-analogous of (1.14). We point out that the 1d-model does not account for the "volume conservation" constraint (1.15) derived in the 2d model.

The energy functional associated to the system (2.1),(2.2) is given by

$$J(y_0) = \int_0^{x^*} |\partial_x u|^2\,dx + \eta \int_{y_0}^1 |\partial_y w|^2\,dy \tag{2.5}$$

with a parameter $\eta > 0$. The one-dimensional shape optimization problem consists in finding the reference point $y_0 \in I_0$ that minimizes

$$\min_{y_0 \in I_0} J(y_0). \tag{2.6}$$

where $I_0 = \{y_0 \in (0,1) \text{ such that } x^* = x^*(y_0) \in (0,1)\}$.

2.1. **Well-posedness.** In this section, we show that for $y_0 \in (0,1)$ and for f and g *small enough*, the problem (2.1)–(2.4) admits a unique solution (u, w, x^*) with $x^* \in (0,1)$. This will be proved by a fixed point argument using the contraction mapping theorem.

Let us fix $y_0 \in (0,1)$, $f \in L^2(0,1)$ and $g \in L^2(0,1)$. We introduce the mapping T:

$$T(s) = y_0 + v(s, y_0) \quad \text{for } s \in (0,1), \tag{2.7}$$

where v is the solution of

$$\begin{aligned} -\partial_{yy}v(s,y) &= g(y),\ y \in (y_0,1) \\ v(s,1) &= 0 \\ \partial_y v(s,y_0) &= \partial_x u(s) \end{aligned} \tag{2.8}$$

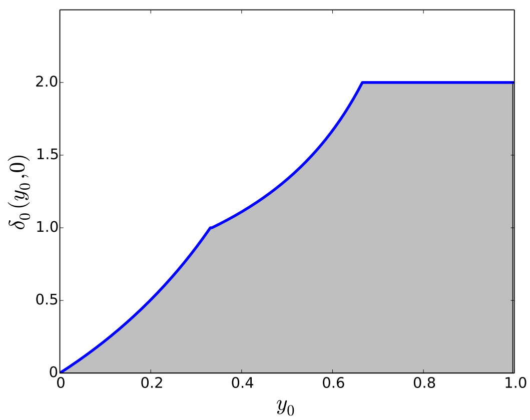

FIGURE 3. The bound $\delta_0(y_0,0)$ on f and g for the well-posedness of (2.1)–(2.4) for $y_0 \in (0,1)$.

$$\begin{aligned} -\partial_{xx}u(x) &= f(x),\ x \in (0,s) \\ u(0) &= u(s) = 0 \end{aligned} \tag{2.9}$$

For any $s \in (0,1)$, Problem (2.9) admits a unique solution $u = u(s,\cdot) \in H_0^1(0,s) \cap H^2(0,s)$. The derivative $\partial_x u$ is then continuous in $[0,s]$ and Problem (2.8) also admits a unique solution $v = v(s,\cdot) \in H^2(y_0,1)$. It is clear that $x^* \in (0,1)$ is a fixed point for T i.e. $x^* = T(x^*)$ if and only if $(u(x^*,\cdot), v(x^*,\cdot), x^*)$ is a solution of Problem (2.1)–(2.4). The following existence result holds.

Proposition 2.1. *Let $0 \le \varepsilon < 1$, $y_0 \in (\varepsilon,1)$ and $f,g \in L^\infty(0,1)$. There exists $\delta_0 = \delta_0(y_0,\varepsilon) > 0$ such that if $\|f\|_\infty + \|g\|_\infty \le \delta_0$ then Problem (2.1)–(2.4) admits a unique solution (u,w,x^*) with $u \in H^2(0,x^*)$, $w \in H^2(y_0,1)$ and $x^* \in (\varepsilon,1)$ which satisfies the following relation*

$$x^* = y_0 + \int_{y_0}^1 (1-y)g(y)\,dy + \frac{(1-y_0)}{x^*}\int_0^{x^*} xf(x)\,dx. \tag{2.10}$$

Moreover, δ_0 can be choosen as a non-decreasing function of y_0 with

$$\delta_0(y_0,\varepsilon) = 2\min(1, \frac{y_0-\varepsilon}{1-y_0}, \frac{1}{3(1-y_0)}) > 0. \tag{2.11}$$

Proof. Let $\varepsilon \in [0,1)$. We prove that for sufficiently small f and g, the mapping T defined by (2.7) maps the interval $(\varepsilon,1)$ into itself and T is a contraction mapping on $(\varepsilon,1)$. This ensures the existence and the uniqueness of a fixed point $x^* \in (\varepsilon,1)$ for T.

According to (2.7), if $|v(s,y_0)| < \min(y_0-\varepsilon, 1-y_0)$ for all $s \in (\varepsilon,1)$ then $T(s) \in (\varepsilon,1)$ for all $s \in (\varepsilon,1)$. Let $s \in (\varepsilon,1)$ be fixed. We estimate $v(s,y_0)$

with respect to f and g. To this end, let us write

$$v(s,y_0) = -\int_{y_0}^{1} \partial_y v(s,y)\, dy = -\int_{y_0}^{1} \partial_y v(s,y) \partial_y \varphi(y)\, dy,$$

with $\varphi(y) = y - 1$. Since $\varphi(1) = 0$ and $\partial_y \varphi \equiv 1$ in $(y_0, 1)$, we obtain by integrating by parts

$$\begin{aligned} v(s,y_0) &= \int_{y_0}^{1} \partial_{yy} v(s,y)\,(y-1)dy + \partial_y v(s,y_0)\,(y_0-1) \\ &= -\int_{y_0}^{1} g(y)\,(y-1)dy + \partial_y v(s,y_0)\,(y_0-1) \\ &= -\int_{y_0}^{1} g(y)\,(y-1)dy + \partial_x u(s)\,(y_0-1), \end{aligned} \tag{2.12}$$

thanks to the boundary condition in (2.8). In addition, starting from (2.9) we have

$$-\int_0^s \partial_{xx} u(x)\, \phi(x) dx = \int_0^s f(x)\, \phi(x) dx,$$

with $\phi(x) = x$. Integrating by parts, using $\phi(0) = 0$ and $\partial_x \phi \equiv 1$ in $(0, s)$ together with the boundary conditions for u in (2.9), we get

$$\partial_x u(s) = -\frac{1}{s}\int_0^s x f(x)\, dx. \tag{2.13}$$

Combining (2.12) with (2.13), we finally obtain

$$v(s,y_0) = \int_{y_0}^{1} (1-y) g(y)\, dy + \frac{(1-y_0)}{s}\int_0^s x f(x)\, dx. \tag{2.14}$$

We are now in position to estimate $v(s, y_0)$:

$$\begin{aligned} |v(s,y_0)| &\leq \|g\|_\infty \int_{y_0}^{1} (1-y)\, dy + \frac{(1-y_0)}{s}\|f\|_\infty \int_0^s x\, dx \\ &\leq \frac{(1-y_0)^2}{2}\|g\|_\infty + \frac{s}{2}(1-y_0)\|f\|_\infty \\ &\leq \frac{(1-y_0)}{2}\left(\|g\|_\infty + \|f\|_\infty\right) \end{aligned} \tag{2.15}$$

We choose f and g such that

$$\|g\|_\infty + \|f\|_\infty \leq 2\min(\frac{y_0-\varepsilon}{1-y_0}, 1) \tag{2.16}$$

so that we have $|v(s, y_0)| < \min(y_0 - \varepsilon, 1 - y_0)$ and thus $T(s) \in (\varepsilon, 1)$.

Now, we prove that T is a contraction mapping on $(0, 1)$. According to (2.14), we have, for any $s_1, s_2 \in (0, 1)$, $s_1 \neq s_2$,

$$\begin{aligned} T(s_1) - T(s_2) &= v(s_1,y_0) - v(s_2,y_0) \\ &= (1-y_0)\left(\frac{1}{s_1}\int_0^{s_1} x f(x)\, dx - \frac{1}{s_2}\int_0^{s_2} x f(x)\, dx\right). \end{aligned}$$

Without loss of generality we assume that $s_1 > s_2$ and we write

$$T(s_1) - T(s_2) = (1-y_0)\left(\left(\frac{1}{s_1} - \frac{1}{s_2}\right)\int_0^{s_2} xf(x)\,dx + \frac{1}{s_1}\int_{s_2}^{s_1} xf(x)\,dx\right).$$

This leads to

$$\begin{aligned} |T(s_1) - T(s_2)| &\leq (1-y_0)\|f\|_\infty \left(\left|\frac{1}{s_1} - \frac{1}{s_2}\right| \frac{{s_2}^2}{2} + \frac{1}{s_1}\left|\frac{{s_1}^2}{2} - \frac{{s_2}^2}{2}\right|\right) \\ &\leq \frac{(1-y_0)}{2}\|f\|_\infty \left(\frac{s_2}{s_1} + \frac{s_1+s_2}{s_1}\right)|s_1 - s_2|\,. \end{aligned}$$

Since $s_1 > s_2$, we obtain

$$|T(s_1) - T(s_2)| < \frac{3}{2}(1-y_0)\|f\|_\infty |s_1 - s_2| \tag{2.17}$$

We choose f such that

$$\|f\|_\infty \leq \frac{2}{3(1-y_0)}, \tag{2.18}$$

so that $|T(s_1) - T(s_2)| < |s_1 - s_2|$ and thus T is a contraction mapping on $(0,1)$.

Let $\delta_0 = \delta_0(y_0, \varepsilon) = 2\min(1, \frac{y_0-\varepsilon}{1-y_0}, \frac{1}{3(1-y_0)}) > 0$. Combining (2.16) with (2.18), we conclude that if $\|g\|_\infty + \|f\|_\infty \leq \delta_0$ then T admits a unique fixed point $x^* \in (\varepsilon, 1)$ which thus satisfies (2.10). □

2.2. **A fixed domain formulation.** In this section we transform the 1d fluid-elastic system (2.1)-(2.4) in a nonlinear problem posed in reference intervals. Let us fix two reference points $\hat{x}_0$, $\hat{y}_0 \in (0,1)$. For given s and $t \in (0,1)$, we introduce the one-to-one regular mappings φ_s and ϕ_t defined in $[0,1]$ such that

$$\begin{aligned} &\varphi_s([0,\hat{x}_0]) = [0,s] \ \text{ with } \ \varphi_s(0) = 0,\ \varphi_s(\hat{x}_0) = s \\ &\phi_t([\hat{y}_0,1]) = [t,1] \ \text{ with } \ \phi_t(\hat{y}_0) = t,\ \phi_t(1) = 1, \end{aligned} \tag{2.19}$$

with

$$\varphi_{\hat{x}_0} \equiv I_d, \quad \phi_{\hat{y}_0} \equiv I_d. \tag{2.20}$$

We suppose that $\varphi_s \in C^2([0,1])$ for all $s \in (0,1)$ and $s \mapsto \varphi_s(x)$ belongs to $C^1(0,1)$ for all $x \in [0,1]$. Similarly, we suppose $\phi_t \in C^2([0,1])$ for all $t \in (0,1)$ and $t \mapsto \phi_t(y)$ belongs to $C^1(0,1)$ for all $y \in [0,1]$. We have that $\varphi'_s > 0$ in $[0,\hat{x}_0]$, for all $s \in (0,1)$ and $\phi'_t > 0$ in $[\hat{y}_0,1]$, for all $t \in (0,1)$.

Let (u, w, x^*) be the solution of (2.1)-(2.4). Then we define the following changes of variables

$$\begin{aligned} &\hat{u}(\hat{x}) = u(x),\ \hat{f}(\hat{x}) = f(x) \text{ with } x = \varphi_{x^*}(\hat{x}) \text{ for } \hat{x} \in [0,\hat{x}_0], \\ &\hat{w}(\hat{y}) = w(y),\ \hat{g}(\hat{y}) = g(y) \text{ with } y = \phi_{y_0}(\hat{y}) \text{ for } \hat{y} \in [\hat{y}_0,1]. \end{aligned} \tag{2.21}$$

The functions $(\hat{u}, \hat{w})$ satisfy the following nonlinear problem posed in the reference intervals $[0, \hat{x}_0]$ and $[\hat{y}_0, 1]$:

$$\begin{array}{rcl} -\partial_{\hat{x}}\left(\frac{1}{\varphi'_{x^*}(\hat{x})}\partial_{\hat{x}}\hat{u}(\hat{x})\right) & = & \varphi'_{x^*}(\hat{x})\,\hat{f}(\hat{x}), \quad \hat{x} \in (0, \hat{x}_0) \\ \hat{u}(0) & = & \hat{u}(\hat{x}_0) \;=\; 0 \end{array} \tag{2.22}$$

$$\begin{array}{rcl} -\partial_{\hat{y}}\left(\frac{1}{\phi'_{y_0}(\hat{y})}\partial_{\hat{y}}\hat{w}(\hat{y})\right) & = & \phi'_{y_0}(\hat{y})\,\hat{g}(\hat{y}), \quad \hat{y} \in (\hat{y}_0, 1) \\ \hat{w}(1) & = & 0 \end{array} \tag{2.23}$$

$$\frac{1}{\varphi'_{x^*}(\hat{x}_0)}\partial_{\hat{x}}\hat{u}(\hat{x}_0) \;=\; \frac{1}{\phi'_{y_0}(\hat{y}_0)}\partial_{\hat{y}}\hat{w}(\hat{y}_0). \tag{2.24}$$

The mappings φ_{x^*} and ϕ_{y_0} can be chosen for instance, as the unique solutions of the two problems

$$\begin{array}{cc} \varphi''_{x^*} = 0 \text{ in } (0, \hat{x}_0) & \phi''_{y_0} = 0 \text{ in } (\hat{y}_0, 1) \\ \varphi_{x^*}(0) = 0,\ \varphi_{x^*}(\hat{x}_0) = x^* & \phi_{y_0}(\hat{y}_0) = y_0,\ \phi_{y_0}(1) = 1 \end{array} \tag{2.25}$$

that is

$$\begin{array}{l} \varphi_{x^*}(\hat{x}) = \dfrac{x^*}{\hat{x}_0}\hat{x} = \dfrac{y_0 + \hat{w}(\hat{y}_0)}{\hat{x}_0}\hat{x} \quad \text{for } \hat{x} \in [0, \hat{x}_0] \\ \phi_{y_0}(\hat{y}) = \dfrac{(y_0 - 1)}{(\hat{y}_0 - 1)}(\hat{y} - 1) + 1 \quad \text{for } \hat{y} \in [\hat{y}_0, 1] \end{array} \tag{2.26}$$

With that choices for φ_{x^*} and ϕ_{y_0}, the unknows $(\hat{u}, \hat{w})$ satisfy

$$\begin{array}{rcl} -\partial_{\hat{x}\hat{x}}\hat{u}(\hat{x}) & = & (\frac{y_0 + \hat{w}(\hat{y}_0)}{\hat{x}_0})^2 \hat{f}(\hat{x}), \quad \hat{x} \in (0, \hat{x}_0) \\ \hat{u}(0) & = & \hat{u}(\hat{x}_0) \;=\; 0 \\ -\partial_{\hat{y}\hat{y}}\hat{w}(\hat{y}) & = & (\frac{y_0 - 1}{\hat{y}_0 - 1})^2 \hat{g}(\hat{y}), \quad \hat{y} \in (\hat{y}_0, 1) \\ \hat{w}(1) & = & 0 \\ (\frac{\hat{x}_0}{y_0 + \hat{w}(\hat{y}_0)})\partial_{\hat{x}}\hat{u}(\hat{x}_0) & = & (\frac{\hat{y}_0 - 1}{y_0 - 1})\partial_{\hat{y}}\hat{w}(\hat{y}_0) \end{array} \tag{2.27}$$

2.3. **Existence of an optimal interval.** We shall prove that the optimal problem (2.5),(2.6) admits an optimal reference point y_0. More precisely, we have the following result

Proposition 2.2. *Let* $0 < \varepsilon_1 < \varepsilon_2 < 1$ *and* $f,\, g \in L^\infty(0,1)$*. There exists* $\eta_0 = \eta_0(\varepsilon_1) > 0$ *such that if* $\|f\|_\infty + \|g\|_\infty \le \eta_0$ *then there exists* $y_0^* \in [\varepsilon_1, \varepsilon_2]$ *that realizes* $\displaystyle\min_{y_0 \in [\varepsilon_1, \varepsilon_2]} J(y_0)$.

Proof. We fix $0 < \varepsilon_1 < \varepsilon_2 < 1$. We define $\eta_0(\varepsilon_1) = \delta_0(\varepsilon_1, \varepsilon_1/2) > 0$ where δ_0 is given by (2.11) in Proposition 2.1. We choose $f,\, g \in L^\infty(0,1)$ such that $\|f\|_\infty + \|g\|_\infty \le \eta_0(\varepsilon_1) = \delta_0(\varepsilon_1, \varepsilon_1/2)$. Since δ_0 is a non-decreasing function of y_0, we have $\eta_0(\varepsilon_1) \le \delta_0(y_0, \varepsilon_1/2)$ for all $y_0 \in [\varepsilon_1, \varepsilon_2]$. According to Proposition 2.1, Problem (2.1)–(2.4) admits a unique solution for all $y_0 \in [\varepsilon_1, \varepsilon_2]$, with $x^* \in [\varepsilon_1/2, 1)$. Thus, J is well-defined in $[\varepsilon_1, \varepsilon_2]$. Let $(y_n)_{n\ge 1} \in [\varepsilon_1, \varepsilon_2]$ be a minimizing sequence of J i.e. $\lim_{n\to+\infty} J(y_n) = \inf_{y_0 \in [\varepsilon_1, \varepsilon_2]} J(y_0)$. There exists a subsequence still denoted y_n and $y_0^* \in [\varepsilon_1, \varepsilon_2]$ such that

$\lim_{n\to+\infty} y_n = y_0^*$. We have to prove that $\lim_{n\to+\infty} J(y_n) = J(y_0^*)$. We denote by $(u_n, w_n, x_n^*) \in H^2(0, x_n^*) \times H^2(y_n, 1) \times [\varepsilon_1/2, 1)$ the solution of

$$
\begin{aligned}
-\partial_{xx} u_n(x) &= f(x),\ x \in (0, x_n^*) \\
u_n(0) &= u_n(x_n^*) = 0 \\
-\partial_{yy} w_n(y) &= g(y),\ y \in (y_n, 1) \\
w_n(1) &= 0 \\
\partial_x u_n(x_n^*) &= \partial_y w_n(y_n) \\
x_n^* &= y_n + w_n(y_n)
\end{aligned}
\tag{2.28}
$$

According to Section 2.2, we transform the system (2.28) on a fixed domain independent of n by setting $\hat{u}_n(\hat{x}) = u_n(x)$ with $x = \varphi(\hat{x})$ for $\hat{x} \in [0, \hat{y}_0^*]$ and $\hat{w}_n(\hat{y}) = w_n(y)$ with $y = \phi(\hat{y})$ for $\hat{y} \in [\hat{y}_0^*, 1]$. The functions φ and ϕ (see (2.26)) are given by

$$
\begin{aligned}
\varphi(\hat{x}) &= \frac{y_n + \hat{w}_n(y_0^*)}{y_0^*}\hat{x} \quad \text{for } \hat{x} \in [0, \hat{y}_0^*] \\
\phi(\hat{y}) &= \frac{(y_n - 1)}{(y_0^* - 1)}(\hat{y} - 1) + 1 \quad \text{for } \hat{y} \in [\hat{y}_0^*, 1]
\end{aligned}
\tag{2.29}
$$

The functions $(\hat{u}_n, \hat{w}_n)$ satisfy

$$
\begin{aligned}
-\partial_{\hat{x}\hat{x}} \hat{u}_n(\hat{x}) &= \left(\tfrac{y_n + \hat{w}_n(y_0^*)}{y_0^*}\right)^2 \hat{f}(\hat{x}), \quad \hat{x} \in (0, y_0^*) \\
\hat{u}_n(0) &= \hat{u}_n(y_0^*) = 0
\end{aligned}
\tag{2.30}
$$

$$
\begin{aligned}
-\partial_{\hat{y}\hat{y}} \hat{w}_n(\hat{y}) &= \left(\tfrac{y_n - 1}{y_0^* - 1}\right)^2 \hat{g}(\hat{y}), \quad \hat{y} \in (y_0^*, 1) \\
\hat{w}_n(1) &= 0 \\
\left(\tfrac{y_0^*}{y_n + \hat{w}_n(y_0^*)}\right)\partial_{\hat{x}} \hat{u}_n(y_0^*) &= \left(\tfrac{y_0^* - 1}{y_n - 1}\right)\partial_{\hat{y}} \hat{w}_n(y_0^*)
\end{aligned}
\tag{2.31}
$$

Since $x_n^* = y_n + w_n(y_n) = y_n + \hat{w}_n(y_0^*) \in [\varepsilon_1/2, 1)$, we deduce from (2.30) that $\|\hat{u}_n\|_{H^2(0, y_0^*)} \le C$ where $C > 0$ is a constant independent of n. Then there exists a subsequence still denoted $\hat{u}_n$ and $\hat{u}_0 \in H^2(0, y_0^*)$ such that $\hat{u}_n \rightharpoonup_{n\to+\infty} \hat{u}_0$ weakly in $H^2(0, y_0^*)$. From (2.31), we deduce that $\|\hat{w}_n\|_{H^2(y_0^*, 1)} \le C'$ where $C' > 0$ is a constant independent of n. Then there exists a subsequence still denoted $\hat{w}_n$ and $\hat{w}_0 \in H^2(y_0^*, 1)$ such that $\hat{w}_n \rightharpoonup_{n\to+\infty} \hat{w}_0$ weakly in $H^2(y_0^*, 1)$ and $\hat{w}_0$ satisfies

$$
\begin{aligned}
-\partial_{\hat{y}\hat{y}} \hat{w}_0(\hat{y}) &= \hat{g}(\hat{y}), \quad \hat{y} \in (y_0^*, 1) \\
\hat{w}_0(1) &= 0.
\end{aligned}
\tag{2.32}
$$

Since $x_n^* = y_n + \hat{w}_n(y_0^*)$ and due to the compactness of the embedding $H^2(y_0^*, 1) \hookrightarrow C^1([y_0^*, 1])$, we deduce that $\lim_{n\to+\infty} x_n^* = x_0^*$ with

$$
x_0^* = y_0^* + \hat{w}_0(y_0^*) \in [\varepsilon_1/2, 1] \subset (0, 1]. \tag{2.33}
$$

In addition, we obtain that $\hat{u}_0$ satisfies

$$
\begin{aligned}
-\partial_{\hat{x}\hat{x}} \hat{u}_0(\hat{x}) &= \left(\tfrac{y_0^* + \hat{w}_0(y_0^*)}{y_0^*}\right)^2 \hat{f}(\hat{x}), \quad \hat{x} \in (0, y_0^*) \\
\hat{u}_0(0) &= \hat{u}_0(y_0^*) = 0
\end{aligned}
\tag{2.34}
$$

and due to the compactness of the embedding $H^2(0,y_0^*) \hookrightarrow C^1([0,y_0^*])$ we have

$$(\frac{y_0^*}{y_0+\hat{w}_0(y_0^*)})\partial_{\hat{x}}\hat{u}_0(y_0^*) = \partial_{\hat{y}}\hat{w}_0(y_0^*) \tag{2.35}$$

We transform the problem (2.34), (2.35) on the interval $(0,x_0^*)$ by using the change of variables $\hat{u}_0(\hat{x}) = u_0(x)$ with (see Section 2.2)

$$x = \frac{x_0^*}{y_0^*}\hat{x} = \frac{y_0^*+\hat{w}_0(y_0^*)}{y_0^*}\hat{x} \quad \text{ for } \hat{x}\in[0,y_0^*]. \tag{2.36}$$

Thus the function u_0 satisfies

$$\begin{array}{rcl} -\partial_{xx}u_0(x) &=& f(x),\ x\in(0,x_0^*) \\ u_0(0) &=& u_0(x_0^*) = 0 \\ \partial_x u_0(x_0^*) &=& \partial_{\hat{y}}\hat{w}_0(y_0^*) \end{array} \tag{2.37}$$

Moreover, using the change of variable (2.29) we have

$$\begin{array}{rcl} J(y_n) &=& \displaystyle\int_0^{x_n^*}|\partial_x u_n|^2\,dx + \eta\int_{y_n}^1|\partial_y w_n|^2\,dy \\ &=& \displaystyle(\frac{y_0^*}{y_n+\hat{w}_n(y_0^*)})\int_0^{y_0^*}|\partial_{\hat{x}}\hat{u}_n|^2\,d\hat{x} + \eta(\frac{y_0^*-1}{y_n-1})\int_{y_0^*}^1|\partial_{\hat{y}}\hat{w}_n|^2\,d\hat{y} \end{array}$$

We deduce that

$$\lim_{n\to+\infty}J(y_n) = (\frac{y_0^*}{y_0^*+\hat{w}_0(y_0^*)})\int_0^{y_0^*}|\partial_{\hat{x}}\hat{u}_0|^2\,d\hat{x} + \eta\int_{y_0^*}^1|\partial_{\hat{y}}\hat{w}_0|^2\,d\hat{y} \tag{2.38}$$

Using the change of variable (2.36) with (2.33) in the right hand side of (2.38), we obtain

$$\lim_{n\to+\infty}J(y_n) = \int_0^{x_0^*}|\partial_x u_0|^2\,dx + \eta\int_{y_0^*}^1|\partial_{\hat{y}}\hat{w}_0|^2\,d\hat{y} = J(y_0^*) \tag{2.39}$$

where $(u_0,\hat{w}_0)$ satisfies (2.32),(2.37). The proof is then complete. □

2.4. **Shape differentiability.** In this section, we prove the existence of the material derivatives associated to the solution (u,w) of the coupled problem (2.1)-(2.4). A full characterization of the material derivatives is given as the solution of an adjoint problem.

For a given $t\in(0,1)$, we consider the following problem for (u_t,w_t,x_t^*):

$$\begin{array}{rcl} -\partial_{xx}u_t(x) &=& f(x),\ x\in(0,x_t^*) \\ u_t(0) &=& u_t(x_t^*) = 0 \\ -\partial_{yy}w_t(y) &=& g(y),\ y\in(t,1) \\ w_t(1) &=& 0 \\ \partial_x u_t(x_t^*) &=& \partial_y w_t(t) \\ x_t^* &=& t+w_t(t). \end{array} \tag{2.40}$$

Let $y_0 \in (0,1)$ and $\gamma > 0$ given. We choose $t \in (y_0-\gamma, y_0+\gamma) \cap (0,1)$. We assume that the functions $f,\, g \in W^{1,\infty}(0,1)$ and

$$\begin{aligned}\|f\|_{L^\infty(0,1)} + \|g\|_{L^\infty(0,1)} \leq & \delta_0(y_0-\gamma, y_0-2\gamma) \\ & = 2\min\left(1, \frac{\gamma}{1-y_0+\gamma}, \frac{1}{3(1-y_0+\gamma)}\right)\end{aligned} \tag{2.41}$$

where δ_0 is given by (2.11). Since $\delta_0(y_0,\varepsilon)$ is a non-decreasing function of y_0, choosing $\varepsilon = y_0 - 2\gamma$ we have $\delta_0(y_0-\gamma, y_0-2\gamma) \leq \delta_0(t, y_0-2\gamma)$ for all $t \in (y_0-\gamma, y_0+\gamma)$. Then, according to Proposition 2.1, Problem (2.40) admits a unique solution $(u_t, w_t, x_t^*) \in H^2(0,x_t^*) \times H^2(t,1) \times (y_0-2\gamma, 1)$, for all $t \in (y_0-\gamma, y_0+\gamma) \cap (0,1)$.

We emphasize that the solution (u,w,x^*) of (2.1)-(2.4) coincides with the solution of (2.40) with $t = y_0$, i.e. $(u,w,x^*) = (u_{y_0}, w_{y_0}, x_{y_0}^*)$. Moreover, since we choose $f,\, g \in W^{1,\infty}(0,1)$, the solution of (2.1)-(2.4) has the additionnal regularity

$$(u,w) \in H^3(0,x^*) \times H^3(y_0,1). \tag{2.42}$$

We are dealing with a fixed domain formulation by using the one-to-one regular mappings φ_s and ϕ_t defined on $[0,1]$ such that (see Section 2.2) :

$$\begin{aligned}&\varphi_s([0,x^*]) = [0,s] \ \text{ with } \ \varphi_s(0)=0,\ \varphi_s(x^*)=s \\ &\phi_t([y_0,1]) = [t,1] \ \text{ with } \ \phi_t(y_0)=t,\ \phi_t(1)=1,\end{aligned} \tag{2.43}$$

with

$$\varphi_{x^*} \equiv I_d, \quad \phi_{y_0} \equiv I_d. \tag{2.44}$$

We suppose that $\varphi_s \in C^2([0,1])$ for all $s \in (0,1)$ and $s \mapsto \varphi_s(x)$ belongs to $C^1(0,1)$ for all $x \in [0,1]$. Similarly, we suppose $\phi_t \in C^2([0,1])$ for all $t \in (0,1)$ and $t \mapsto \phi_t(y)$ belongs to $C^1(0,1)$ for all $y \in [0,1]$. We have that $\varphi_s' > 0$ in $[0,x^*]$, for all $s \in (0,1)$ and $\phi_t' > 0$ in $[y_0,1]$, for all $t \in (0,1)$.

Following [5, p.13-14], we shall say that a map $\mathcal{F} : t \in \mathbb{R} \mapsto f(t) \in X$ where X is a Banach space, is weakly continous at $t = t_0$ if for any sequence $t_n \to t_0$ as $n \to \infty$, we have $f(t_n) \rightharpoonup f(t_0)$ weakly in X. The map $\mathcal{F}$ is weakly-differentiable at $t = t_0$ if for any sequence $t_n \to t_0$, there exists $f'(t_0) \in X$ such that $\frac{f(t_n)-f(t_0)}{t_n-t_0} \rightharpoonup f'(t_0)$ weakly in X as $n \to \infty$.

Proposition 2.3. *Let $y_0 \in (0,1)$ and $\gamma \in (0,1/4)$ given. We assume that $f,\, g \in W^{1,\infty}(0,1)$ satisfy (2.41). For all $t \in (y_0-\gamma, y_0+\gamma) \cap (0,1)$, we consider the solution (u_t, w_t, x_t^*) of (2.40) and let (φ_s, ϕ_t) be the mappings defined by (2.43),(2.44). Then, the map $\mathcal{F} : t \mapsto (u_t \circ \varphi_{x_t^*}, w_t \circ \phi_t, x_t^*) \in H^2(0,x^*) \times H^2(y_0,1) \times (0,1)$ defined for $t \in (y_0-\gamma, y_0+\gamma) \cap (0,1)$, is weakly-continuous and weakly-differentiable at $t = y_0$ and the associated material* *derivative* $(\dot{u}, \dot{w}, \dot{x}^*) \in H^2(0,x^*) \times H^2(y_0,1) \times \mathbb{R}$ *is the solution of*

$$(2.45)\qquad \begin{array}{rcl} -\partial_{xx}\dot{u} &=& -\dot{x}^*\,\partial_{xx}\left((\partial_x u)\dfrac{d\varphi_s}{ds}_{|s=x^*}\right) \quad \textit{in } (0,x^*) \\ \dot{u}(0) &=& \dot{u}(x^*) \;=\; 0 \\ -\partial_{yy}\dot{w} &=& -\partial_{yy}\left((\partial_y w)\dfrac{d\phi_t}{dt}_{|t=y_0}\right) \quad \textit{in } (y_0,1) \\ \dot{w}(1) &=& 0 \end{array}$$

$$(2.46)\qquad \partial_x\dot{u}(x^*) - \dot{x}^*\partial_x u(x^*)\frac{d}{ds}\big(\varphi'_s(x^*)\big)_{|s=x^*} = \partial_y\dot{w}(y_0) - \partial_y w(y_0)\frac{d}{dt}\big(\phi'_t(y_0)\big)_{|t=y_0}$$

Moreover, the derivative $\dot{x}^*$ *is given by*

$$(2.47)\qquad \dot{x}^* = \frac{1-d^*-(1-y_0)g(y_0)}{1+(1-y_0)(d^*/x^*-f(x^*))}$$

with $d^* = \dfrac{1}{x^*}\displaystyle\int_0^{x^*} xf(x)\,dx.$

Proof. We first prove that the map $\mathcal{F} : t \mapsto (u_t(\varphi_{x_t^*}), w_t(\phi_t), x_t^*)$ is weakly-continuous at $t = y_0$. More precisely, we shall prove that $x_t^* \to x^*$ and $u_t(\varphi_{x_t^*}) \rightharpoonup u$ weakly in $H^2(0,x^*)$, $w_t(\phi_t) \rightharpoonup w$ weakly in $H^2(y_0,1)$ as $t \to y_0$.

According to (2.10), for all $t \in (y_0-\gamma, y_0+\gamma) \cap (0,1)$, x_t^* satisfies

$$(2.48)\qquad x_t^* = t + \int_t^1 (1-y)g(y)\,dy + \frac{(1-t)}{x_t^*}\int_0^{x_t^*} xf(x)\,dx.$$

Since $x_t^* \in (0,1)$, there exists a subsequence $t_n \to y_0$ such that $x_{t_n}^* \to \tilde{x} \in [0,1]$ which satisfies

$$(2.49)\qquad \tilde{x} = y_0 + \int_{y_0}^1 (1-y)g(y)\,dy + \frac{(1-y_0)}{\tilde{x}}\int_0^{\tilde{x}} xf(x)\,dx.$$

Since x^* is the unique point satisfying (2.49) (see (2.10)), we have $\tilde{x} = x^*$. We can also prove that the whole sequence x_t^* is converging with $t \to y_0$. Thus, we have

$$(2.50)\qquad x_t^* \to x^* \quad \text{as } t \to y_0.$$

Now, we turn to the convergence of u_t and w_t. Using the changes of variables $\hat{u} = u_t(\varphi_{x_t^*})$ and $\hat{w} = w_t(\phi_t)$ (see (2.21)) with $x = \varphi_{x_t^*}(\hat{x})$ and $y = \phi_t(\hat{y})$, the system (2.40) becomes (see (2.22), (2.23) and (2.24)):

$$(2.51)\qquad \begin{array}{rcl} -\partial_x\left(\frac{1}{\varphi'_{x_t^*}}\partial_x\hat{u}\right) &=& \varphi'_{x_t^*}\,f(\varphi_{x_t^*}) \quad \text{in } (0,x^*) \\ \hat{u}(0) &=& \hat{u}(x^*) \;=\; 0 \end{array}$$

(2.52)
$$\begin{aligned} -\partial_y\left(\frac{1}{\phi_t'}\partial_y\hat{w}\right) &= \phi_t'\, g(\phi_t) \quad \text{in } (y_0,1) \\ \hat{w}(1) &= 1 \end{aligned}$$

(2.53)
$$\frac{1}{\varphi'_{x_t^*(x^*)}}\partial_x\hat{u}(x^*) = \frac{1}{\phi_t'(y_0)}\partial_y\hat{w}(y_0)$$

We introduce

(2.54)
$$c_{1,t} = \hat{u} - u = u_t(\varphi_{x_t^*}) - u \in H^2(0,x^*)$$

and substracting (2.51) with (2.40) at $t = y_0$ for u, we get

(2.55)
$$\begin{aligned} -\partial_x\left(\frac{1}{\varphi'_{x_t^*}}\partial_x c_{1,t}\right) - \partial_x\left(\left(\frac{1}{\varphi'_{x_t^*}} - 1\right)\partial_x u\right) &= \left(\varphi'_{x_t^*}\, f(\varphi_{x_t^*}) - f\right) \text{ in } (0,x^*) \\ c_{1,t}(0) &= c_{1,t}(x^*) \;=\; 0 \end{aligned}$$

Due to (2.50) and the fact that $\|\varphi_s'\|_{L^\infty(0,x^*)} \to 1$ as $s \to x^*$, we have

(2.56)
$$\left\|\frac{1}{\varphi'_{x_t^*}} - 1\right\|_{L^\infty(0,x^*)} \xrightarrow[t\to y_0]{} 0, \quad \|\varphi'_{x_t^*} f(\varphi_{x_t^*}) - f\|_{L^\infty(0,x^*)} \xrightarrow[t\to y_0]{} 0.$$

As a result, we deduce from (2.55) that for $|t - y_0|$ small enough,

$$\|c_{1,t}\|_{H^2(0,x^*)} \le C$$

where $C > 0$ does not depend on t. Thus, there exists a subsequence $t_n \to y_0$ and $c_1 \in H^2(0,x^*)$ such that $c_{1,t_n} \rightharpoonup c_1$ weakly in H^2 and c_1 satisfies

$$\begin{aligned} -\partial_{xx}c_1 &= 0 \text{ in } (0,x^*) \\ c_1(0) &= c_1(x^*) \;=\; 0 \end{aligned}$$

Thus, we have $c_1 \equiv 0$ in $(0,x^*)$ and in addition we can prove that the whole sequence $c_{1,t}$ is converging to 0. Thus,

(2.57)
$$u_t(\varphi_{x_t^*}) \rightharpoonup u \quad \text{weakly in } H^2(0,x^*) \text{ as } t \to y_0.$$

Moreover, from the compactness of the embedding of $H^2(0,x^*)$ into $C^1([0,x^*])$, we deduce that

(2.58)
$$\partial_x c_{1,t}(x^*) \to 0 \quad \text{as } t \to y_0,$$

that is

(2.59)
$$\partial_x u_t(\varphi_{x_t^*})(x^*) \to \partial_x u(x^*) \quad \text{as } t \to y_0.$$

Now, we introduce

(2.60)
$$c_{2,t} = \hat{w} - w = w_t(\phi_t) - w \in H^2(y_0,1).$$

Substracting (2.52) and (2.53) with (2.40) at $t = y_0$ for w, we get
(2.61)
$$\begin{aligned} -\partial_y\left(\tfrac{1}{\phi_t'}\partial_y c_{2,t}\right) - \partial_y\left((\tfrac{1}{\phi_t'}-1)\partial_y w\right) &= (\phi_t'\, g(\phi_t) - g) \text{ in } (y_0,1) \\ c_{2,t}(1) &= 0 \\ \tfrac{1}{\varphi'_{x_t^*}(x^*)}\partial_x c_{1,t}(x^*) + (\tfrac{1}{\varphi'_{x_t^*}(x^*)}-1)\partial_x u(x^*) &= \tfrac{1}{\phi_t'(y_0)}\partial_y c_{2,t}(y_0) + (\tfrac{1}{\phi_t'(y_0)}-1)\partial_y w(y_0) \end{aligned}$$

We deduce that for all $v \in H^1(y_0,1)$ with $v(1)=0$, we have

(2.62)
$$\begin{aligned} \int_{y_0}^1 \frac{1}{\phi_t'(y)}\partial_y c_{2,t}(y)\partial_y v(y)\,dy + \left(\frac{1}{\varphi'_{x_t^*}(x^*)}\partial_x c_{1,t}(x^*) + (\frac{1}{\varphi'_{x_t^*}(x^*)}-1)\partial_x u(x^*)\right) v(y_0) \\ = \int_{y_0}^1 \left(\frac{1}{\phi_t'(y)}-1\right)\partial_y w(y)\partial_y v(y)\,dy + \int_{y_0}^1 \big(\phi_t'(y) g(\phi_t(y)) - g(y)\big)\, v(y)\,dy, \end{aligned}$$

We recall that $\phi_t' > 0$ in $[y_0,1]$ and $\|\phi_t'\|_{L^\infty(y_0,1)} \to 1$ as $t \to y_0$. Then,

$$\|\frac{1}{\phi_t'} - 1\|_{L^\infty(y_0,1)} \xrightarrow[t\to y_0]{} 0, \quad \|\phi_t' g(\phi_t) - g\|_{L^\infty(0,x^*)} \xrightarrow[t\to y_0]{} 0. \tag{2.63}$$

We take $v = c_{2,t}$ in (2.62). Using (2.58) and the trace inequality $|v(y_0)| \le C\|\partial_y v\|_{L^2(y_0,1)}$ for all $v \in H^1(y_0,1)$ with $v(1)=0$, where C is independent of v, we obtain that for $|t-y_0|$ small enough,

$$\|c_{2,t}\|_{H^1(y_0,1)} \le C$$

where $C>0$ does not depend on t. Going back to the strong form (2.61), we get a uniform bound for $\|\partial_{xx} c_{2,t}\|_{L^2(y_0,1)}$ with respect to t and thus for $|t-y_0|$ small enough, we have

$$\|c_{2,t}\|_{H^2(y_0,1)} \le C \tag{2.64}$$

where $C>0$ does not depend on t. Thus, there exists a subsequence $t_n \to y_0$ and $c_2 \in H^2(y_0,1)$ such that $c_{2,t_n} \rightharpoonup c_2$ weakly in H^2 and c_2 satisfies

$$\begin{aligned} -\partial_{yy} c_2 &= 0 \text{ in } (y_0,1) \\ c_2(1) &= 0 \end{aligned} \tag{2.65}$$

We can prove that the whole sequence $c_{2,t}$ is converging. We have $c_{2,t}(y_0) \to c_2(y_0)$ as $t \to y_0$. Furthermore, since $c_{2,t}(y_0) = w(y_0) - w_t(\phi_t(y_0)) = -y_0 + x^* + t - x_t^*$, we deduce that $c_{2,t}(y_0) \to 0$ as $t \to y_0$ thanks to (2.50). Hence, we obtain that $c_2(y_0) = 0$ and using (2.65) we conclude that $c_2 \equiv 0$ in $(y_0,1)$. We have proved that

$$w_t(\phi_t) \rightharpoonup w \quad \text{weakly in } H^2(y_0,1) \text{ as } t \to y_0. \tag{2.66}$$

The properties (2.50),(2.57),(2.66) show that the map $\mathcal{F} : t \mapsto (u_t \circ \varphi_{x_t^*}, w_t \circ \phi_t, x_t^*)$ is weakly-continuous at $t = y_0$.

Now, let us prove the weak-differentiability of $\mathcal{F}$ at $t = y_0$. We first prove that the map $t \mapsto x_t^*$ is differentiable at $t = y_0$. Let us introduce

$$\tau_t = \frac{x_t^* - x^*}{h} \quad \text{with} \quad h = t - y_0. \tag{2.67}$$

Starting from (2.10) and (2.48), we obtain that τ_t satisfies the relation

$$\left(1 - \frac{(1-t)}{x_t^*}\big(S_t - d^*\big)\right)\tau_t = 1 + R_t - d^* \tag{2.68}$$

with

$$\begin{aligned} d^* &= \frac{1}{x^*}\int_0^{x^*} x f(x)\,dx \\ R_t &= \frac{1}{t - y_0}\int_t^{y_0} (1-y)g(y)\,dy \\ S_t &= \frac{1}{x_t^* - x^*}\int_{x^*}^{x_t^*} x f(x)\,dx \end{aligned}$$

We clearly have $\dfrac{|S_t|}{x_t^*} \le \|f\|_\infty$ and $\dfrac{|d^*|}{x_t^*} \le \dfrac{x^*}{2x_t^*}\|f\|_\infty$ and then

$$\left|\frac{(1-t)}{x_t^*}\big(S_t - d^*\big)\right| \le (1 - y_0 + \gamma)\left(1 + \frac{x^*}{2x_t^*}\right)\|f\|_\infty$$

for all $t \in (y_0 - \gamma, y_0 + \gamma)$. Since $x_t^* \to x^*$ as $t \to y_0$, we deduce that for $|t - y_0|$ small enough, we have

$$\left|\frac{(1-t)}{x_t^*}\big(S_t - d^*\big)\right| \le 2(1 - y_0 + \gamma)\|f\|_\infty.$$

The assumption (2.41) ensures that $\|f\|_\infty \le \dfrac{2\gamma}{1 - y_0 + \gamma}$ and then we obtain

$$\left|\frac{(1-t)}{x_t^*}\big(S_t - d^*\big)\right| \le 4\gamma$$

and therefore, for $|t - y_0|$ small enough,

$$1 - \frac{(1-t)}{x_t^*}\big(S_t - d^*\big) \ge 1 - 4\gamma > 0. \tag{2.69}$$

Hence τ_t is well defined by (2.68) for $|t - y_0|$ small enough. Moreover, when $t \to y_0$, we have

$$\begin{aligned} R_t &\to -(1 - y_0)g(y_0) \\ S_t &\to x^* f(x^*) \end{aligned} \tag{2.70}$$

Thus, there exists $\dot{x}^* \in \mathbb{R}$ such that

$$\tau_t \to \dot{x}^* \quad \text{as} \quad t \to y_0 \tag{2.71}$$

and we deduce from (2.68) and (2.70) that $\dot{x}^*$ satisfies

$$\dot{x}^* = \frac{1 - d^* - (1 - y_0)g(y_0)}{1 + (1 - y_0)(d^*/x^* - f(x^*))} \tag{2.72}$$

Now, we turn to the differentiability of $\hat{u}$ and $\hat{w}$. We define

$$
(2.73)\qquad \begin{aligned} d_{1,t} &= \frac{\hat{u}-u}{h} = \frac{u_t(\varphi_{x_t^*})-u}{h}, \\ d_{2,t} &= \frac{\hat{w}-w}{h} = \frac{w_t(\phi_t)-w}{h}, \qquad \text{with} \quad h = t-y_0. \end{aligned}
$$

The function $d_{1,t} \in H^2(0,x^*)$ satisfies

$$
(2.74)\qquad \begin{aligned} -\partial_x\left(\tfrac{1}{\varphi'_{x_t^*}}\partial_x d_{1,t}\right) - \partial_x\left(\tfrac{1}{h}(\tfrac{1}{\varphi'_{x_t^*}}-1)\partial_x u\right) &= \tfrac{1}{h}\left(\varphi'_{x_t^*}\, f(\varphi_{x_t^*}) - f\right) \text{ in } (0,x^*) \\ d_{1,t}(0) &= d_{1,t}(x^*) \;=\; 0 \end{aligned}
$$

From (2.50), (2.71), we deduce that

$$
(2.75)\qquad \left\| \frac{1}{h}\left(\frac{1}{\varphi'_{x_t^*}}-1\right) + \dot{x}^*\frac{d\varphi'_s}{ds}_{|s=x^*} \right\|_{L^\infty(0,x^*)} \longrightarrow 0 \quad \text{as } t\to y_0
$$

$$
(2.76)\qquad \left\| \frac{1}{h}\left(\varphi'_{x_t^*} f(\varphi_{x_t^*}) - f\right) - \dot{x}^*\frac{d}{ds}\left(\varphi'_s\, f(\varphi_s)\right)_{|s=x^*} \right\|_{L^\infty(0,x^*)} \longrightarrow 0 \quad \text{as } t\to y_0
$$

As a result, we deduce from (2.74) that for $|t-y_0|$ small enough,

$$
\|d_{1,t}\|_{H^2(0,x^*)} \le C
$$

where $C>0$ does not depend on t. Thus, there exists a subsequence $t_n \to y_0$ and $\dot{u} \in H^2(0,x^*)$ such that $d_{1,t_n} \rightharpoonup \dot{u}$ weakly in H^2 and $\dot{u}$ satisfies

$$
(2.77)\qquad \begin{aligned} -\partial_{xx}\dot{u} + \dot{x}^*\partial_x\left(\frac{d\varphi'_s}{ds}_{|s=x^*}\partial_x u\right) &= \dot{x}^*\frac{d}{ds}\left(\varphi'_s\, f(\varphi_s)\right)_{|s=x^*} \text{ in } (0,x^*) \\ \dot{u}(0) &= \dot{u}(x^*) \;=\; 0 \end{aligned}
$$

Using the fact that $u \in H^3(0,x^*)$ and $-\partial_{xxx}u = \partial_x f$ in $(0,x^*)$, we obtain by straightforward calculations that

$$
\partial_x\left(\frac{d\varphi'_s}{ds}_{|s=x^*}\partial_x u\right) - \frac{d}{ds}\left(\varphi'_s\, f(\varphi_s)\right)_{|s=x^*} = \partial_{xx}\left((\partial_x u)\frac{d\varphi_s}{ds}_{|s=x^*}\right) \quad \text{in } (0,x^*)
$$

Then (2.77) becomes

$$
(2.78)\qquad \begin{aligned} -\partial_{xx}\dot{u} &= -\dot{x}^*\,\partial_{xx}\left((\partial_x u)\frac{d\varphi_s}{ds}_{|s=x^*}\right) \quad \text{in } (0,x^*) \\ \dot{u}(0) &= \dot{u}(x^*) \;=\; 0 \end{aligned}
$$

In addition it can be proved that the whole sequence $d_{1,t}$ is converging to $\dot{u}^*$.

The function $d_{2,t} \in H^2(y_0,1)$ satisfies

$$
(2.79)\qquad \begin{aligned} -\partial_y\left(\tfrac{1}{\phi'_t}\partial_y d_{2,t}\right) - \partial_y\left(\tfrac{1}{h}(\tfrac{1}{\phi'_t}-1)\partial_y w\right) &= \tfrac{1}{h}\left(\phi'_t\, g(\phi_t) - g\right) \text{ in } (y_0,1) \\ d_{2,t}(1) &= 0 \end{aligned}
$$

$$\frac{1}{\varphi'_{x_t^*}(x^*)}\partial_x d_{1,t}(x^*) + \frac{1}{h}(\frac{1}{\varphi'_{x_t^*}(x^*)} - 1)\partial_x u(x^*) = \frac{1}{\phi'_t(y_0)}\partial_y d_{2,t}(y_0) + \frac{1}{h}(\frac{1}{\phi'_t(y_0)} - 1)\partial_y w(y_0) \tag{2.80}$$

Moreover, we have that

$$\left\| \frac{1}{h}\left(\frac{1}{\phi'_t} - 1\right) + \frac{d\phi'_t}{dt}_{|t=y_0} \right\|_{L^\infty(y_0,1)} \longrightarrow 0 \quad \text{as } t \to y_0 \tag{2.81}$$

$$\left\| \frac{1}{h}\left(\phi'_t g(\phi_t) - g\right) - \frac{d}{dt}\left(\phi'_t\, g(\phi_t)\right)_{|t=y_0} \right\|_{L^\infty(y_0,1)} \longrightarrow 0 \quad \text{as } t \to y_0 \tag{2.82}$$

Proceeding as for the proof of the continuity of $c_{2,t}$ (see (2.61)–(2.64)), we deduce from (2.79) that there exists $\dot{w} \in H^2(y_0, 1)$ such that $d_{2,t} \rightharpoonup \dot{w}$ weakly in H^2 as $t \to y_0$ and $\dot{w}$ satisfies

$$\begin{array}{rcl} -\partial_{yy}\dot{w} + \partial_y\left(\dfrac{d\phi'_t}{dt}_{|t=y_0}\partial_y w\right) & = & \dfrac{d}{dt}(\phi'_t\, g(\phi_t))_{|t=y_0} \text{ in } (y_0, 1) \\ \dot{w}(1) & = & 0 \end{array} \tag{2.83}$$

Using the fact that $w \in H^3(y_0, 1)$ and $-\partial_{yyy}w = \partial_y g$ in $(y_0, 1)$, we obtain by straightforward calculations that

$$\partial_y\left(\frac{d\phi'_t}{dt}_{|t=y_0}\partial_y w\right) - \frac{d}{dt}\left(\phi'_t\, g(\phi_t)\right)_{|t=y_0} = \partial_{yy}\left((\partial_y w)\frac{d\phi_t}{dt}_{|t=y_0}\right) \quad \text{in } (y_0, 1)$$

Then (2.83) becomes

$$\begin{array}{rcl} -\partial_{yy}\dot{w} & = & -\partial_{yy}\left((\partial_y w)\dfrac{d\phi_t}{dt}_{|t=y_0}\right) \quad \text{in } (y_0, 1) \\ \dot{w}(1) & = & 0 \end{array} \tag{2.84}$$

Finally, (2.80) leads to

$$\partial_x\dot{u}(x^*) - \dot{x}^*\frac{d}{ds}\left(\varphi'_s(x^*)\right)_{|s=x^*}\partial_x u(x^*) = \partial_y\dot{w}(y_0) - \frac{d}{dt}\left(\phi'_t(y_0)\right)_{|t=y_0}\partial_y w(y_0) \tag{2.85}$$

The proof of Proposition 2.3 is then complete. □

Remark 2.4. Due to the compactness of the embedding of $H^2(0, x^*) \times H^2(y_0, 1)$ into $C^1([0, x^*]) \times C^1([y_0, 1])$, Proposition 2.3 ensures that the map $t \mapsto (u_t \circ \varphi_{x_t^*}, w_t \circ \phi_t) \in C^1([0, x^*]) \times C^1([y_0, 1])$ is (strongly) differentiable at $t = y_0$.

Now, we are in position to compute the shape derivative of the solution of (2.1)-(2.4). We first extend the solution (u_t, w_t) of (2.40) to the whole real line : $u_t \in H^1_0(0, x^*)$ is extended by 0 outside the interval $(0, x^*)$, so that we consider $u_t \in H^1(\mathbb{R})$. In the same way, $w_t \in H^1(y_0, 1)$ is extended to 0 outside $(y_0, 1)$ so that we consider $w_t \in L^2(\mathbb{R})$.

Proposition 2.5. *Under the hypothesis of Proposition 2.3, the map* $t \mapsto (u_t, w_t) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ *is differentiable at* $t = y_0$. *The shape derivatives* $(u', w') \in H^2(0, x^*) \times H^2(y_0, 1)$ *are given by*

$$(2.86)\qquad \begin{aligned} u' &= \dot{u} - \dot{x}^* (\partial_x u) \frac{d\varphi_s}{ds}_{|s=x^*} \quad \text{in } (0, x^*) \\ w' &= \dot{w} - (\partial_y w) \frac{d\phi_t}{dt}_{|t=y_0} \quad \text{in } (y_0, 1) \end{aligned}$$

and satisfy

$$(2.87)\qquad \begin{aligned} \partial_{xx} u' &= 0 \quad \text{in } (0, x^*) \\ u'(0) &= 0 \end{aligned}$$

$$(2.88)\qquad u'(x^*) = -\dot{x}^* \, \partial_x u(x^*)$$

$$(2.89)\qquad \begin{aligned} \partial_{yy} w' &= 0 \quad \text{in } (y_0, 1) \\ w'(1) &= 0 \end{aligned}$$

$$(2.90)\qquad w'(y_0) = \dot{x}^* - 1 - \partial_y w(y_0)$$

$$(2.91)\qquad \partial_y w'(y_0) - g(y_0) = \partial_x u'(x^*) - \dot{x}^* f(x^*)$$

Proof. The proof is a direct consequence of the derivability of $\hat{u}$ and $\hat{w}$ stated in Proposition (2.3) (see also [10, Proposition 2.32] and [8, Lemme 5.3.3]. We start from the relations

$$(2.92)\qquad \begin{aligned} u_t &= (u_t \circ \varphi_{x_t^*}) \circ \varphi_{x_t^*}^{-1} = \hat{u} \circ \varphi_{x_t^*}^{-1} \\ w_t &= (w_t \circ \phi_t) \circ \phi_t^{-1} = \hat{w} \circ \phi_t^{-1}. \end{aligned}$$

The derivability of u_t and w_t with respect to t at $t = y_0$ is a direct consequence of the derivability of $t \mapsto (\hat{u}, \hat{w}, x_t^*)$ established in Proposition 2.3. We denote by (u', w') the derivative of $t \mapsto (u_t, w_t)$ at $t = y_0$. Differentiating (2.92) with t, we obtain at $t = y_0$:

$$\begin{aligned} u' &= \dot{u} - \dot{x}^* (\partial_x u) \frac{d\varphi_s}{ds}_{|s=x^*} \in H^1(0, x^*) \\ w' &= \dot{w} - (\partial_y w) \frac{d\phi_t}{dt}_{|t=y_0} \in H^1(y_0, 1) \end{aligned}$$

According to Proposition 2.3, we have that $(u', w') \in H^2(0, x^*) \times H^2(y_0, 1)$ and from (2.45),we deduce that (2.87), (2.88), (2.89) hold. Relation (2.46) yields (2.91). Finally, differentiating the relation $x_t^* = t + w_t(t)$ in (2.40) leads to (2.90). □

We define the energy functional J associated to the solution (u_t, w_t, x_t^*) of (2.40) by

$$(2.93)\qquad J(t) = \int_0^{x_t^*} |\partial_x u_t|^2 \, dx + \eta \int_t^1 |\partial_y w_t|^2 \, dy.$$

From Proposition 2.5, we deduce the following differentiability result for the function J.

Proposition 2.6. *Under the hypothesis of Proposition 2.3, the functional* $t \mapsto J(t)$ *is differentiable at* $t = y_0$ *and its derivative at* $t = y_0$ *is given by*

$$J'(y_0) = \big(\partial_y w(y_0)\big)^2 (1 + \partial_y w(y_0) - \eta) + \partial_y w'(y_0) \Big((y_0 - 1)\big(\partial_y w(y_0)\big)^2 - 2\eta w(y_0)\Big) \tag{2.94}$$

with

$$\partial_y w'(y_0) = \frac{x^* g(y_0) - \big(1 + \partial_y w(y_0)\big)\big(\partial_y w(y_0) + x^* f(x^*)\big)}{(x^* + y_0 - 1)}. \tag{2.95}$$

Proof. From (2.1) and (2.2), we deduce that

$$J(t) = \int_0^{x_t^*} f u_t \, dx + \eta \int_t^1 g w_t \, dy - \eta w_t(t) \partial_y w_t(t). \tag{2.96}$$

According to the differentiability result established in Proposition 2.5 (see also Remark 2.4), we deduce that J is differentiable at $t = y_0$ and differentiating (2.96) at $t = y_0$ leads to

$$J'(y_0) = \int_0^{x^*} f u' \, dx + \dot{x}^* \underbrace{[fu]_0^{x^*}}_{=0} + \eta \int_{y_0}^1 g w' \, dy - \eta g(y_0) w(y_0) - \eta \frac{d}{dt}\Big(w_t(t) \partial_y w_t(t)\Big)_{|t=y_0}. \tag{2.97}$$

Moreover, we have

$$\begin{aligned}
\frac{d}{dt}\Big(w_t(t)\partial_y w_t(t)\Big)_{|t=y_0} &= \frac{d}{dt}\Big(w_t(t)\Big)_{|t=y_0} \partial_y w(y_0) + w(y_0) \frac{d}{dt}\Big(\partial_y w_t(t)\Big)_{|t=y_0} \\
&= \left(w'(y_0) + \partial_y w(y_0)\right) \partial_y w(y_0) \\
&\quad + w(y_0) \left(\partial_y w'(y_0) + \partial_{yy} w(y_0)\right) \\
&= w'(y_0) \partial_y w(y_0) + w(y_0) \partial_y w'(y_0) \\
&\quad + (\partial_y w(y_0))^2 - w(y_0) g(y_0)
\end{aligned} \tag{2.98}$$

Combining (2.97) and (2.98), we obtain

$$J'(y_0) = \int_0^{x^*} f u' \, dx + \eta \int_{y_0}^1 g w' \, dy - \eta \big(\partial_y w(y_0)\big)^2 - \eta \Big(w'(y_0) \partial_y w(y_0) + w(y_0) \partial_y w'(y_0)\Big). \tag{2.99}$$

Moreover, using the regularity of u and u' with (2.88), we get

$$\begin{aligned}\int_0^{x^*} fu'\,dx &= -\int_0^{x^*}(\partial_{xx}u)u'\,dx\\ &= \int_0^{x^*}\partial_x u\,\partial_x u'\,dx - u'(x^*)\partial_x u(x^*)\\ &= \int_0^{x^*}\partial_x u\,\partial_x u'\,dx + \dot{x}^*\big(\partial_x u(x^*)\big)^2\\ &= -\int_0^{x^*} u\,\underbrace{\partial_{xx}u'}_{=0}\,dx + \big[\underbrace{u}_{=0}\partial_x u'\big]_0^{x^*} + \dot{x}^*\big(\partial_x u(x^*)\big)^2\end{aligned}$$

Then, we have

$$\int_0^{x^*} fu'\,dx = \dot{x}^*\big(\partial_x u(x^*)\big)^2. \tag{2.100}$$

Similarly, we obtain

$$\begin{aligned}\int_{y_0}^1 gw'\,dy &= -\int_{y_0}^1(\partial_{yy}w)w'\,dy\\ &= \int_{y_0}^1\partial_y w\,\partial_y w'\,dy + w'(y_0)\partial_y w(y_0)\\ &= -\int_{y_0}^1 w\,\underbrace{\partial_{yy}w'}_{=0}\,dy + \big[w\partial_y w'\big]_{y_0}^1 + w'(y_0)\partial_y w(y_0)\end{aligned}$$

Then, we have

$$\int_{y_0}^1 gw'\,dy = -w(y_0)\partial_y w'(y_0) + w'(y_0)\partial_y w(y_0). \tag{2.101}$$

Relations (2.100), (2.101) with (2.4) in (2.99) lead to

$$J'(y_0) = (\dot{x}^* - \eta)\big(\partial_y w(y_0)\big)^2 - 2\eta w(y_0)\partial_y w'(y_0) \tag{2.102}$$

From (2.90), we have $\dot{x}^* = 1 + w'(y_0) + \partial_y w(y_0)$ and then we can express the derivative $J'(y_0)$ as follows

$$J'(y_0) = \big(1 + w'(y_0) + \partial_y w(y_0) - \eta\big)\,\big(\partial_y w(y_0)\big)^2 - 2\eta w(y_0)\partial_y w'(y_0) \tag{2.103}$$

Now, we derive a relation between $w'(y_0)$ and $\partial_y w'(y_0)$. For $y \in [y_0, 1]$, we introduce the function $\psi(y) = y - 1$ which satisfies $\partial_y\psi \equiv 1$ in $[y_0, 1]$ and $\psi(1) = 0$. Then, we write

$$\begin{aligned}w'(y_0) &= -\int_{y_0}^1\partial_y w'(y)\partial_y\psi(y)\,dy\\ &= \int_{y_0}^1\underbrace{\partial_{yy}w'(y)}_{=0}\partial_y\psi(y)\,dy - \big[\partial_y w'(y)\psi(y)\big]_{y_0}^1\\ &= \partial_y w'(y_0)\psi(y_0)\end{aligned}$$

and thus we get

$$w'(y_0) = (y_0 - 1)\partial_y w'(y_0). \tag{2.104}$$

Combining (2.103) with (2.104), we obtain the desired formula (2.94).

Finally, we turn to the expression of $\partial_y w'(y_0)$ with respect to $\partial_y w(y_0)$. For $x \in [0, 1]$, we introduce the function $\psi(x) = x$ which satisfies $\partial_x \psi \equiv 1$ in $[0, 1]$ and $\psi(0) = 0$. Then, we write

$$\begin{aligned} u'(x^*) &= \int_0^{x^*} \partial_x u'(x) \partial_x \psi(x)\, dx \\ &= -\int_0^{x^*} \underbrace{\partial_{xx} u'(x)}_{=0} \partial_x \psi(x)\, dx + \left[\partial_x u'(x)\psi(x)\right]_0^{x^*} \\ &= \partial_x u'(x^*)\psi(x^*) \end{aligned}$$

and thus we have

$$u'(x^*) = x^* \partial_x u'(x^*). \tag{2.105}$$

Combining (2.91) with (2.104),(2.105),(2.88) and(2.90), we obtain the desired formula (2.95) for $\partial_y w'(y_0)$. □

3. An explicit one-dimensional optimal solution

In this section, we study in details the particular case where the functions f and g are two constants. These constants have to be chosen small enough for ensuring the well-posedness of (2.1)–(2.4) (see Proposition 2.1). We choose $f \equiv 1$ and $g \equiv \alpha \in \mathbb{R}$ a constant. The solution of Problem (2.1)–(2.4) is then given by

$$u(x) = -\frac{1}{2}x(x - x^*), \quad x \in (0, x^*) \tag{3.1}$$

$$w(y) = \left(c_0 - \frac{\alpha}{2}(y-1)\right)(y - 1), \quad y \in (y_0, 1) \tag{3.2}$$

with

$$c_0 = -\frac{\alpha(1 - y_0)(3 + y_0) + 2y_0}{2(1 + y_0)} \tag{3.3}$$

$$x^* = 2\left(\alpha(y_0 - 1) - c_0\right) = \frac{\alpha(1 - y_0)^2 + 2y_0}{1 + y_0}. \tag{3.4}$$

The constant α must be chosen small enough. In order to make certain that x^* lies in the interval $(0, 1)$, we shall see that we have to restrict the

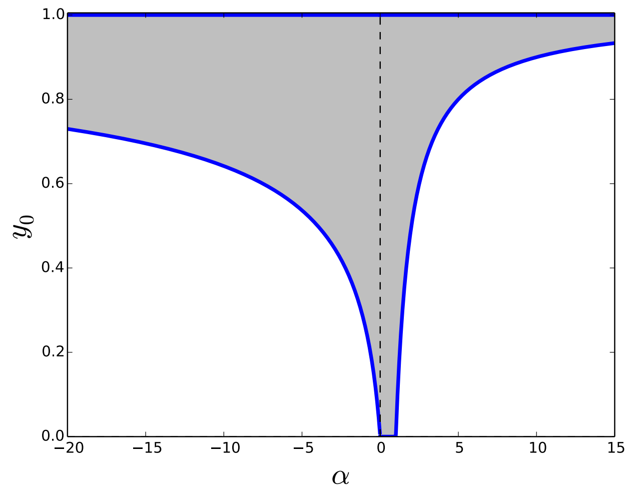

FIGURE 4. The admissible domain $\mathcal{D}$ (gray region) for the 1d case with $f \equiv 1$ and $g \equiv \alpha$.

values of α and y_0. Indeed, we have that

$$x^* \in (0,1) \quad \Leftrightarrow \quad \alpha \in I_0 = (\frac{-2y_0}{(1-y_0)^2}, \; \frac{1}{1-y_0})$$

$$\Leftrightarrow \quad y_0 \in I_\alpha = \begin{cases} (\dfrac{\alpha - 1 + \sqrt{1-2\alpha}}{\alpha}, \; 1) & \text{if } \alpha < 0 \\ (\,\max(0, \, 1 - \dfrac{1}{\alpha}), \; 1) & \text{if } \alpha \geq 0 \end{cases}$$

Then, we introduce the admissible domain $\mathcal{D}$ where the parameters (y_0, α) are allowed to lie for ensuring $x^* \in (0,1)$:

$$\mathcal{D} = \{(y_0, \alpha) \in (0,1) \times \mathbb{R}, \; y_0 \in I_\alpha\}. \tag{3.5}$$

The admissible domain $\mathcal{D}$ is drawn in Figure 4.

We recall that the energy functional J is given by

$$J(y_0) = \int_0^{x^*} |\partial_x u|^2 \, dx + \eta \int_{y_0}^1 |\partial_y w|^2 \, dy \tag{3.6}$$

with a parameter $\eta > 0$. Let $\alpha \in \mathbb{R}$ be fixed. The shape optimization problem consists in finding the reference point y_0 that minimizes

$$\min_{y_0 \in I_\alpha} J(y_0). \tag{3.7}$$

Using the explicit formula (3.1)–(3.4), we obtain

$$J(y_0) = \left(2 - \frac{\eta}{\alpha}\right) \frac{(x^*)^3}{24} - \frac{\eta}{3\alpha} {c_0}^3 \tag{3.8}$$

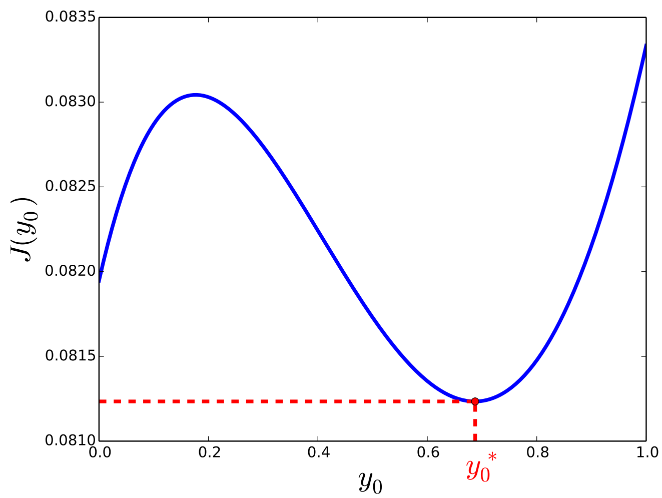

FIGURE 5. The energy functional $y_0 \mapsto J(y_0)$

where x^* and c_0 are given by (3.3) and (3.4). This formula provides a fully explicit expression of the functional J with y_0. The derivative $J'(y_0)$ of the functional with respect to y_0 can be computed exactly as well as the optimal value y_0^* that minimizes J. It can be checked that this direct calculation coincides with the general formula (2.94),(2.95) given in Proposition 2.6. In the sequel, we do not give this expression for $J'(y_0)$, we only consider a numerical example of an optimal solution.

Numerical example. We choose $\alpha = 0.4$ and $\eta = 0.442$. The energy functional $J(y_0)$ is depicted on Figure 5. The minimum of $J(y_0)$ is reached at $y_0^* \simeq 0.6868$. The corresponding optimal point x^* is equal to $x^* \simeq 0.8376$. The optimal solutions u and w are drawn on Figure 6. We point out that the functional J has a nontrivial behaviour with respect to y_0, in particular J is a nonconvex function of y_0. This indicates the difficulty and the pertinence of the two-dimensional shape optimization problem (1.22),(1.23) introduced at the beginning of this paper.

4. Conclusion

We introduced a shape optimization problem for a fluid-structure interaction system coupling the Stokes equations with the linear elasticity equation. We have shown that a shape optimization problem for a simplified model in one spatial dimension is well-posed and we are able to fully characterize the shape derivatives associated to this one-dimensional free-boundary problem. All the (variational) technical tools we have employed for the study of the one-dimensional free-boundary problem have been made in the spirit to tackle and solve the two-dimensional shape optimization problem presented

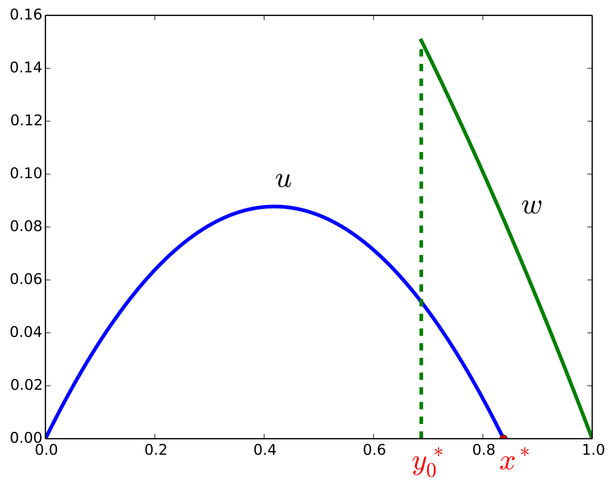

FIGURE 6. The optimal solutions u and w.

in the introduction of this paper. We aim to extend our 1d technics to the two dimensional problem for getting a rigorous statement of the 2d shape derivatives.

REFERENCES

[1] V. Barbu, Optimal control of variational inequalities, Research Notes in Mathematics 100, Pitman (Advanced Publishing Program), Boston, MA, 1984, iv+298 pages.

[2] Avner Friedman, Variational principles and free-boundary problems, Second Edition, Robert E. Krieger Publishing Co., Inc., Malabar, FL, 1988, x+710 pages.

[3] D. Kinderlehrer, G., Stampacchia, An introduction to variational inequalities and their applications, Classics in Applied Mathematics 31, Reprint of the 1980 original, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000, xx+313 pages.

[4] G. P. Galdi, M. Kyed, Steady flow of a Navier-Stokes liquid past an elastic body, Arch. Ration. Mech. Anal. 194 (2009), no. 3, 849–875.

[5] V.I. Gorbachuk, M.L. Gorbachuk, Boundary Value Problems for Operator Differential Equations, Kluwer Academic Publ., Dordrecht, 1991.

[6] C. Grandmont, Existence for a three-dimensional steady state fluid-structure interaction problem, J. Math. Fluid Mech., 4 (2002), no. 1, 76–94.

[7] A. Halanay, C.M. Murea, D. Tiba, Existence of a Steady Flow of Stokes Fluid Past a Linear Elastic Structure Using Fictitious Domain, J. Math. Fluid Mech., 18 (2016), no. 2, 397–413.

[8] A. Henrot, M. Pierre, Variation et optimisation de formes: Une analyse géométrique. Mathématiques et Applications, vol. 48. Springer Berlin Heidelberg, 2005.

[9] I. Lasiecka, K. Szulc, A. Żochowski, Linearised coupling of elasticity and Navier-Stokes equations, 20th International Conference on Methods and Models in Automation and Robotics (MMAR) (2015), 208–210.

[10] J. Sokołowski, J.P. Zolésio, Introduction to shape optimization: shape sensitivity analysis, Springer-Verlag, 1992.

[11] C. Surulescu, On the stationary motion of a Stokes fluid in a thick elastic tube: a 3D/3D interaction problem, Acta Math. Univ. Comenianae, Vol. LXXV, 1 (2006), 95–106.

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