Shape Design in Aorto-Coronaric Bypass Anastomoses using Perturbation Theory

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Summary

In this paper we present a new approach in the study of Aorto-Coronaric bypass anastomoses configurations based on small perturbation theory. The theory of optimal control based on adjoint formulation is applied in order to optimize the shape of the zone of the incoming branch of the bypass (the toe) into the coronary (see Figure (1)). The aim is to provide design indications in the perspective of future development for prosthetic bypasses.

Key words: *Optimal Control, Shape Optimization, Small Perturbation Theory, Finite Elements, Haemodynamics, Aorto-Coronaric Bypass Anastomoses, Design of Improved Medical Devices.*

1 Introduction

We consider the application of optimal control approaches to shape optimization of aorto-coronaric bypass anastomoses ([22]). We analyze the "first correction" method which is derived by applying a perturbation method to the initial problem in a domain $\Omega \subset \mathbb{R}^2$ whose boundary $\partial\Omega$ is parameterized by a suitable

*This work has been prepared when the first author was visiting Bernoulli Center of the Swiss Federal Institute of Technology Lausanne in the framework of the special semester on "The Mathematical Modelling of the Cardiovascular System"

function f. Then we propose numerical methods for its solution.

The surgical realization of a bypass to overcome a critically stenosed artery is a very common practice in everyday cardiovascular clinic.
Improvement in the understanding of the genesis of coronary diseases is very important as it allows to reduce surgical and post-surgical failures. It may also suggest new means in bypass surgical procedures with less invasive methods and to devise new shape in bypass configuration ([19]).
Generally speaking, mathematical modelling and numerical simulation can allow better understanding of phenomena involved in vascular diseases ([24], [23] and [6]).
When a coronary artery is affected by a stenosis, the heart muscle can't be properly oxygenated through blood. Aorto-coronaric anastomosis restores the oxygen amount through a bypass surgery downstream an occlusion.
At present, different kind and shape for aorto-coronaric bypass anastomoses are available and consequently different surgery procedures are used to set up a bypass.
A bypass can be made up either by organic material (e.g. the saphena vein taken from patient's legs or the mammary artery) or by prosthetic material. The current saphenous bypass solution requires the extraction of saphena vein with possible complications. In this respect, prosthetic bypasses are less invasive. They may feature very different shape for bypass anastomoses, such as, e.g., cuffed arteriovenous access grafts. Different cuffed models are used such as Taylor Patch [2] and Miller Cuff Bypass, [4], but also standard end-to-side anastomoses at different graft angle [3] or other shaped carbon-fiber prostheses. In the cardiovascular system altered flow conditions such as separation, flow reversal, low and oscillatory shear stress areas and abnormal pulse pattern are all recognized as potentially important factors in the development of arterial diseases (see [15] and [18]). For all these different aspects the design of artificial arterial bypass is a very complex problem. Carbon fiber and Collagen cuffed grafts instead of natural saphenous vein can be used for studying new shape design without needing "in loco" reconstruction. In this framework, Optimal Control (Lions [12]) by perturbation theory (Van Dyke [31]) provide a new approach to the problem, with the goal of improving arterial bypass graft on the basis of a better understanding of fluid dynamics aspects involved in the bypass studying.

2 Notation and Problem Statement

Let Ω be a bounded domain of $\mathbb{R}^2$, $\Gamma \equiv \partial\Omega$ is the boundary of Ω, $\overline{\Omega} = \Omega \cup \partial\Omega$, $\underline{x} := (x, y)$ is a point of $\overline{\Omega}$. For every scalar function ϕ and a vector function $\underline{v}$, whose components are u, v, we recall the definition of the following operators:

$$\nabla\phi = (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}),$$

$$\nabla \cdot \underline{v} := div(\underline{v}) := \mathcal{D}(\underline{v}) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y},$$

$$\nabla \times \underline{v} := rot(\underline{v}) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

$$\underline{rot}(\phi) = (\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}).$$

We remind that:

$$\underline{rot}(\nabla \times \underline{v}) = -\Delta \underline{v} + \nabla(\nabla \cdot \underline{v}),\ \Delta \phi = \nabla \cdot (\nabla \phi).$$

In the sequel vectors are marked with an underlined notation $\underline{v}$, aggregation of vector quantities $\underline{v}$ with scalar quantities p are indicated with $\underline{Q}$ ($\underline{Q} = (\underline{v}, p)$), $\underline{\Phi}$ or $\underline{\hat{\Phi}}$.
Consider an idealized, two-dimensional bypass bridge configuration in Fig.(1) and the domain on Fig.(2), where the dotted line represents the geometry of the complete anastomosis; Γ_{w_2} is the section of the original artery, Γ_{in} is the new anastomosis inflow after bypass surgery, Γ_{out} is the anastomosis outflow.
We consider the following boundary value problem for the Stokes equations

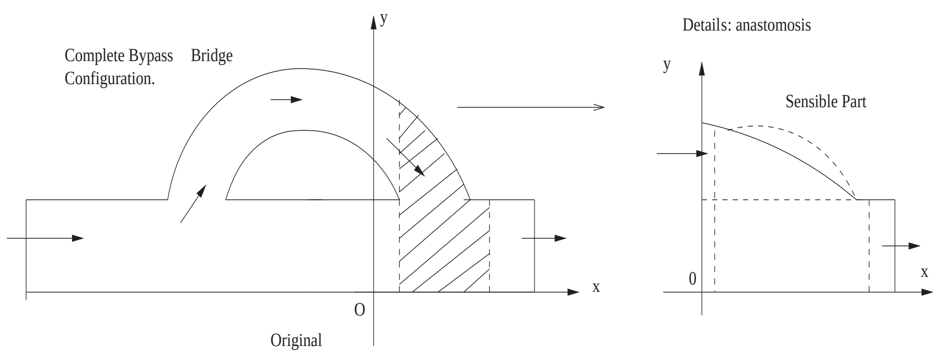

Figure 1: Idealized, 2-D bypass bridge configuration (top) and detail of the sensible part for the optimization process (bottom). The dotted curve represents a possible shape variation.

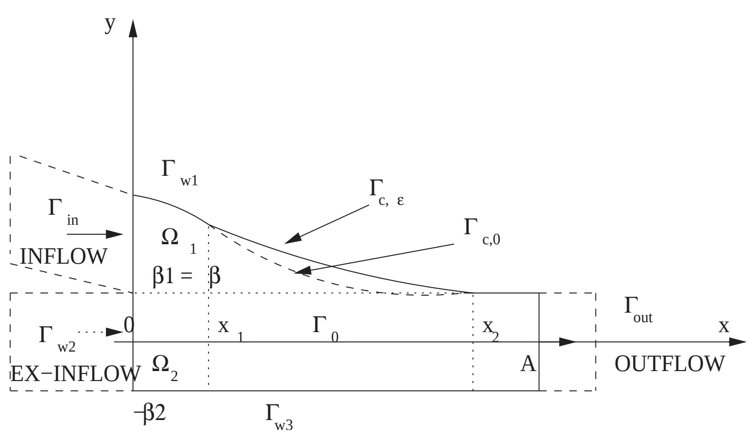

Figure 2: $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2, \Gamma_w = \Gamma_{w_1} \cup \Gamma_{w_2} \cup \Gamma_{w_3}, \Gamma_0 = \partial\Omega_1 \cap \partial\Omega_2$.

[33], used to model low Reynolds blood flow in this study. For mathematical aspects in fluid mechanics see, for example, [14]. The problem reads: find $\underline{v}, p$

s.t.

$$\begin{cases} -\nu\Delta \underline{v} + \nabla p = \underline{F} \quad in\ \Omega, \\ \nabla \cdot \underline{v} = 0 \quad in\ \Omega, \\ \underline{v} = \underline{v}_{in}\ on\ \Gamma_{in},\ \underline{v} = 0\ on\ \Gamma_{w_1} \cup \Gamma_{w_3}, \\ -p \cdot \underline{n} + \nu \frac{\partial v}{\partial n} = \underline{g}_{out} on\ \Gamma_{out} \cup \Gamma_{w_2}, \end{cases} \tag{1}$$

where $\underline{n} = (n_1, n_2)$ is the outward unit normal vector on Γ, $\underline{F} = \underline{F}(x, y)$, $\underline{v}_{in} = \underline{v}_{in}(x, y)$, $\underline{g}_{out} = \underline{g}_{out}(x, y)$ are given vector functions, $\nu = const > 0$ and $v_f = \{\underline{v}_{in}\ on\ \Gamma_{in}; \underline{0}\ on\ \Gamma_{w_1} \cup \Gamma_{w_3}\}$. In the following we may need to impose some additional restriction on p (for example $\int_\Omega p d\Omega{=}0$ if $\Gamma_{in} = \Gamma$).
The subset $\Gamma_{c,\varepsilon}$ of Γ_{w_1} is parametrized by a function $f(x, \varepsilon)$ of $\underline{x} \in [x_1, x_2]$ and of small parameter $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, $\varepsilon_0 = const$. More precisely we assume that $f(x, \varepsilon)$ can be developed as follows:

$$f(x, \varepsilon) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \ldots, \tag{2}$$

where $f_k \in \mathbb{W}^{1,\infty}(x_1, x_2)$, for $k = 0$, (we recall that $\mathbb{W}^{1,\infty}(x_1, x_2)$ is the space of functions $f_k \in \mathbb{L}^\infty(x_1, x_2)$ such that all the distribution derivatives of first order of f_k are functions of $\mathbb{L}^\infty(x_1, x_2)$) and $f_k \in \mathbb{W}_0^{1,\infty}(x_1, x_2)$, for $k \geq 1$, so that $f_k(x_1) = f_k(x_2) = 0, k \geq 1$. Here the function $f_0(x) > 0$ describes the original subset $\Gamma_{c,0}$ of the boundary of "unperturbed domain", $\Gamma_{w0} \equiv \partial\Omega_0$ of the domain Ω_0 (see Fig. 3(left)), while $f_k(x), k \geq 1$, could be unknown when dealing with control problem (see Section 4).

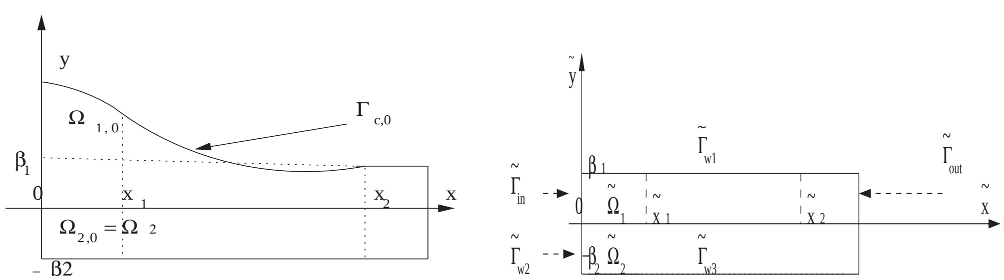

Figure 3: "Unperturbed domain" Ω_0, $\overline{\Omega}_0 = \overline{\Omega}_{1,0} \cup \overline{\Omega}_{2,0}$ (left). The "simple" domain $\tilde{\Omega}$ (right).

The weak statement of (1) reads: find $\underline{v} \in (\mathbb{H}^1(\Omega))^2$, $p \in \mathbb{L}^2(\Omega)$ s.t.

$$\begin{cases} a(\underline{v}, \underline{\hat{v}}) = b(p, \underline{\hat{v}}) + G(\underline{\hat{v}})\ \forall \underline{\hat{v}} \in \mathbb{X}, \\ b(\hat{p}, \underline{v}) = 0\ \forall \hat{p} \in \mathbb{L}^2(\Omega), \\ \underline{v} = \underline{v}_f\ on\ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3}, \end{cases} \tag{3}$$

where with $\underline{\hat{v}}$ we indicate test functions and:

$$a(\underline{v}, \underline{\hat{v}}) = \int_\Omega \nu \nabla \underline{v} \cdot \nabla \underline{\hat{v}} d\Omega$$

$$b(p, \underline{\hat{v}}) = \int_\Omega p \nabla \cdot \underline{\hat{v}} d\Omega,\ G(\underline{\hat{v}}) = \int_\Omega \underline{F} \cdot \underline{\hat{v}} d\Omega + \int_{\Gamma_{out} \cup \Gamma_{w_2}} \underline{g}_{out} \cdot \underline{\hat{v}} d\Gamma,$$

$$\mathbb{X} := \{\underline{\hat{v}} : \underline{\hat{v}} \in (\mathbb{H}^1(\Omega))^2, \underline{\hat{v}} = 0\ on\ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3}.\}$$

Although $a(.,.), b(.,.)$ and $G(.)$ depend on the parametrization f of the part $\Gamma_{c,\varepsilon}$, this dependence will be understood for simplicity of notations.

3 The problem for the perturbation functions

Let us introduce the reference (simple-shaded) domains $\tilde{\Omega}_1 = \{0 \le \tilde{x} < A, 0 < \tilde{y} < \beta_1 \equiv \beta\}$, $\tilde{\Omega}_2 = \{0 < \tilde{x} < A, -\beta_2 < \tilde{y} < 0\}$, and $\overline{\tilde{\Omega}} = \overline{\tilde{\Omega}}_1 \cup \overline{\tilde{\Omega}}_2$ (see Fig.3 (right). Then we assume that $f(x, \varepsilon) > 0$ and consider the following variables transformation:

$$\mathrm{T}_f : \overline{\Omega}_1 \cup \overline{\Omega}_2 \to \overline{\tilde{\Omega}},\ \underline{\tilde{x}} = \mathrm{T}_f(\underline{x}),$$

such as T_f is the identity in Ω_2, while $\mathrm{T}_f(x,y) = (x, \frac{\beta}{f(x,y)}y)$ in Ω_1
We set $\underline{\tilde{x}} = (\tilde{x}, \tilde{y})$ and define

$$\underline{\tilde{v}}(\underline{\tilde{x}}) := \underline{v} \circ \mathrm{T}_f^{-1}(\underline{\tilde{x}}) = \underline{v}(\tilde{x}, \tilde{y} f(\tilde{x}, \varepsilon)/\beta).$$

where $\underline{\tilde{v}} = (\tilde{u}, \tilde{v})$. Then,

$$dxdy = \frac{f(\tilde{x}, \epsilon)}{\beta} d\tilde{x} d\tilde{y}$$

and the following relations hold:

$$\begin{cases} \frac{\partial \phi}{\partial y}(\underline{\tilde{x}}) = \frac{\beta}{f(\tilde{x},\varepsilon)} \frac{\partial \tilde{\phi}(\underline{\tilde{x}})}{\partial \tilde{y}}, \\ \frac{\partial \phi}{\partial x}(\underline{\tilde{x}}) = \frac{\partial \tilde{\phi}(\underline{\tilde{x}})}{\partial \tilde{x}} - \tilde{y} \frac{f_x(\tilde{x},\varepsilon)}{f(\tilde{x},\varepsilon)} \frac{\partial \tilde{\phi}(\underline{\tilde{x}})}{\partial \tilde{y}} \quad \big(with\ f_x := \frac{df}{dx}\big), \end{cases} \tag{4}$$

$$\begin{cases} \tilde{\mathcal{D}}(f)\underline{\tilde{v}}(\underline{\tilde{x}}) := ((\nabla \cdot \underline{v}) \circ \mathrm{T}_f^{-1})(\underline{\tilde{x}}) = \frac{\partial \tilde{u}}{\partial \tilde{x}} - \tilde{y} \frac{f_x(\tilde{x},\epsilon)}{f(\tilde{x},\epsilon)} \frac{\partial \tilde{u}}{\partial \tilde{y}} + \frac{\beta}{f(\tilde{x},\varepsilon)} \frac{\partial \tilde{v}}{\partial \tilde{y}}, \\ \tilde{\mathcal{R}}(f)\underline{\tilde{v}}(\underline{\tilde{x}}) := ((\nabla \times \underline{v}) \circ \mathrm{T}_f^{-1})(\underline{\tilde{x}}) = \frac{\partial \tilde{v}}{\partial \tilde{x}} - \tilde{y} \frac{f_x(\tilde{x},\epsilon)}{f(\tilde{x},\varepsilon)} \frac{\partial \tilde{v}}{\partial \tilde{y}} - \frac{\beta}{f(\tilde{x},\varepsilon)} \frac{\partial \tilde{u}}{\partial \tilde{y}}. \end{cases} \tag{5}$$

Then in $\tilde{\Omega}$ we have:

$$\tilde{\mathcal{D}}(f)\underline{\tilde{v}} = m_2 \tilde{\nabla} \cdot \underline{\tilde{v}} + m_1 \tilde{\mathcal{D}}(f)\underline{\tilde{v}},\ \tilde{\mathcal{R}}(f)\underline{\tilde{v}} = m_2 \tilde{\nabla} \times \underline{\tilde{v}} + m_1 \tilde{\mathcal{R}}(f)\underline{\tilde{v}},$$

where $\tilde{\nabla}\phi := \big(\frac{\partial \phi}{\partial \tilde{x}}, \frac{\partial \phi}{\partial \tilde{y}}\big)$, while m_s is the characteristic function of Ω_s $(s = 1, 2)$. To simplify the notations from now on we will set (unless otherwise specified):

$$\underline{\tilde{x}} = \underline{x},\ \underline{\tilde{v}}(\tilde{x}, \tilde{y}) := \underline{v}(x, y),\ \tilde{u} = u,\ \tilde{v} = v, \ldots, \tilde{\mathcal{D}} = \mathcal{D}, \tilde{\mathcal{R}} = \mathcal{R},\ \tilde{\Omega} \equiv \Omega,\ \tilde{\Gamma}_{w_k} \equiv \Gamma_{w_k}.$$

Then problem (3) in the new variables reads as follows:

$$\begin{cases} a(f; \underline{v}, \underline{\hat{v}}) = b(f; p, \underline{\hat{v}}) + G(f; \underline{\hat{v}})\ \forall \underline{\hat{v}} \in \mathbb{X}, \\ b(f; \underline{\hat{p}}, \underline{v}) = 0\ \forall \hat{p} \in \mathbb{L}^2(\Omega), \\ \underline{v} = \underline{v}_f\ on\ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3} \end{cases} \tag{6}$$

We have emphasized the dependence of $a(f; ., .), b(f; ., .)$, and $G(f; .)$ on f. Precisely, (with $\Omega_1 \equiv \tilde{\Omega}_1, \Omega_2 \equiv \tilde{\Omega}_2$):

$$a(f; \underline{v}, \underline{\hat{v}}) = a_1(f; \underline{v}, \underline{\hat{v}}) + a_2(\underline{v}, \underline{\hat{v}}),$$

$$a_1(f; \underline{v}, \underline{\hat{v}}) = \int_{\Omega_1} \frac{f\nu}{\beta} \big(\big(\frac{\partial \underline{v}}{\partial x} - \frac{y f_x}{f} \frac{\partial \underline{v}}{\partial y} \big) \cdot \big(\frac{\partial \underline{\hat{v}}}{\partial x} - \frac{y f_x}{f} \frac{\partial \underline{\hat{v}}}{\partial y} \big) + \frac{\beta^2}{f^2} \frac{\partial \underline{v}}{\partial y} \cdot \frac{\partial \underline{\hat{v}}}{\partial y} \big) dxdy,$$

$$a_2(\underline{v}, \underline{\hat{v}}) = \int_{\Omega_2} \nu \big(\frac{\partial \underline{v}}{\partial x} \cdot \frac{\partial \underline{\hat{v}}}{\partial x} + \frac{\partial \underline{v}}{\partial y} \cdot \frac{\partial \underline{\hat{v}}}{\partial y} \big) dxdy,$$

$$b(f; p, \underline{\hat{v}}) = b_1(f; p, \underline{\hat{v}}) + b_2(p, \underline{\hat{v}}),$$

$$b_1(f;p,\underline{\hat{v}}) = \int_{\Omega_1} \frac{f}{\beta} p \mathcal{D}(f) \underline{\hat{v}} dxdy, \quad b_2(p,\underline{\hat{v}}) = \int_{\Omega_2} p \nabla \cdot \underline{\hat{v}} dxdy,$$

$$G(f;\underline{\hat{v}}) = G_1(f;\underline{\hat{v}}) + G_2(\underline{\hat{v}}),$$

$$G_1(f;\underline{\hat{v}}) = \int_{\Omega_1} \frac{f}{\beta} \underline{F} \cdot \underline{\hat{v}} dxdy + \int_{(\Gamma_{out}\cup\Gamma_{w2})\cap\partial\Omega_1} \underline{g}_{out} \cdot \underline{\hat{v}} d\Gamma,$$

$$G_2(\underline{\hat{v}}) = \int_{\Omega_2} \underline{F} \cdot \underline{\hat{v}} dxdy + \int_{(\Gamma_{out}\cup\Gamma_{w2})\cap\partial\Omega_2} \underline{g}_{out} \cdot \underline{\hat{v}} d\Gamma.$$

Note that the functions $\underline{\hat{v}}$, $\hat{p}$ on (6) can be assumed to be independent of ε in the sequel.
Assume that the problem (6) has a solution $\underline{v}, p$ that is infinitely differentiable with respect to ε:

$$\begin{cases} \underline{v} = \underline{v}_0 + \varepsilon \underline{v}_1 + \varepsilon^2 \underline{v}_2 + \dots \\ p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots \end{cases} \tag{7}$$

where $p_k \in \mathbb{L}^2, \underline{v}_k \in \mathbb{X},\ k \geq 1$. Using (2), (7) and small perturbation techniques we can derive the equations for $\underline{v}_k, p_k,\ k \geq 0$. In particular for $k = 0$ $\underline{v}_0$ and p_0 satisfy

$$\begin{cases} a(f_0;\underline{v}_0,\underline{\hat{v}}) = b(f_0;p_0,\underline{\hat{v}}) + G(f_0;\underline{\hat{v}})\ \forall \underline{\hat{v}} \in \mathbb{X}, \\ b(f_0;\underline{\hat{p}},\underline{v}_0) = 0\ \forall \hat{p} \in \mathbb{L}^2(\Omega), \\ \underline{v}_0 = \underline{v}_f\ on\ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3}. \end{cases} \tag{8}$$

Correspondingly we define:

$$\mathcal{R}_{obs,0} := \mathcal{R}(f_0)\underline{v}_0 \tag{9}$$

For $k = 1$ the functions $\underline{v}_1$ and p_1 are the solution of the equations:

$$\begin{cases} a(f_0;\underline{v}_1,\underline{\hat{v}}) = b(f_0;p_1,\underline{\hat{v}}) + \frac{\partial}{\partial\varepsilon} b(f;p_0,\underline{\hat{v}})|_{\varepsilon=0} + \\ + \frac{\partial}{\partial\varepsilon} G(f;\underline{\hat{v}})|_{\varepsilon=0} - \frac{\partial}{\partial\varepsilon} a(f;\underline{v}_0,\underline{\hat{v}})|_{\varepsilon=0}\ \forall \underline{\hat{v}} \in \mathbb{X}, \\ b(f_0;\underline{\hat{p}},\underline{v}_1) + \frac{\partial}{\partial\varepsilon} b(f;\hat{p},\underline{v}_0)|_{\varepsilon=0} = 0\ \forall \hat{p} \in \mathbb{L}^2(\Omega), \\ \underline{v}_1 = 0\ on\ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3}, \end{cases} \tag{10}$$

where

$$\frac{\partial}{\partial\varepsilon} b(f;p_0,\underline{\hat{v}})|_{\varepsilon=0} := b_f(f_1,p_0,\underline{\hat{v}}) = \int_{\Omega_1} \frac{f_1}{\beta} p_0 \mathcal{D}(f_0)\underline{\hat{v}} dxdy + \int_{\Omega_1} \frac{f_0}{\beta} p_0 \mathcal{D}_f(f_1,\underline{\hat{v}}) dxdy,$$

$$\mathcal{D}_f(f_1,\underline{\hat{v}}) := \frac{\partial}{\partial\varepsilon} \mathcal{D}(f)\underline{\hat{v}}|_{\varepsilon=0} = -\big[y\big(\frac{f_{1,x}f_0 - f_{0,x}f_1}{f_0^2}\big)\frac{\partial \hat{u}}{\partial y} + \frac{\beta f_1}{f_0^2}\frac{\partial \hat{v}}{\partial y}\big]$$

$$\mathcal{D}_f(f_1,\underline{v}_0) := \frac{\partial}{\partial\varepsilon} \mathcal{D}(f)\underline{v}_0|_{\varepsilon=0} (:= \mathcal{D}_f f_1\ in\ the\ sequel),$$

$$\frac{\partial}{\partial\varepsilon} G(f;\underline{\hat{v}})|_{\varepsilon=0} := G_1(f_1;\underline{\hat{v}}) = \int_{\Omega_1} \frac{f_1}{\beta} F \cdot \underline{\hat{v}} dxdy,$$

$$\frac{\partial}{\partial\varepsilon} a(f;\underline{v}_0,\underline{\hat{v}})|_{\varepsilon=0} := a_f(f_1;\underline{v}_0,\underline{\hat{v}}) = \int_{\Omega_1} \frac{f_1\nu}{\beta}\big(\big(\frac{\partial \underline{v}_0}{\partial x} - \frac{yf_{0,x}}{f_0}\frac{\partial \underline{v}_0}{\partial y}\big)\cdot\big(\frac{\partial \underline{\hat{v}}}{\partial x} - \frac{yf_{0,x}}{f_0}\frac{\partial \underline{\hat{v}}}{\partial y}\big) + \frac{\beta^2}{f_0^2}\frac{\partial \underline{v}_0}{\partial y}\cdot\frac{\partial \underline{\hat{v}}}{\partial y}\big) dxdy +$$

$$- \int_{\Omega_1} \frac{f_0\nu}{\beta} y \frac{(f_{1,x}f_0 - f_{0,x}f_1)}{f_0^2}\big(\frac{\partial \underline{v}_0}{\partial y}\cdot\big(\frac{\partial \underline{\hat{v}}}{\partial x} - \frac{yf_{0,x}}{f_0}\frac{\partial \underline{\hat{v}}}{\partial y}\big) + \big(\frac{\partial \underline{v}_0}{\partial x} - \frac{yf_{0,x}}{f_0}\frac{\partial \underline{v}_0}{\partial y}\big)\cdot\frac{\partial \underline{\hat{v}}}{\partial y}\big) dxdy$$

$$-\int_{\Omega_1} \frac{f_0 \nu}{\beta} (\frac{2\beta^2 f_1}{f_0^3}) \frac{\partial \underline{v}_0}{\partial y} \cdot \frac{\partial \hat{\underline{v}}}{\partial y} dxdy.$$

So the problem for $\underline{v}_1,\ p_1$ reads as follows: find $\underline{v}_1 \in \mathbb{X},\ p_1 \in \mathbb{L}^2(\Omega)$ s.t.:

$$\begin{cases} a(f_0; \underline{v}_1, \hat{\underline{v}}) - b(f_0; p_1, \hat{\underline{v}}) = b_f(f_1; p_0, \hat{\underline{v}}) + G_1(f_1; \hat{\underline{v}}) - a_f(f_1; \underline{v}_0, \hat{\underline{v}})\ \forall \hat{\underline{v}} \in \mathbb{X}, \\ b(f_0; \hat{\underline{p}}, \underline{v}_1) + b_f(f_1; \hat{p}, \underline{v}_0) = 0\ \forall \hat{p} \in \mathbb{L}^2(\Omega), \end{cases} \tag{11}$$

This is a generalized Stokes Problem [7]. By a similar technique we can derive the equations for $\underline{v}_k,\ p_k$ with $k \geq 2$. However we will not further carry on this development in this work.

4 The Shape Optimization Problem

Suppose now that the function $f_1(x)$ in (10) is unknown as well as $\underline{v}_1,\ p_1$. To complete problem (10) we will have to formulate some "additional equations"; otherwise we should require that f_1 be determined by minimizing a suitable "cost functional".
Problem (3) can be supplemented by the "additional equations":

$$\mathcal{C}(f, \underline{v}, p) = 0 \tag{12}$$

where $\mathcal{C}$ is an operator (linear or nonlinear) defined on $\mathbb{H}^1_0(x_1, x_2) \times \mathbb{X} \times \mathbb{L}^2(\Omega)$. (We consider now $f \in \mathbb{H}^1_0$ for convenience). We assume $\mathcal{C}$ to be smooth with respect to its variables $f, \underline{v}, p$. Using the representations (2) and (7) we derive from (12) the following equation:

$$\mathcal{C}(f, \underline{v}, p) = \mathcal{C}(f_0, \underline{v}_0, p_0) + \varepsilon \mathcal{C}_1(f_1, \underline{v}_1, p_1) + \mathcal{O}(\varepsilon^2) = 0, \quad \forall \varepsilon \in [-\varepsilon_0, \varepsilon_0] \tag{13}$$

where

$$\mathcal{C}_1(f_1, \underline{v}_1, p_1) := \frac{\partial \mathcal{C}}{\partial \varepsilon}(f, \underline{v}, p)|_{\varepsilon=0}. \tag{14}$$

If we assume that the data of our problems are such that $\mathcal{C}(f_0, \underline{v}_0, p_0) = 0$, then we can use

$$\mathcal{C}_1(f_1, \underline{v}_1, p_1) = 0 \tag{15}$$

as additional equation to complete (10). An alternative approach would consist in replacing the exact controllability equation (15) by the following minimization problem:

$$\inf_{f_1} \int_\Omega \frac{f_0}{\beta} |\mathcal{C}_1(f_1, \underline{v}_1, p_1)|^2 dxdy \tag{16}$$

where we assume that $\mathcal{C}_1$ has image in $\mathbb{L}^2(\Omega)$. Note that (16) is a weak statement of (15).
In the next sections we apply the approach described above for the completion of (10) and we will use the following special choice of (12):

$$\mathcal{C}(f, \underline{v}) := ((\nabla \times \underline{v}) \circ \mathrm{T}_f^{-1})(x, y) - \mathcal{R}_{obs,\varepsilon}(x, y)\ in\ \Omega_{wd} \subseteq \Omega, \tag{17}$$

where Ω_{wd} is a suitable subset of Ω in which we want our additional equation (or our "control") to take place. Morever

$$\mathcal{R}_{obs,\varepsilon} = \mathcal{R}_{obs,0} + \varepsilon \mathcal{R}_{obs,1} + \varepsilon^2 \mathcal{R}_{obs,2} + \dots\ ,\ \mathcal{R}_{obs,0} := ((\nabla \times \underline{v}_0) \circ \mathrm{T}_{f_0}^{-1}). \tag{18}$$

Then we have: $\mathcal{C}(f_0, \underline{v}_0) = 0$, while the equation (15) reads:

$$\mathcal{C}(f_1, \underline{v}_1) = \mathcal{R}(f_0)\underline{v}_1 + m_1 \mathcal{R}_f f_1 - \mathcal{R}_{obs,1} = 0 \ in \ \Omega_{wd}, \tag{19}$$

where

$$\mathcal{R}(f_0)\underline{v}_1 = (\nabla \times \underline{v}_1) \circ \mathrm{T}_{f_0}^{-1}(x,y) = \frac{\partial v_1}{\partial x} - \frac{y f_{0,x}}{f_0}\frac{\partial v_1}{\partial y} - \frac{\beta}{f_0}\frac{\partial u_1}{\partial y},$$

$$\mathcal{R}_f f_1 := \mathcal{R}_f(f_1, \underline{v}_0) = -y\frac{(f_{1,x} f_0 - f_{0,x} f_1)}{f_0^2}\frac{\partial v_0}{\partial y} + \frac{\beta f_1}{f_0^2}\frac{\partial u_0}{\partial y}.$$

Therefore we have the problem: find $\underline{v}_1 \in \mathbb{X},\ p_1 \in \mathbb{L}^2(\Omega),\ f_1 \in \mathbb{H}_0^1(x_1, x_2)$ s.t.

$$\begin{cases} a(f_0; \underline{v}_1, \hat{\underline{v}}) = b(f_0; p_1, \hat{\underline{v}}) + b_f(f_1; p_0, \hat{\underline{v}}) + G_1(f_1; \hat{\underline{v}}) - a_f(f_1; \underline{v}_0, \hat{\underline{v}}) \ \forall \hat{\underline{v}} \in \mathbb{X}, \\ b(f_0; \hat{\underline{p}}, \underline{v}_1) + b_f(f_1; \hat{p}, \underline{v}_0) = 0 \ \forall \hat{p} \in \mathbb{L}^2(\Omega), \\ \mathcal{R}(f_0)\underline{v}_1 + m_1 \mathcal{R}_f f_1 - \mathcal{R}_{obs,1} = 0 \ in \ \Omega_{wd}, \end{cases} \tag{20}$$

where $\mathcal{R}_{obs,1}$ is a given function. Problem (20) is an "exact controllability problem". These problems have solutions in some particular cases only. For this reason we replace (20) by the following optimal control problem: find $\underline{v}_1 \in \mathbb{X},\ p_1 \in \mathbb{L}^2(\Omega),\ f_1 \in \mathbb{H}_0^1(x_1, x_2)$ s.t.

$$\begin{cases} a(f_0; \underline{v}_1, \hat{\underline{v}}) - b(f_0; p_1, \hat{\underline{v}}) = b_f(f_1; p_0, \hat{\underline{v}}) + G_1(f_1; \hat{\underline{v}}) - a_f(f_1; \underline{v}_0, \hat{\underline{v}}) \ \forall \hat{\underline{v}} \in \mathbb{X}, \\ b(f_0; \hat{\underline{p}}, \underline{v}_1) + b_f(f_1; \hat{p}, \underline{v}_0) = 0 \ \forall \hat{p} \in \mathbb{L}^2(\Omega), \\ \inf_{f_1} = \frac{\alpha}{2}||f_1||^2_{\mathbb{H}_0^1(x_1,x_2)} + \gamma_1 J_1(f_1, \underline{v_1}), \end{cases} \tag{21}$$

where

$$J_1(f_1, \underline{v}_1) = \frac{1}{2}\int_\Omega m_{wd}\frac{f_0}{\beta}(\mathcal{R}(f_0)\underline{v}_1 + m_1 \mathcal{R}_f f_1 - \mathcal{R}_{obs,1})^2 dxdy,$$

$\alpha = const \geq 0$ is a small regularization parameter, $\gamma_1 > 0$ is a weight coefficient, m_{wd} is the characteristic function of Ω_{wd}.

Note that the third equation from (20) is considered in (21) in the least square sense; then (21) for $\alpha = 0$ provides the weak statement of problem (20). Otherwise the solution $v_1 = v_1(\alpha),\ p_1 = p_1(\alpha),\ f_1 = f_1(\alpha)$ of (21) represents an approximate (regularized) solution of (20).

We will also consider a generalized optimal control problem still given by (21) however now instead of J_1 we use

$$J(f_1, \underline{v}_1, p_1) = \gamma_1 J_1(f_1, \underline{v}_1) + \gamma_2 J_2(f_1, \underline{v}_1, p_1).$$

Here $\gamma_2 = const \geq 0$ is a weight coefficient, while $J_2(f_1, \underline{v}_1, p_1)$ is an additional functional assumed to be quadratic. An example of $J_2(f_1, \underline{v}_1, p_1)$ follows.

Example 1.

$$J_2(f_1, \underline{v}_1, p_1) := J_2(\underline{v}_1, p_1) = \frac{1}{2}(||p_1 - p_{out,1}||^2_{\mathbb{L}^2(\Gamma_{out})} + \int_{\Gamma_{out}} |\underline{v}_1 - \underline{v}_{out,1}|^2 d\Gamma) \tag{22}$$

where $p_{out}, \underline{v}_{out}$ are given.

5 The variational equations of the optimal control problem

While considering (21) we can still consider the simple domain Ω of Fig. 3(right). Another possibility consists of using the new variable transformation

$$T_{f_0}^{-1}(\underline{\tilde{x}}) = \underline{x},\ \underline{\tilde{x}} \in \Omega,\ \underline{x} \in \Omega_0, \tag{23}$$

which is the identity in $\tilde{\Omega}_2$, while $T_{f_0}^{-1}(\tilde{x}, \tilde{y}) = (\tilde{x}, \frac{f_0(\tilde{x})}{\beta}\tilde{y})$ in $\tilde{\Omega}_1$. After applying (23) we will work in the "unperturbed" domain Ω_0 (see Fig. 4) where the expressions for the bilinear forms in (21) become simpler. Let us use the variable transformation (23). Indeed problem (21) reads upon its reformulation in Ω_0:

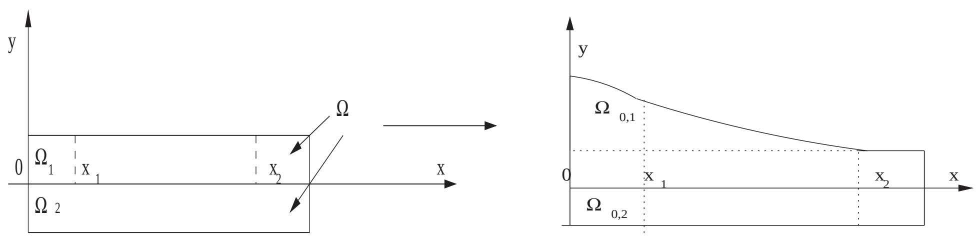

Figure 4: "Simple" domain $\Omega \to \Omega_0$.

find $\underline{v} := \underline{v}_1$, $p := p_1$,$f := f_1$[1] s.t.

$$\begin{cases} a_0(\underline{v}, \underline{\hat{v}}) - b_0(p, \underline{\hat{v}}) = b_f(f; p_0, \underline{\hat{v}}) + G_1(f; \underline{\hat{v}}) - a_f(f; \underline{v}_0, \underline{\hat{v}})\ \forall \hat{v} \in \mathbb{X} \\ b_0(\hat{p}, \underline{v}) + b_f(f; \hat{p}, \underline{v}_0) = 0\ \forall \hat{p} \in \mathbb{L}^2(\Omega), \\ \inf_f = \frac{\alpha}{2}||f||^2_{\mathbb{H}^1_0(x_1,x_2)} + J(f, \underline{v}, p), \end{cases} \tag{24}$$

where

$$a_0(\underline{v}, \underline{\hat{v}}) = \int_{\Omega_0} \nu\Big(\frac{\partial \underline{v}}{\partial x} \cdot \frac{\partial \underline{\hat{v}}}{\partial x} + \frac{\partial \underline{v}}{\partial y} \cdot \frac{\partial \underline{\hat{v}}}{\partial y}\Big) dxdy.$$

$$b_0(p, \underline{\hat{v}}) = \int_{\Omega_0} p\nabla \cdot \underline{\hat{v}} dxdy,$$

$$b_f(f, p_0, \underline{\hat{v}}) = \int_{\Omega_{0,1}} p_0 \mathcal{D}_f(f, \underline{\hat{v}}) dxdy + \int_{\Omega_{0,1}} \frac{f}{f_0} p_0 \nabla \cdot \underline{\hat{v}} dxdy,$$

$$\mathcal{D}_f(f, \underline{\hat{v}}) = -\Big[y\Big(\frac{f_x f_0 - f_{0,x} f}{f_0^2}\Big)\frac{\partial \hat{u}}{\partial y} + \frac{f}{f_0}\frac{\partial \hat{v}}{\partial y}\Big],$$

$$\mathcal{D}_f(f, v_0) := \mathcal{D}_f f,$$

$$G_1(f; \underline{\hat{v}}) = \int_{\Omega_{0,1}} \frac{f}{f_0} \underline{F} \cdot \underline{\hat{v}} dxdy,$$

[1] From now on we denote $\underline{v}_1 = \underline{v},\ p_1 = p,\ f_1 = f$ however we should keep in mind that now $\underline{v}, p, f$ represent the "first corrections" of $\underline{v}_0,\ p_0,\ f_0$ on the unperturbed domain.

$$a_f(f;\underline{v}_0,\hat{\underline{v}}) = \int_{\Omega_{0,1}} \frac{f\nu}{f_0}\nabla\underline{v}_0\cdot\nabla\hat{\underline{v}}dxdy - \int_{\Omega_{0,1}} \nu y\frac{(f_x f_0 - f_{0,x}f)}{f_0^2}(\frac{\partial\underline{v}_0}{\partial y}\cdot\frac{\partial\hat{\underline{v}}}{\partial x}+\frac{\partial\underline{v}_0}{\partial x}\cdot\frac{\partial\hat{\underline{v}}}{\partial y})dxdy+$$

$$-\int_{\Omega_{0,1}} \frac{2f\nu}{f_0}\frac{\partial\underline{v}_0}{\partial y}\cdot\frac{\partial\hat{\underline{v}}}{\partial y}dxdy.$$

$$J(f,\underline{v},p) = \gamma_1 J_1(f,\underline{v}) + \gamma_2 J_2(f,\underline{v},p),$$

$$J_1(f,\underline{v}) = \frac{1}{2}\int_{\Omega_0} m_{wd}|\nabla\times\underline{v} + m_1\mathcal{R}_f f - \mathcal{R}_{obs,1}|^2 dxdy,$$

$$\mathcal{R}_f f := \mathcal{R}_f(f,\underline{v}_0) = -y\frac{(f_x f_0 - f_{0,x}f)}{f_0^2}\frac{\partial v_0}{\partial y} + \frac{f}{f_0}\frac{\partial u_0}{\partial y}.$$

$$\nabla\times\underline{v} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},\ \nabla\cdot\underline{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

and $J_2(f,\underline{v},p)$ are given by corresponding expressions. In order to derive the operator form of problem (24) we introduce the following real spaces :

$$\mathbb{X} \subseteq (\mathbb{L}^2(\Omega))^2 \subseteq \mathbb{X}^*, \mathbb{H}^p \subseteq \mathbb{L}^2(\Omega) \subseteq \mathbb{H}^{p*},$$

$$\mathbb{H}_f \subseteq \mathbb{L}^2(x_1,x_2) \subseteq \mathbb{H}_f^*,$$

$$\mathbb{W} := \mathbb{X}\times\mathbb{H}^p \subseteq \mathbb{H}_0 := (\mathbb{L}^2(\Omega))^2\times\mathbb{L}^2(\Omega) \subseteq \mathbb{W}^*,$$

Let us reformulate (24) in the following form: find $\underline{\Phi} := (\underline{v},p) \in \mathbb{W} = (\mathbb{X}\times\mathbb{H}^p)$, $f\in\mathbb{H}^f$, s.t

$$\begin{cases} \mathcal{L}(\underline{\Phi},\hat{\underline{\Phi}}) = B(f,\hat{\underline{\Phi}})\ \forall\hat{\underline{\Phi}} = (\hat{\underline{v}},\hat{p})\in\mathbb{W}, \\ \inf_{f\in\mathbb{H}_f} = \frac{\alpha}{2}||f||^2_{\mathbb{H}^1} + J(f,\underline{\Phi}), \end{cases} \tag{25}$$

where

$$\mathcal{L}(\underline{\Phi},\hat{\underline{\Phi}}) := a_0(\underline{v},\hat{\underline{v}}) - b_0(p,\hat{\underline{v}}) + b_0(\hat{p},\underline{v}),$$

$$B(f,\hat{\underline{\Phi}}) := b_f(f,p_0,\hat{\underline{v}}) + G_1(f,\hat{\underline{v}}) - a_f(f,\underline{v}_0,\hat{\underline{v}}) - b_f(f,\hat{p},\underline{v}_0)$$

Should $\underline{\Phi}$ be a solution of (25), then

$$\alpha(f,\hat{f})_{\mathbb{H}_f} + \langle J'_\Phi(f,\underline{\Phi}),\underline{\Phi}_{\hat{f}}\rangle + \langle J'_f(f,\underline{\Phi}),\hat{f}\rangle = 0, \tag{26}$$

for any $\hat{f}\in\mathbb{H}_f$ ($\hat{f}$ is the independent variation), where $\underline{\Phi}_{\hat{f}}\in\mathbb{W}$ satisfies the following equation:

$$\mathcal{L}(\underline{\Phi}_{\hat{f}},\hat{\underline{\Phi}}) = B(\hat{f},\hat{\underline{\Phi}})\ \forall\hat{\underline{\Phi}}\in\mathbb{W}. \tag{27}$$

In (26), $J'_\Phi = \frac{\partial J}{\partial\underline{\Phi}}$ and $J'_f = \frac{\partial J}{\partial f}$ are partial derivatives of J, while $\langle Q,\underline{\Phi}\rangle$ is the duality between $\mathbb{W}$ and $\mathbb{W}^*$ and $\langle g,f\rangle$ the duality between $\mathbb{H}_f$ and $\mathbb{H}_f^*$. Then we can rewrite (25) as a system of "optimality conditions":

$$\begin{cases} \mathcal{L}(\underline{\Phi},\hat{\underline{\Phi}}) = B(f,\hat{\underline{\Phi}})\ \forall\hat{\underline{\Phi}}\in\mathbb{W}, \\ \alpha(f,\hat{f})_{\mathbb{H}^f} + \langle J'_\Phi(f,\underline{\Phi}),\underline{\Phi}_{\hat{f}}\rangle + \langle J'_f(f,\underline{\Phi}),\hat{f}\rangle = 0\ \forall\hat{f}\in\mathbb{H}_f \end{cases} \tag{28}$$

The element $\underline{\Phi}_{\hat{f}}$ can be eliminated from (28) by introducing the adjoint problem: find $\underline{Q} := (\underline{q},\sigma)\in\mathbb{W}$ s.t.

$$\mathcal{L}^*(\underline{Q},\hat{\underline{W}}) := \mathcal{L}(\hat{\underline{W}},\underline{Q}) = \langle J'_\Phi(f,\underline{\Phi}),\hat{\underline{W}}\rangle\ \forall\hat{\underline{W}}\in\mathbb{W}. \tag{29}$$

Since $\underline{\Phi}_{\hat{f}} \in \mathbb{W}$ we can choose $\underline{\hat{W}} = \underline{\Phi}_{\hat{f}}$ in (29), yielding

$$\langle J'_\Phi(f, \underline{\Phi}), \underline{\Phi}_{\hat{f}} \rangle = \mathcal{L}(\underline{\Phi}_{\hat{f}}, \underline{Q}) = B(\hat{f}, \underline{Q}) \tag{30}$$

and the system of variational equations (28) reads now as follows:

$$\begin{cases} \mathcal{L}(\underline{\Phi}, \underline{\hat{\Phi}}) = B(f, \underline{\hat{\Phi}}) \ \forall \underline{\hat{\Phi}} \in \mathbb{W}, \\ \mathcal{L}^*(\underline{Q}, \underline{\hat{W}}) = \langle J'_\Phi(f, \underline{\Phi}), \underline{\hat{W}} \rangle \ \forall \underline{\hat{W}} \in \mathbb{W}, \\ \alpha(f, \hat{f})_{\mathbb{H}_f} + B(\hat{f}, \underline{Q}) + \langle J'_f(f, \underline{\Phi}), \hat{f} \rangle = 0 \ \forall \hat{f} \in \mathbb{H}_f. \end{cases} \tag{31}$$

The first equation is the state equation. Let us define the following operators. See [13], [12], [1].

$$L : \mathbb{W} \to \mathbb{W}^*, \ (L\underline{\Phi}, \underline{\hat{\Phi}})_{\mathbb{H}_0} := \mathcal{L}(\underline{\Phi}, \underline{\hat{\Phi}}), \ \forall \underline{\Phi}, \ \underline{\hat{\Phi}} \in \mathbb{W},$$

$$L^* : \mathbb{W} \to \mathbb{W}^*, \ (\underline{\hat{W}}, L^*\underline{Q})_{\mathbb{H}_0} = (L\underline{\hat{W}}, \underline{Q})_{\mathbb{H}_0}, \forall \underline{Q}, \underline{\hat{W}} \in \mathbb{W},$$

$$B : \mathbb{H}_f \to \mathbb{W}^*, \ (Bf, \underline{\Phi})_{\mathbb{H}_0} = B(f, \underline{\Phi}) \ \forall f, \underline{\Phi},$$

$$\Lambda_w : \mathbb{W}^* \to \mathbb{W}^*, \ (\Lambda_w J_\Phi(f, \underline{\Phi}), \underline{\hat{W}})_{\mathbb{H}_0} := \langle J'_\Phi(f, \underline{\Phi}), \underline{\hat{W}} \rangle,$$

$$\Lambda_f : \mathbb{H}_f^* \to \mathbb{H}_f^*, \ (\Lambda_f J_f(f, \underline{\Phi}), \hat{f})_{\mathbb{L}^2(x_1, x_2)} := \langle J'_f(f, \underline{\Phi}), \hat{f} \rangle.$$

Now the system (31) can be written in operator form as follows:

$$\begin{cases} L\underline{\Phi} = Bf \quad (in \ \mathbb{W}^*), \\ L^*\underline{Q} = \Lambda_w J_\Phi(f, \underline{\Phi}) \quad (in \ \mathbb{W}^*), \\ \alpha\Lambda_c f + B^*\underline{Q} + \Lambda_f J_f(f, \underline{\Phi}) = 0 \quad (in \ (\mathbb{H}_f)^*), \end{cases} \tag{32}$$

where Λ_c is the extension to $\mathbb{H}_f$ of the following operator $\Lambda_{c,0}$:

$$\Lambda_{c,0} f := -f_{xx} + f, \quad \mathcal{D}(\Lambda_{c,0}) = \mathbb{H}^2 \cap \mathbb{H}_f$$

Remark. The system (32) with a cost functional $J = \|C\underline{\Phi} - \underline{\Psi}\|^2_{\mathbb{H}_{ob}}$, where $C : \mathbb{W} \to \mathbb{H}_{ob}$ is a given operator and $\underline{\Psi} \in \mathbb{H}_{ob}$ a given observation function analyzed in [1]. In this case $J'_f = 0$ and $\Lambda_w J'_\Phi(f, \underline{\Phi}) = C^*(C\underline{\Phi} - \underline{\Psi})$.

6 Uniqueness and existence results

We analyze the particular cases where the cost functional J is chosen as outlined by **Example 1** of Section 4.

6.1. Let J be the functional J_2 of in Example 1. Then

$$J(f, \underline{\Phi}) = J(f, \underline{v}, p) = \frac{\gamma_1}{2} \int_{\Omega_0} m_{wd} |\nabla \times \underline{v} + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}|^2 d\Omega + \tag{33}$$

$$+ \frac{\gamma_2}{2} \int_{\Gamma_{out}} \left(|p - p_{out}|^2 + |\underline{v} - \underline{v}_{out}|^2 \right) d\Gamma$$

To study the problem in this case we assume that $\Omega_{wd} = \Omega_0$ and we put here:

$$\mathbb{X} := \{ \underline{v} : \underline{v} \in (\mathbb{H}^2(\Omega))^2, \underline{v} = 0 \ on \ \Gamma_{in} \cup \Gamma_{w_1} \cup \Gamma_{w_3} \},$$

$$\mathbb{H}^p := \mathbb{H}^1(\Omega_0),\ \mathbb{H}_f := \mathbb{H}^2(x_1, x_2) \cap \mathbb{H}^1_0(x_1, x_2).$$

Here we consider $\mathbb{H}^2$ for velocity in order to use the uniqueness continuation theorem. The derivatives $J'_\Phi(f, \underline{\Phi})$ and $J'_f(f, \underline{\Phi})$ become

$$\langle J'_\Phi(f, \underline{\Phi}), \underline{\hat{\Phi}}\rangle = \gamma_1 \int_{\Omega_0} m_{wd}(\nabla \times \underline{v} + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}) \cdot (\nabla \times \underline{\hat{v}}) d\Omega +$$

$$+\gamma_2 \int_{\Gamma_{out}} (p - p_{out})\hat{p} d\Gamma + \gamma_2 \int_{\Gamma_{out}} (\underline{v} - \underline{v}_{out}) \cdot \underline{\hat{v}} d\Gamma,$$

$$\langle J'_f(f, \underline{\Phi}), \hat{f}\rangle = \gamma_1 \int_{\Omega_0} m_{wd}(\nabla \times \underline{v} + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}) \mathcal{R}_f \hat{f} d\Omega,$$

$$\forall\ \underline{\hat{\Phi}} = (\underline{\hat{v}}, \hat{p}))\ and \forall\ \hat{f}.$$

The system of variational equations (28) reads: find $\underline{v}_f \in \mathbb{X}, p_f \in \mathbb{H}^p$

$$\begin{cases} a_0(\underline{v}_f, \underline{\hat{v}}) = b_0(p_f, \underline{\hat{v}}) + F(f, \underline{\hat{v}})\ \forall \underline{\hat{v}} \in \mathbb{X}, \\ b_0(\underline{\hat{p}}, \underline{v}_f) + b_f(f; \hat{p}, \underline{v}_0) = 0\ \forall \hat{p} \in \mathbb{H}^p(\Omega), \\ \alpha(f, \hat{f})_{\mathbb{H}_f} + \gamma_1 \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}_f + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}) \cdot (\nabla \times \underline{v}_{\hat{f}} + m_1 \mathcal{R}_f \hat{f}) d\Omega + \\ +\gamma_2 \int_{\Gamma_{out}} ((p_f - p_{out}) p_{\hat{f}} + (\underline{v}_f - \underline{v}_{out}) \cdot \underline{v}_{\hat{f}}) d\Gamma = 0\ \forall \hat{f} \in \mathbb{H}_f, \end{cases} \tag{34}$$

where

$$F(f, \underline{\hat{v}}) := b_f(f, p_0, \underline{\hat{v}}) + G_1(f, \underline{\hat{v}}) - a_f(f, \underline{v}_0, \underline{\hat{v}}),$$

and for every $\hat{f}$, $\underline{v}_{\hat{f}} = \underline{v}_f(\hat{f})$, $p_{\hat{f}} = p_f(\hat{f})$ denote the solution of the system given by the first and second equations in (34) corresponding to a right end side $f = \hat{f}$. The system (31) is: find $\underline{v}_f \in \mathbb{X}, p_f \in \mathbb{H}^p$

$$\begin{cases} a_0(\underline{v}_f, \underline{\hat{v}}) = b_0(p_f, \underline{\hat{v}}) + F(f, \underline{\hat{v}})\ \forall \underline{\hat{v}} \in \mathbb{X}, \\ b_0(\underline{\hat{p}}, \underline{v}_f) + b_f(f; \hat{p}, \underline{v}_0) = 0\ \forall \hat{p} \in \mathbb{H}^p(\Omega), \\ a_0(\underline{\hat{q}}, \underline{q}) = -b_0(\sigma, \underline{\hat{q}}) + \gamma_1 \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}_f + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}) \cdot (\nabla \times \underline{\hat{q}}) d\Omega + \\ +\gamma_2 \int_{\Gamma_{out}} (\underline{v}_f - \underline{v}_{out}) \cdot \underline{\hat{q}} d\Gamma\ \forall \underline{\hat{q}} \in \mathbb{X}, \\ -b_0(\hat{\sigma}, \underline{q}) = \gamma_2 \int_{\Gamma_{out}} (p_f - p_{out}) \hat{\sigma} d\Gamma\ \forall \hat{\sigma} \in \mathbb{H}^p, \\ \alpha(f, \hat{f})_{\mathbb{H}_f} + F(\hat{f}, \underline{q}) - b_f(\hat{f}; \sigma, \underline{v}_0) + \\ +\gamma_1 \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}_f + m_1 \mathcal{R}_f f - \mathcal{R}_{obs,1}) m_1 \mathcal{R}_f \hat{f} d\Omega = 0\ \forall \hat{f} \in \mathbb{H}_f. \end{cases} \tag{35}$$

In the sequel we assume that the generalized Stokes problem (10) (see ref. [7]) has a unique solution for any given $\underline{v}_0$, p_0 (the solution in the unperturbed domain Ω_0) and for each $f \in \mathbb{H}_f$. (See [8]).
Consider now the problem (35) for $\alpha > 0$.

Proposition I. *For any $\alpha > 0$ problem (35) has a unique solution for each given $\underline{\mathcal{R}}_{obs,1}$.*
Proof. Following [1], we formally invert L and L^* in the first and second equations of (32) then we substitute $\underline{\Phi}$, $\underline{Q}$ into the third equation and we obtain the following weak problem: $f \in \mathbb{H}_f$ satisfies:

$$\alpha(f, \hat{f})_{\mathbb{H}_f} + (Af, A\hat{f})_{\mathbb{L}^2(x_1,x_2)} = (G, A\hat{f})_{\mathbb{L}^2(x_1,x_2)}\ \forall \hat{f} \in \mathbb{H}_f, \tag{36}$$

where A is a linear operator, which depends on previous operators from variational equations, while G will depend on the data more precisely from (34) we obtain:

$$(f,\hat{f})_{\mathbb{H}_f} = (\Lambda_f f,\hat{f})_{\mathbb{L}^2(x_1,x_2)},$$

$$(Af,A\hat{f})_{\mathbb{L}^2(x_1,x_2)} = \gamma_1\int_\Omega m_{wd}(\nabla\times\underline{v}+m_1\mathcal{R}_f f)\cdot(\nabla\times\underline{v}_{\hat{f}}+m_1\mathcal{R}_f\hat{f})d\Omega+$$

$$+\gamma_2\int_{\Gamma_{out}}(pp_{\hat{f}}+\underline{v}\cdot\underline{v}_{\hat{f}})d\Gamma,$$

$$(G,A\hat{f})_{\mathbb{L}^2(x_1,x_2)} = \gamma_1\int_\Omega m_{wd}\mathcal{R}_{obs,1}\cdot(\nabla\times\underline{v}_{\hat{f}}+m_1\mathcal{R}_f\hat{f})d\Omega+\gamma_2\int_{\Gamma_{out}}(p_{out}p_{\hat{f}}+\underline{v}_{out}\cdot\underline{v}_{\hat{f}})d\Gamma,$$

where $\underline{\Phi}=(\underline{v},p)=L^{-1}Bf$, $\underline{\Phi}_{\hat{f}}=(\underline{v}_{\hat{f}},p_{\hat{f}})=L^{-1}B\hat{f},\forall\hat{f}\in\mathbb{H}_f$.
We see, that if $\alpha>0$ then the problem (36) has unique solution which satisfies and: $\|f\|^2_{\mathbb{H}_f}\le\|G\|^2/(2\alpha)<\infty$. Correspondingly we can construct $\underline{v}$, p, $\underline{q}$, σ, which jointly with f provides the unique solution of (35).

Consider now the problem (35) with $\alpha=0$.

Proposition II. *Assume that: i) The solution of the generalized Stokes problem satisfies* $\left(\frac{\partial v_0}{\partial y}\right)^2+\left(\frac{\partial u_0}{\partial y}\right)^2>0$ *at* $y=0,\ x\in(x_1,x_2)$ *ii) problem (35) has a solution. Then this solution is unique in the class* $(\mathbb{H}^2(\Omega))^2\times\mathbb{H}^1(\Omega)\times\mathbb{W}^{1,\infty}(x_1,x_2)$.
Proof. Let $(\underline{v}_1,\dots,f_1)$ and $(\underline{v}_2,\dots,f_2)$ be two solutions of (35). Then for $\underline{v}=\underline{v}_1-\underline{v}_2,\dots,f=f_1-f_2$ from (34) we obtain:

$$\left\{\begin{array}{l} a_0(\underline{v},\hat{\underline{v}})=b_0(p,\hat{\underline{v}})+F(f,\hat{\underline{v}})\ \forall\hat{\underline{v}}\in\mathbb{X},\\ b_0(\hat{\underline{p}},\underline{v})+b_f(f;\hat{p},\underline{v}_0)=0\ \forall\hat{p}\in\mathbb{H}^p(\Omega),\\ \nabla\times\underline{v}+m_1\mathcal{R}_f f=0\ in\ \Omega,\\ p=0,\ \underline{v}=0\ on\ \Gamma_{out}.\end{array}\right. \tag{37}$$

Consider the second and third equation from (37) in $\Omega_{2,0}$

$$\nabla\cdot\underline{v}=0,\ \nabla\times\underline{v}=0\ in\ \Omega_{2,0}.$$

Then $\Delta\underline{v}=0$ in $\Omega_{2,0}$. Considering $\hat{\underline{v}}$ with $supp(\hat{\underline{v}})\subseteq\Omega_{2,0}$ from the first equation of (37) we find $\nabla p=0$, then $p=const$ in $\Omega_{2,0}$ and $-p\cdot\underline{n}+\nu\frac{\partial\underline{v}}{\partial\underline{n}}=0$ *on* Γ_{out}. Since $p=0$ *on* Γ_{out} then $p=0$ *in* $\Omega_{2,0}$ and $\nu\frac{\partial\underline{v}}{\partial\underline{n}}=0$ *on* Γ_{out} too. Consequently, $\underline{v}$ satisfies:

$$\Delta\underline{v}=0\ in\ \Omega_{2,0},\ \underline{v}=\nu\frac{\partial\underline{v}}{\partial\underline{n}}=0\ on\ \Gamma_{out}.$$

This problem has only the trivial solution $\underline{v}=0$ *in* $\Omega_{2,0}$. Since $\underline{v}\in(\mathbb{H}^2(\Omega))^2$ then

$$\underline{v}=\frac{\partial\underline{v}}{\partial\underline{n}}=0\ on\ \Gamma_0:=\{(x,y):y=0,x_1<x<x_2\}.$$

Consider now the second and third equations from (37) in $\Omega_{1,0}$:

$$\left\{\begin{array}{l}\nabla\cdot\underline{v}-\left[y\left(\frac{f_xf_0-f_{0,x}f}{f_0^2}\right)\frac{\partial u_0}{\partial y}+\frac{f}{f_0}\frac{\partial v_0}{\partial y}\right]=0\ in\ \Omega_{1,0},\\ \nabla\times\underline{v}-\left[y\left(\frac{f_xf_0-f_{0,x}f}{f_0^2}\right)\frac{\partial v_0}{\partial y}-\frac{f}{f_0}\frac{\partial u_0}{\partial y}\right]=0\ in\ \Omega_{1,0}.\end{array}\right. \tag{38}$$

On Γ_0 we have:

$$\nabla \cdot \underline{v} - \frac{f}{f_0}\frac{\partial v_0}{\partial y} = 0,\ \nabla \times \underline{v} + \frac{f}{f_0}\frac{\partial u_0}{\partial y} = 0,$$

$$|f(x)| = f_0 \frac{\left[(\nabla \cdot \underline{v})^2 + (\nabla \times \underline{v})^2\right]^{1/2}}{\left[\left(\frac{\partial v_0}{\partial y}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2\right]^{1/2}} \ on\ \Gamma_0,$$

(the dependence of the right end side on x and y is understood). Since $\underline{v} = \frac{\partial \underline{v}}{\partial \underline{n}} = \frac{\partial \underline{v}}{\partial y} = 0$ *on* Γ_0, then

$$\nabla \cdot \underline{v}|_{y=0} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}|_{y=0} = 0,\ \nabla \times \underline{v}|_{y=0} = \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}|_{y=0} = 0,\ x \in (x_1, x_2).$$

i.e. $f(x) = 0$. Therefore, $\underline{v} = 0,\ p = 0$ too.

Let us once more note, that if $\gamma_2 > 0$ and we introduce into considerations the cost functional J_2, then we overdeterminate the problem (15) for $\alpha = 0$ and the initial problem. Therefore in this case we have usually uniqueness results, however not existence results generally. But in some physical problems the above overdeterminations (and the term $\alpha\|f\|^2_{\mathbb{H}_f}$ also) are reasonable and have a physical sense, therefore in these cases we can consider the optimal control problems like (16) as the problems to be independent of the initial problem (where we have only J_1). Here, we have also existence results and can name these optimal control problems as the "optimal shape design problems". Nevertheless, it is interesting to study solvability results of above variational problems as $\alpha = \gamma_2 = 0$.

7 Iterative Processes

In this section we propose some iterative processes which are well suited for solving the variational equations obtained in the previous sections.

7.1. Consider the problem (32); if for $k = 0, 1, \dots$ $f^{(k)}$ is known then $f^{(k+1)}$ can be determinated by solving the following equations ([1]):

$$\begin{cases} L\Phi^{(k)} = Bf^{(k)}, \\ L^*\underline{Q}^{(k)} = \Lambda_w J_\Phi(f^{(k)}, \underline{\Phi}^{(k)}), \\ \Lambda_c w^{(k)} = B^*\underline{Q}^{(k)} + \Lambda_f J_f(f^{(k)}, \underline{\Phi}^{(k)}), \\ f^{(k+1)} = f^{(k)} - \tau_k(\alpha f^{(k)} + w^{(k)}), \end{cases} \tag{39}$$

where $\{\tau_k\}$ is a family of parameters whose determination follows from the theory of extremal problems ([32]), the general theory of iterative processes ([16], [25], [27]), and the ill-posed problems theory ([28] and [30]). The step (39) would read as follows for the variational form (31) of problem (32):

$$\begin{cases} \mathcal{L}(\underline{\Phi}^{(k)}, \hat{\underline{\Phi}}) = B(f^{(k)}, \hat{\underline{\Phi}})\ \forall \hat{\underline{\Phi}} \in \mathbb{W}, \\ \mathcal{L}(\hat{\underline{W}}, \underline{Q}^{(k)}) = \langle J'_\Phi(f^{(k)}, \underline{\Phi}^{(k)}), \hat{\underline{W}}\rangle\ \forall \hat{\underline{W}} \in \mathbb{W}, \\ (w^{(k)}, \hat{f})_{\mathbb{H}_f} = B(\hat{f}, \underline{Q}^{(k)}) + \langle J'_f(f^{(k)}, \underline{\Phi}^{(k)}), \hat{f}\rangle\ \forall \hat{f} \in \mathbb{H}_f, \\ f^{(k+1)} = f^{(k)} - \tau_k(\alpha f^{(k)} + w^{(k)}). \end{cases} \tag{40}$$

7.2. Consider now problem (34) (with $\Omega_{wd} \subseteq \Omega$). The iterative process (40) for this problem read as follows:

$$\begin{cases} a_0(\underline{v}^{(k)}, \underline{\hat{v}}) = b_0(p^{(k)}, \underline{\hat{v}}) + F(f^{(k)}, \underline{\hat{v}}) \ \forall \underline{\hat{v}} \in \mathbb{X}, \\ b_0(\hat{p}, \underline{v}^{(k)}) + b_f(f^{(k)}; \hat{p}, \underline{v}_0) = 0 \ \forall \hat{p} \in \mathbb{H}^p(\Omega), \\ a_0(\underline{\hat{q}}, \underline{q}^{(k)}) = -b_0(\sigma^{(k)}, \underline{\hat{q}}) + \gamma_1 \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}^{(k)} + m_1 \mathcal{R}_f f^{(k)} - \mathcal{R}_{obs,1}) \cdot \\ \cdot (\nabla \times \underline{\hat{q}}) d\Omega + \gamma_2 \int_{\Gamma_{out}} (\underline{v}^{(k)} - \underline{v}_{out}) \cdot \underline{\hat{q}} d\Gamma \ \forall \underline{\hat{q}} \in \mathbb{X}, \\ -b_0(\hat{\sigma}, \underline{q}^{(k)}) = \gamma_2 \int_{\Gamma_{out}} (p^{(k)} - p_{out}) \hat{\sigma} d\Gamma \ \forall \hat{\sigma} \in \mathbb{H}^p, \\ (w^{(k)}, \hat{f})_{\mathbb{H}_f} = F(\hat{f}, \underline{\hat{q}}) - b_f(\hat{f}; \hat{\sigma}^{(k)}, \underline{v}_0) + \\ +\gamma_1 \int_{\Omega_0} m_{wd}(\nabla \times \underline{v}^{(k)} + m_1 \mathcal{R}_f f^{(k)} - \mathcal{R}_{obs,1}) m_1 \mathcal{R}_f \hat{f} d\Omega \ \forall \hat{f} \in \mathbb{H}_f, \\ f^{(k+1)} = f^{(k)} - \tau_k(\alpha f^{(k)} + w^{(k)}), \quad k = 0, 1, \dots . \end{cases} \tag{41}$$

Consider now the *finite dimensional case* in which the function $f, \{f^{(k)}\}, \hat{f}$ all are sought for in a finite-dimensional subspace $\mathbb{H}_{f,N} \subset \mathbb{H}_f$ of dimension $N < \infty$, whose basis $\varphi_i \in \mathbb{W}^{1,\infty}(x_1, x_2), i = 1, 2, \dots, N$. Then the following theorem holds true.

Theorem 1. *Assume that* $\Omega_{wd} = \Omega$, $\left(\frac{\partial v_0}{\partial y}\right)^2 + \left(\frac{\partial u_0}{\partial y}\right)^2 > 0$ *at* $y = 0, x \in (x_1, x_2)$. *Then:*

1. *The problem (34) is correctly solvable for* $\alpha \geq 0$ *and all* $N < \infty$*;*

2. *The iterative process (41) is convergent for any* $\alpha > 0$, $N < \infty$ *and provided the parameters* $\tau_k > 0$, $k = 0, 1, 2, \dots .$ *are small enough;*

3. *If* α *is sufficiently small while* k *is sufficiently large, then* $\{\underline{v}^{(k)}, p^{(k)}, f^{(k)}\}$ *can be taken as an approximate solution of problem (34).*

Proof:

1. The existence of the solution for $\alpha > 0$ has been proved early. Let us consider the case $\alpha = 0$. Since $f = \Sigma_{i=1}^N a_i \varphi_i \in \mathbb{H}_{f,N}$ then in the form (36) with $\alpha = 0$ we conclude that this equation is correctly solvable (because the problem (34) can have only unique solution in $\mathbb{X} \times \mathbb{H}^p \times \mathbb{H}_f$, see **Proposition II**). We assume the generalized Stokes problem to be correctly solvable for given $f \in \mathbb{H}_f$. Hence the problem (34) is correctly solvable too.

2. If $\alpha > 0$ then the bilinear form on the left hand side of (36) is coercive and continuous with respect to the norm $\|f\|_{A,\alpha} = \sqrt{\alpha \|f\|^2_{\mathbb{H}_f} + \|Af\|^2_{\mathbb{L}^2(x_1,x_2)}}$. Then according to the general theory of iterative algorithm the process given by

$$(f^{(k+1)}, \hat{f})_{\mathbb{H}_f} = (f^{(k)}, \hat{f})_{\mathbb{H}_f} - \tau(\alpha(f^{(k)}, \hat{f})_{\mathbb{H}_f} + (Af^{(k)}, A\hat{f})_{\mathbb{L}^2(x_1,x_2)}) -$$

$$-(G, A\hat{f})_{\mathbb{L}^2(x_1,x_2)}, \ k = 0, 1, \dots$$

is convergent for small $\tau > 0$. Hence the process (41) is convergent also and

$$\|\underline{v}^{(k)} - \underline{v}\|_{\mathbb{X}} + \|p^{(k)} - p\|_{\mathbb{H}^p} + \|f - f^{(k)}\|_{\mathbb{H}_f} \to 0, \ k \to \infty. \tag{42}$$

If $\Lambda_C^{-1} A^* A \in [C_1, C_2], C_1, C_2 = const$, and $\tau_k = 2/(2\alpha + C_1 + C_2)$ then (42) becomes (see [1]):

$$\|\underline{v}^{(k)} - \underline{v}\|_{\mathbb{X}} + \|p^{(k)} - p\|_{\mathbb{H}^p} + \|f - f^{(k)}\|_{\mathbb{H}_f} \leq C\Big(\frac{C_2 - C_1}{2\alpha + C_1 + C_2}\Big)^k \to 0,\ k \to \infty. \tag{43}$$

3. Let $\underline{v}_0,\ p_0,\ f_0$ be a solution of (34) when $\alpha = 0$. According to the theory of ill-posed problem ([28] and [30]) we have: $\|f_0 - f_\alpha\|_{\mathbb{H}^p} \to 0$ as $\alpha \to +0$, where $(f_\alpha, \underline{v}_\alpha, p_\alpha)$ is the solution of (34) for $\alpha > 0$. Hence

$$\|\underline{v}_0 - \underline{v}_\alpha\|_{\mathbb{X}} + \|p_0 - p_\alpha\|_{\mathbb{H}^p} \to 0,\ as\ \alpha \to +0.$$

Then owing to (42) we conclude that the conclusions of our theorem holds true also.
The simple schemes in Fig.(5) can be considered as examples of the above problems when $f \in \mathbb{H}_{f,N}$ for small N (the dimension of $\mathbb{H}_{f,N}$).

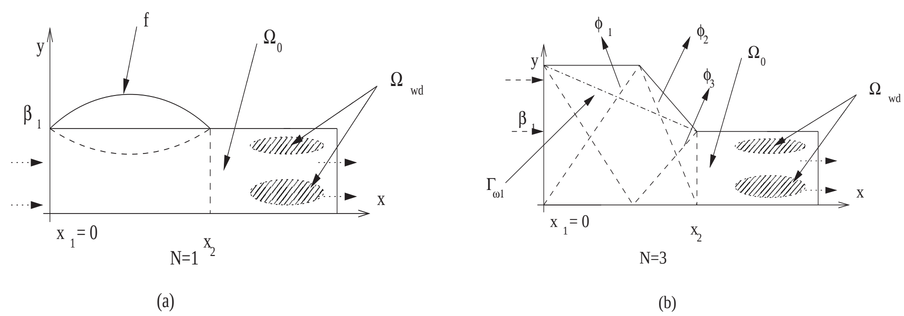

Figure 5: Domain Ω with N shape functions: (a) $N = 1,\ f = \beta_1 + a\varphi_0(x)$, $\varphi_0 = x(x_2 - x)$; (b) $N = 3,\ f = \beta_1 + \Sigma_{i=1}^3 a_i\varphi_i$.

8 Test Problem and Numerical Results

To test our method we consider some test problems on simplified configurations. Numerical simulations have been carried out using *Bamg* [11], a Bi-dimensional Anisotropic Mesh Generator and *FreeFem*, a finite element Library developed at INRIA [10], the French National Institute for Research in Computer Science and Control, with the development of algorithms based on control theory and adjoint formulation for generalized Stokes problem. For application of finite element method to incompressible flow see [9]. In this section we present numerical results using as cost functional the $\mathbb{L}^2$ norm of the vorticity in the downfield zone of the new incoming branch of the bypass.
Wall curvature was considered only in the zone of the incoming branch of the bypass where we set $f_0 = sin(x)$; in other parts we used piecewise constant function. The graft angle of the bypass incoming branch (which influences vorticity) is equal to zero (between the artery and the new incoming branch there isn't a relative angle).
Velocity values $\underline{v}_{in}$ at the inflow are chosen in such a way that the Reynolds

number $Re = \frac{\bar{v} * D}{\nu}$ has order 10^3. Blood kinematic viscosity $\nu = \frac{\mu}{\rho}$ is equal to $4 * 10^{-6}\ m^2\ s^{-1}$, blood density $\rho = 1\ g\ cm^{-3}$ and dynamic viscosity $\mu = 4 * 10^{-2}\ g\ cm^{-1} s^{-1}$; $\bar{v}$ is a mean inflow velocity related with $\underline{v}_{in}$, while D is the arterial diameter (3.5 mm). [23].
Fig. (6)-(8) provide a preliminary account of numerical results and show how the shape of the bypass using generalized steady Stokes equations in an optimal control problem is smoothed out at the corner. Fig. (6) refers to the original configuration; whereas Fig. (7) to the configuration obtained after 25 iterations of the optimization algorithm (the vorticity has been reduced by about the 30%)

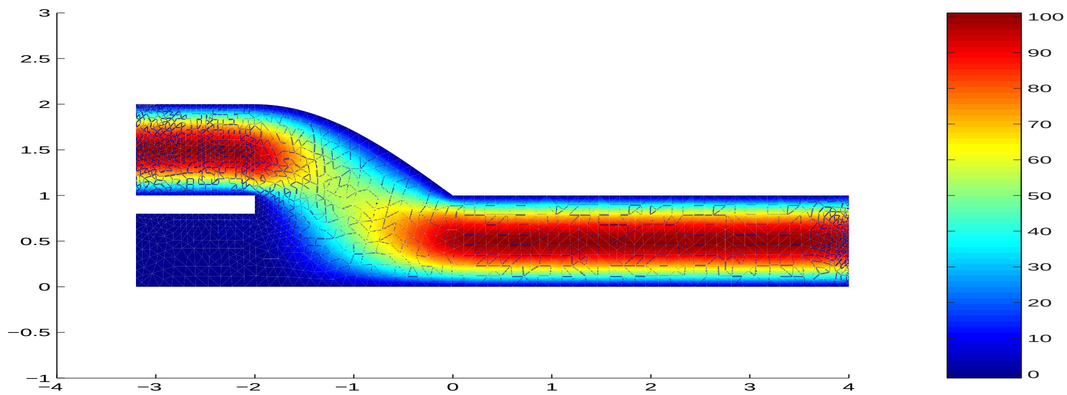

Figure 6: Idealized 2-D bypass configuration before optimal shape design process: iso-velocity $[cms^{-1}]$.

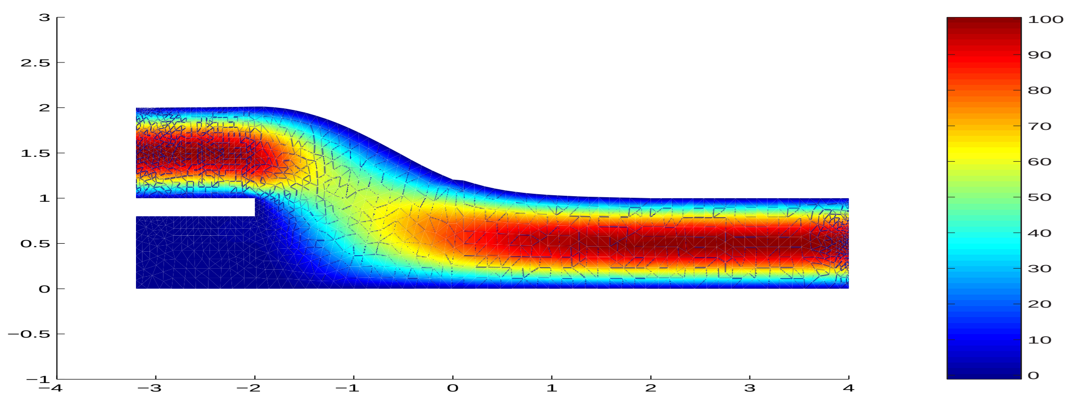

Figure 7: Bypass configuration at the end of shape optimization using first corrections: iso-velocity.

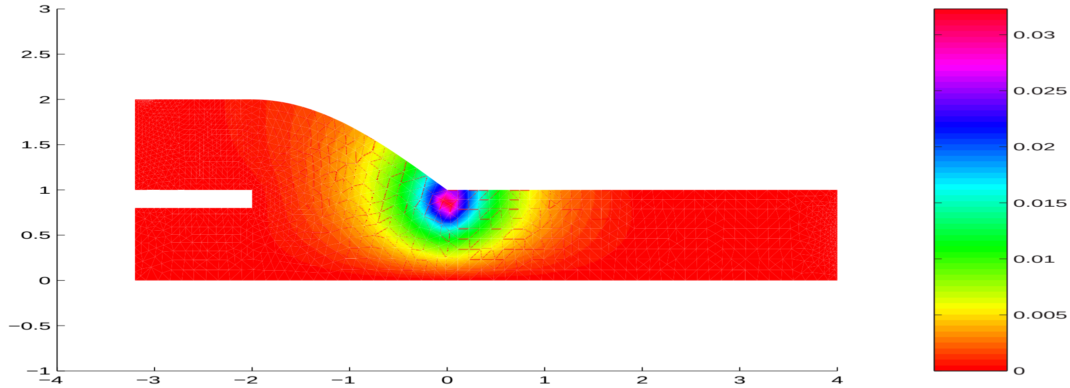

Figure 8: Adjoint solution $\underline{q}$ in Bypass configuration in the reference domain.

9 Future Developments

The development of tools for geometry reconstruction from medical data (medical imaging and other non-invasive means) and their integration with numerical simulation could provide improvements in disease diagnosis procedures.
In this study we have focused on the problem of determining the first corrections for the shape design of simplified two-dimensional bypass configurations.
Using the numerical method developed in this paper it is possible to realize the iterative process for solving initial nonlinear problems. For that it is sufficient to consider $f = f_0 + \varepsilon f_1$, where f_0 is the initial configuration and f_1 the computed first correction, as the new f_0, then to calculate a new first correction and so on.
Optimal control and shape optimization applied to fully unsteady incompressible Stokes and Navier-Stokes equations and possibly the coupled fluid-structure problem and the setting of the problem in a three-dimensional geometry will provide more realistic design indications concerning surgical prosthesis realizations.
A further development will be devoted to build domain decomposition methods ([26]) based on optimal control approaches and efficient schemes for reduced-basis methodology approximations (see for example [20] and [21]) which could be more efficient for use in a repetitive design environment as optimal shape design methodology requires. See [29] for the state of the art of the problem.

Acknowledgements

Bernoulli Center of EPFL is acknowledged for the support of the authors during the special semester on the "Mathematical Modelling of the Cardiovascular System". G.Rozza acknowledges the financial support provided through the European Community's Human Potential Programme under contract HPRN-CT-2002-00270 HaeMOdel. This work has also been supported in part by the Swiss National Science Foundation (Project n. 20-65110.01) and by Italian Cofin2003-MIUR (Italian Research, University and Education Ministry) Project "Numerical Modelling for Scientific Computing and Advanced Applications " and by Indam (Italian Institute of Advanced Mathematics). V. Agoshkov acknowledges the support of Russian Foundation for Basic Research (Project N. 04-01-00615).

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