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A simple analysis of some a posteriori error estimates

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Abstract

We analyze two popular classes of a posteriori error estimates within the abstract framework established by Babuška and Aziz (1972). Within this framework, we find that bounds for the a posteriori error estimates depend on several of the same constants as a priori error estimates, notably the famous *inf–sup* constant. We apply our general theory to some specific finite element approximations for the Poisson equation and Stokes equations. © 1998 Elsevier Science B.V.

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This work is dedicated to Ivo Babuška on the occasion of his seventieth birthday.

1. Introduction

In the classic work [4], Babuška and Aziz established an abstract framework for the mathematical analysis of finite element approximations of partial differential equations. In that work, the *inf–sup* or *LBB* condition was introduced and used to establish existence and uniqueness of solutions, and a priori error estimates for variational approximations.

In this note, our goal is to extend this framework to a posteriori error estimates. The notion of using a posteriori error estimates to measure and control the error in practical finite element calculations was first suggested by Babuška and Rheinboldt [6,7]. Since then, there has been widespread interest in the area; see, for example, [1,2,5,8,14] and the references therein. The survey articles of Verfürth [21,22] are especially useful. Besides providing useful information about the reliability of a given calculation, a posteriori error estimates can also provide the basis for adaptive local mesh refinement, local order refinement and adaptive mesh moving algorithms (h, p and r refinement, respectively).

Let u and u_h be the continuous and discrete solutions, and e_h a *computable* approximate error. Then a posteriori error estimates take the form

$$C_1 |||u - u_h||| \leqslant |||e_h||| \leqslant C_2 |||u - u_h||| \tag{1}$$

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for some appropriate norm $\|\cdot\|$. In our abstract framework, it is easy to derive estimates (1), and to see the form of the constants C_1 and C_2. In particular, we show that these constants depend on continuity and inf–sup constants, a saturation assumption constant, and a strengthened Cauchy inequality. Several of these constants also appear in a priori error estimates. Hence, by treating both in the same abstract framework, it is easy to see some strong connections between a priori and a posteriori estimates. One also obtains some insight about the size of the constants C_1 and C_2, which is clearly important information in practical calculations. In some special cases it can be shown that certain a posteriori error estimates are asymptotically exact, but this topic will not be considered here (see [17,18], for example).

In Section 2, we establish the abstract framework and assumptions, which is essentially that given in [4, Chapters 5 and 6]. In Section 3, we derive and analyze our fundamental a posteriori error estimate. Much of this is similar to [9,11]. In some minor ways our setting is more general (e.g., we allow Petrov–Galerkin type approximations as in [4]). In Section 4, we make an abstract analysis of two popular classes of a posteriori error estimates now in use, those using hierarchical bases [16,25,26], and those based on the solution of local Neumann problems [1,2,12,23]. In Section 5, we present two simple examples, the Poisson equation, and the Stokes system of equations.

2. Abstract setting and assumptions

We consider the nonselfadjoint and possibly indefinite problem: find $u \in \mathcal{H}$ such that

$$B(u,v) = f(v) \quad \forall v \in \mathcal{K}, \tag{2}$$

where $\mathcal{H}$ and $\mathcal{K}$ are appropriate Hilbert spaces, $B(\cdot,\cdot)$ is a bilinear form and $f(\cdot)$ is a linear functional. With respect to the space $\mathcal{H}$, we define an energy norm $\|\cdot\|_{\mathcal{H}}$ associated with the positive definite scalar product $(\cdot,\cdot)_{\mathcal{H}}$

$$\|u\|_{\mathcal{H}}^2 = (u,u)_{\mathcal{H}}. \tag{3}$$

With respect to the space $\mathcal{K}$, we also define an energy norm $\|\cdot\|_{\mathcal{K}}$ associated with the positive definite scalar product $(\cdot,\cdot)_{\mathcal{K}}$ in a similar fashion.

In order to insure that (2) has a unique solution, we assume the bilinear form $B(\cdot,\cdot)$ satisfies the *continuity condition*

$$\left|B(\phi,\eta)\right| \leqslant \nu \|\phi\|_{\mathcal{H}}\|\eta\|_{\mathcal{K}} \quad \forall\phi \in \mathcal{H},\ \forall\eta \in \mathcal{K}. \tag{4}$$

We also assume the *inf–sup conditions*

$$\inf_{\phi\in\mathcal{H},\ \|\phi\|_{\mathcal{H}}=1}\ \sup_{\eta\in\mathcal{K},\ \|\eta\|_{\mathcal{K}}\leqslant 1} B(\phi,\eta) \geqslant \mu > 0, \tag{5}$$

$$\sup_{\phi\in\mathcal{H}} B(\phi,\eta) > 0, \quad \eta \neq 0,\ \eta \in \mathcal{K}. \tag{6}$$

Let $\mathcal{M}_h \subset \mathcal{H}$ and $\mathcal{N}_h \subset \mathcal{K}$ be members of two families of finite dimensional subspaces, characterized by a small parameter h, and consider the approximate problem: find $u_h \in \mathcal{M}_h$ such that

$$B(u_h,v) = f(v) \quad \forall v \in \mathcal{N}_h. \tag{7}$$

To insure a unique solution for (7) we assume the the *inf–sup conditions*

$$\inf_{\phi\in\mathcal{M}_h,\ \|\phi\|_{\mathcal{H}}=1}\ \sup_{\eta\in\mathcal{N}_h,\ \|\eta\|_{\mathcal{K}}\leqslant 1} B(\phi,\eta)\geqslant\mu>0, \tag{8}$$

$$\sup_{\phi\in\mathcal{M}_h} B(\phi,\eta)>0,\quad \eta\neq 0,\ \eta\in\mathcal{N}_h. \tag{9}$$

Babuška and Aziz prove in [4] that with the above assumptions,

$$\|u-u_h\|_{\mathcal{H}}\leqslant\left(1+\frac{\nu}{\mu}\right)\inf_{v\subset\mathcal{M}_h}\|u-v\|_{\mathcal{H}}.$$

We now define larger spaces $\mathcal{M}_h\subset\overline{\mathcal{M}}_h\subset\mathcal{H}$ and $\mathcal{N}_h\subset\overline{\mathcal{N}}_h\subset\mathcal{K}$. With these spaces we have an approximate solution $\overline{u}_h$ satisfying

$$B(\overline{u}_h,v)=f(v)\quad\forall v\in\overline{\mathcal{N}}_h. \tag{10}$$

To insure a unique solution for (10) we assume the *inf–sup conditions*

$$\inf_{\phi\in\overline{\mathcal{M}}_h,\ \|\phi\|_{\mathcal{H}}=1}\ \sup_{\eta\in\overline{\mathcal{N}}_h,\ \|\eta\|_{\mathcal{K}}\leqslant 1} B(\phi,\eta)\geqslant\mu>0, \tag{11}$$

$$\sup_{\phi\in\overline{\mathcal{M}}_h} B(\phi,\eta)>0,\quad \eta\neq 0,\ \eta\in\overline{\mathcal{N}}_h. \tag{12}$$

Although we don't explicitly compute $\overline{u}_h$, it enters into our theoretical analysis of the a posteriori error estimate for u_h. In particular, we assume that the approximate solutions $\overline{u}_h$ converge to u more rapidly than u_h. This is expressed in terms of the *saturation assumption*

$$\|u-\overline{u}_h\|_{\mathcal{H}}\leqslant\beta\,\|u-u_h\|_{\mathcal{H}}, \tag{13}$$

where $\beta<1$ independent of h. In a typical situation, due to the higher degree of approximation for the spaces $\overline{\mathcal{M}}_h$ and $\overline{\mathcal{N}}_h$, one can anticipate that $\beta=\mathrm{O}(h^r)$, for some $r>0$. In this case, $\beta\to 0$ as $h\to 0$, which is stronger than required by our theorems.

We assume that the space $\overline{\mathcal{M}}_h$ has a hierarchical decomposition

$$\overline{\mathcal{M}}_h=\mathcal{M}_h\oplus\overline{\mathcal{V}}_h.$$

Then any function $z\in\overline{\mathcal{M}}_h$ has the unique decomposition $z=v+w$, where $v\in\mathcal{M}_h$ and $w\in\overline{\mathcal{V}}_h$. In a similar fashion, we assume that the space $\overline{\mathcal{N}}_h$ has a hierarchical decomposition

$$\overline{\mathcal{N}}_h=\mathcal{N}_h\oplus\overline{\mathcal{W}}_h.$$

Our final assumption is a *strengthened Cauchy inequality* for this latter decomposition; that is

$$\left|(v,w)_{\mathcal{K}}\right|\leqslant\gamma\,\|v\|_{\mathcal{K}}\|w\|_{\mathcal{K}},\quad\forall v\in\mathcal{N}_h,\ \forall w\in\overline{\mathcal{W}}_h, \tag{14}$$

where $\gamma<1$ independent of h. Some discussion of the role of strengthened Cauchy inequalities in finite element calculations is given in [9,10,15,19,20]. In this work, we use the strengthened Cauchy inequality to obtain the following estimate. Let $v\in\mathcal{N}_h$, $w\in\overline{\mathcal{W}}_h$, with $\|v+w\|_{\mathcal{K}}\leqslant 1$; then

$$\begin{aligned}1&\geqslant\|v+w\|_{\mathcal{K}}^2=\|v\|_{\mathcal{K}}^2+\|w\|_{\mathcal{K}}^2+2(v,w)_{\mathcal{K}}\geqslant\|v\|_{\mathcal{K}}^2+\|w\|_{\mathcal{K}}^2-2\gamma\,\|v\|_{\mathcal{K}}\|w\|_{\mathcal{K}}\\&\geqslant(1-\gamma^2)\,\|w\|_{\mathcal{K}}^2.\end{aligned} \tag{15}$$

3. A fundamental a posteriori error estimate

It is clear that the quantity $|||\overline{u}_h - u_h|||_{\mathcal{H}}$ could be used to estimate $|||u - u_h|||_{\mathcal{H}}$. Indeed, from the saturation assumption (13) and the triangle inequality, we immediately have the estimate

$$(1-\beta)|||u - u_h|||_{\mathcal{H}} \leqslant |||\overline{u}_h - u_h|||_{\mathcal{H}} \leqslant (1+\beta)|||u - u_h|||_{\mathcal{H}}. \tag{16}$$

Although this is a very good a posteriori error estimate in terms of accuracy, the cost of computing $\overline{u}_h$ is usually so high in comparison with that of computing u_h that the method cannot be used in practical computations.

On the other hand, suppose we represent $\overline{u}_h$ using the hierarchical decomposition $\overline{u}_h = \check{u}_h + \check{e}_h$, where $\check{u}_h \in \mathcal{M}_h$ and $\check{e}_h \in \overline{\mathcal{V}}_h$. Intuitively, we might expect $\check{u}_h \approx u_h$. Then $\check{e}_h \in \overline{\mathcal{V}}_h$ could be used as an approximation to the error. This motivates the following a posteriori error estimate. Find $e_h \in \overline{\mathcal{V}}_h$ such that

$$B(e_h, v) = f(v) - B(u_h, v) \quad \forall v \in \overline{\mathcal{W}}_h. \tag{17}$$

To insure a unique solution for (17) we assume the *inf–sup conditions*

$$\inf_{\phi \in \overline{\mathcal{V}}_h,\ |||\phi|||_{\mathcal{H}}=1} \ \sup_{\eta \in \overline{\mathcal{W}}_h,\ |||\eta|||_{\mathcal{K}} \leqslant 1} B(\phi, \eta) \geqslant \mu > 0, \tag{18}$$

$$\sup_{\phi \in \overline{\mathcal{V}}_h} B(\phi, \eta) > 0, \quad \eta \neq 0,\ \eta \in \overline{\mathcal{W}}_h. \tag{19}$$

To see how these estimates are related, it is useful to write (7), (10) and (17) using matrix notation. Suppose $\mathcal{M}_h$ has dimension N and $\overline{\mathcal{M}}_h$ has dimension $\overline{N}$. Let ϕ_i, $1 \leqslant i \leqslant N$, be a basis for $\mathcal{M}_h$ and ϕ_i, $N+1 \leqslant i \leqslant \overline{N}$, be a basis for $\overline{\mathcal{V}}_h$. Similarly define a hierarchical basis ψ_i, $1 \leqslant i \leqslant \overline{N}$, for $\overline{\mathcal{N}}_h$. Let A be the stiffness matrix for (10) computed using these hierarchical bases, with $A_{ji} = B(\phi_i, \psi_j)$. Then the matrix A has a natural block 2×2 decomposition corresponding to the hierarchical decompositions of $\overline{\mathcal{M}}_h$ and $\overline{\mathcal{N}}_h$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Let F be a vector with components $F_i = f(\psi_i)$. Then problem (10) for $\overline{u}_h$ can be written as

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \check{U} \\ \check{E} \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix},$$

where $\check{U}$ is the coefficient vector corresponding to $\check{u}_h \in \mathcal{M}_h$ and $\check{E}$ is the coefficient vector corresponding to $\check{e}_h \in \overline{\mathcal{V}}_h$. Since we are using the hierarchical basis, the original problem (7) for u_h corresponds to the linear system $A_{11}U = F_1$. The linear system for (17) can be written as $A_{22}E = F_2 - A_{21}U$. If we combine these two systems, we can form the block lower triangular system

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} U \\ E \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$$

in which the coefficient vectors for both u_h and e_h are computed. From this we can see the close correspondence of u_h and e_h to $\check{u}_h$ and $\check{e}_h$.

To begin our analysis of the error estimate (17) we note the relations

$$B(u - u_h, v) = 0 \qquad \forall v \in \mathcal{N}_h, \tag{20}$$

$$B(u - \overline{u}_h, v) = 0 \qquad \forall v \in \overline{\mathcal{N}}_h, \tag{21}$$

$$B(\overline{u}_h - u_h, v) = 0 \qquad \forall v \in \mathcal{N}_h, \tag{22}$$

$$B(u - u_h - e_h, v) = 0 \qquad \forall v \in \overline{\mathcal{W}}_h, \tag{23}$$

$$B(\overline{u}_h - u_h - e_h, v) = 0 \qquad \forall v \in \overline{\mathcal{W}}_h. \tag{24}$$

Relations (20)–(24) are proved using various combinations of (2), (7), (10) and (17), restricted to the appropriate subspaces.

Theorem 1. *Let* $u \in \mathcal{H}$, $u_h \in \mathcal{M}_h$, $\overline{u}_h \in \overline{\mathcal{M}}_h$ *and* $e_h \in \overline{\mathcal{V}}_h$ *be defined as above, and assume*

- *the continuity condition* (4),
- *the inf–sup conditions* (5)–(6), (8)–(9), (11)–(12) *and* (18)–(19),
- *the saturation assumption* (13),
- *the strengthened Cauchy inequality* (14).

Then

$$\frac{\mu}{\nu}(1-\beta)\sqrt{1-\gamma^2}\,|||u-u_h|||_{\mathcal{H}} \leqslant |||e_h|||_{\mathcal{H}} \leqslant \frac{\nu}{\mu}\,|||u-u_h|||_{\mathcal{H}}. \tag{25}$$

Proof. First let $w \in \overline{\mathcal{W}}_h$. Using (18), (23) and (4), we have

$$\mu\,|||e_h|||_{\mathcal{H}} \leqslant \sup_{|||w|||_{\mathcal{K}}\leqslant 1} B(e_h, w) = \sup_{|||w|||_{\mathcal{K}}\leqslant 1} B(u-u_h, w) \leqslant \nu\,|||u-u_h|||_{\mathcal{H}},$$

proving the right-hand inequality in (25).

Now let $v \in \mathcal{N}_h$, $w \in \mathcal{W}_h$, with $|||v+w|||_{\mathcal{K}} \leqslant 1$. Then using (11), (22), (24), (4) and (15) we have

$$\begin{aligned}\mu\,|||\overline{u}_h - u_h|||_{\mathcal{H}} &\leqslant \sup_{|||v+w|||_{\mathcal{K}}\leqslant 1} B(\overline{u}_h - u_h, v+w) = \sup_{|||v+w|||_{\mathcal{K}}\leqslant 1} B(\overline{u}_h - u_h, w) \\ &= \sup_{|||v+w|||_{\mathcal{K}}\leqslant 1} B(e_h, w) \leqslant \frac{\nu}{\sqrt{1-\gamma^2}}\,|||e_h|||_{\mathcal{H}}.\end{aligned}$$

This in conjunction with (16) establishes the left inequality in (11). □

4. Two practical a posteriori estimates

Like $\overline{u}_h - u_h$, the error estimate e_h is usually impractical to compute due to the cost of solving a linear system involving the matrix A_{22}. Here we consider two possible enhancements, leading to computationally attractive algorithms. The first is based on the observation that the spaces $\overline{\mathcal{V}}_h$ and $\overline{\mathcal{W}}_h$ typically are made up of functions which must of necessity be quite oscillatory. For typical choices of hierarchical basis functions, this means the matrix A_{22} can be replaced by its diagonal, leading to a very efficient algorithm for computing an a posteriori error estimate. In terms of bilinear forms, this motivates us to consider the following a posteriori error estimate. Let $D(\cdot,\cdot)$ be a bilinear form defined on $\overline{\mathcal{V}}_h \times \overline{\mathcal{W}}_h$. Then our problem is to find $\widetilde{e}_h \in \overline{\mathcal{V}}_h$ such that

$$D(\widetilde{e}_h, v) = f(v) - B(u_h, v) \quad \forall v \in \overline{\mathcal{W}}_h, \tag{26}$$

in analogy with (17). We assume that $D(\cdot,\cdot)$ is chosen such that (26) is easy to solve. To insure a unique solution to (26), we assume the *continuity condition*

$$\left|D(\phi,\eta)\right| \leqslant \widetilde{\nu}\,\|\phi\|_{\mathcal{H}}\|\eta\|_{\mathcal{K}} \quad \forall\phi\subset\overline{\mathcal{V}}_h, \forall\eta\subset\overline{\mathcal{W}}_h, \tag{27}$$

and the *inf–sup conditions*

$$\inf_{\phi\in\overline{\mathcal{V}}_h,\ \|\phi\|_{\mathcal{H}}=1}\ \sup_{\eta\in\overline{\mathcal{W}}_h,\ \|\eta\|_{\mathcal{K}}\leqslant 1} D(\phi,\eta) \geqslant \widetilde{\mu} > 0, \tag{28}$$

$$\sup_{\phi\in\overline{\mathcal{V}}_h} D(\phi,\eta) > 0, \quad \eta\neq 0,\ \eta\in\overline{\mathcal{W}}_h. \tag{29}$$

To analyze this estimate, we note the relations

$$B(u-u_h,v) - D(\widetilde{e}_h,v) = 0 \quad \forall v\in\overline{\mathcal{W}}_h, \tag{30}$$

$$B(\overline{u}_h-u_h,v) - D(\widetilde{e}_h,v) = 0 \quad \forall v\in\overline{\mathcal{W}}_h \tag{31}$$

replace (23)–(24).

Theorem 2. *Let* $u\in\mathcal{H}$, $u_h\in\mathcal{M}_h$, $\overline{u}_h\in\overline{\mathcal{M}}_h$ *and* $\widetilde{e}_h\in\overline{\mathcal{V}}_h$ *be defined as above, and assume*
- *the continuity conditions* (4) *and* (27),
- *the inf–sup conditions* (5)–(6), (8)–(9), (11)–(12) *and* (28)–(29),
- *the saturation assumption* (13),
- *the strengthened Cauchy inequality* (14).

Then

$$\frac{\mu}{\widetilde{\nu}}(1-\beta)\sqrt{1-\gamma^2}\,\|u-u_h\|_{\mathcal{H}} \leqslant \|\widetilde{e}_h\|_{\mathcal{H}} \leqslant \frac{\nu}{\widetilde{\mu}}\,\|u-u_h\|_{\mathcal{H}}. \tag{32}$$

Proof. The proof is exactly analogous to the proof of Theorem 1. First let $w\in\overline{\mathcal{W}}_h$. Using (28), (30) and (4), we have

$$\widetilde{\mu}\,\|\widetilde{e}_h\|_{\mathcal{H}} \leqslant \sup_{\|w\|_{\mathcal{K}}\leqslant 1} D(\widetilde{e}_h,w) = \sup_{\|w\|_{\mathcal{K}}\leqslant 1} B(u-u_h,w) \leqslant \nu\,\|u-u_h\|_{\mathcal{H}},$$

proving the right-hand inequality in (32).

Now let $v\in\mathcal{N}_h$, $w\in\mathcal{W}_h$, with $\|v+w\|_{\mathcal{K}}\leqslant 1$. Then using (11), (22), (31), (27) and (15) we have

$$\begin{aligned}\mu\,\|\overline{u}_h-u_h\|_{\mathcal{H}} &\leqslant \sup_{\|v+w\|_{\mathcal{K}}\leqslant 1} B(\overline{u}_h-u_h,v+w) = \sup_{\|v-w\|_{\mathcal{K}}\leqslant 1} B(\overline{u}_h-u_h,w)\\ &= \sup_{\|v+w\|_{\mathcal{K}}\leqslant 1} D(\widetilde{e}_h,w) \leqslant \frac{\widetilde{\nu}}{\sqrt{1-\gamma^2}}\,\|\widetilde{e}_h\|_{\mathcal{H}}.\end{aligned}$$

This, in conjunction with (16), establishes the left inequality in (32). □

A second approach is to approximate the error in a space of discontinuous piecewise polynomials. Using such nonconforming approximation spaces leads to a system of small element-by-element calculations to compute the error estimate, rather than a global calculation as in the case of e_h.

Thus we consider nonconforming spaces $\mathcal{V}_h\not\subset\mathcal{H}$ and $\mathcal{W}_h\not\subset\mathcal{K}$. We assume that the conforming spaces $\overline{\mathcal{V}}_h\subset\mathcal{H}$ and $\overline{\mathcal{W}}_h\subset\mathcal{K}$ also satisfy $\overline{\mathcal{V}}_h\subset\mathcal{V}_h$ and $\overline{\mathcal{W}}_h\subset\mathcal{W}_h$, and that the solution $u\in\mathcal{H}$ to (2) satisfies

$$B(u,v) = f(v) + G(u,v) \quad \forall v \in \mathcal{K} \cup \mathcal{W}_h, \tag{33}$$

where $G(\cdot,\cdot)$ is a bilinear form on $\mathcal{H} \cup \mathcal{V}_h \times \mathcal{K} \cup \mathcal{W}_h$ arising from the nonforming nature of $\mathcal{V}_h$ and $\mathcal{W}_h$. We note that for all $z \in \mathcal{H}$,

$$G(z,v) = 0 \quad \forall v \in \mathcal{K}. \tag{34}$$

The bilinear form $B(\cdot,\cdot)$ is extended to $\mathcal{H} \cup \mathcal{V}_h \times \mathcal{K} \cup \mathcal{W}_h$ in the usual way (typically involving the summation of contributions from individual elements). The continuity condition (4) should also apply on these enlarged spaces. The scalar products $(\cdot,\cdot)_{\mathcal{H}}$, and $(\cdot,\cdot)_{\mathcal{K}}$ are extended in a similar fashion. The strengthened Cauchy inequality (14) is extended to

$$\left|(v,w)_{\mathcal{K}}\right| \leqslant \gamma \, |\!|\!| v |\!|\!|_{\mathcal{K}} |\!|\!| w |\!|\!|_{\mathcal{K}}, \quad \forall v \in \mathcal{N}_h, \ \forall w \in \mathcal{W}_h, \tag{35}$$

where $\gamma < 1$ is independent of h.

We seek to approximate the error $u - u_h$ in the space $\mathcal{V}_h$. Our a posteriori error estimate is defined by: find $\hat{e}_h \in \mathcal{V}_h$ such that

$$B(\hat{e}_h, v) = f(v) + G(u_h, v) - B(u_h, v) \quad \forall v \in \mathcal{W}_h. \tag{36}$$

We now assume the *inf–sup conditions*

$$\inf_{\phi \in \mathcal{V}_h, \ |\!|\!|\phi|\!|\!|_{\mathcal{H}}=1} \ \sup_{\eta \in \mathcal{W}_h, \ |\!|\!|\eta|\!|\!|_{\mathcal{K}} \leqslant 1} B(\phi,\eta) \geqslant \mu > 0, \tag{37}$$

$$\sup_{\phi \in \mathcal{V}_h} B(\phi,\eta) > 0, \quad \eta \neq 0, \ \eta \in \mathcal{W}_h. \tag{38}$$

Regarding the bilinear form $G(\cdot,\cdot)$, we assume the estimate

$$\left|G(u-u_h,\eta)\right| \leqslant \delta \, |\!|\!| u-u_h |\!|\!|_{\mathcal{H}} |\!|\!| \eta |\!|\!|_{\mathcal{K}} \quad \forall \eta \in \mathcal{V}_h. \tag{39}$$

Now the relations

$$B(u - u_h - \hat{e}_h, v) - G(u-u_h, v) = 0 \quad \forall v \in \mathcal{W}_h, \tag{40}$$

$$B(\overline{u}_h - u_h - \hat{e}_h, v) = 0 \quad \forall v \in \overline{\mathcal{W}}_h \tag{41}$$

replace (23)–(24). Note that $G(\cdot,\cdot)$ does not appear in (41) since $\overline{\mathcal{W}}_h \subset \mathcal{K}$.

Theorem 3. *Let $u \in \mathcal{H}$, $u_h \in \mathcal{M}_h$, $\overline{u}_h \in \overline{\mathcal{M}}_h$, and $\hat{e}_h \in \overline{\mathcal{V}}_h$ be defined as above, and assume*

- *the continuity condition* (4) *(extended to the nonconforming spaces),*
- *the inf–sup conditions* (5)–(6), (8)–(9), (11)–(12) *and* (37)–(38),
- *the saturation assumption* (13),
- *the strengthened Cauchy inequality* (35),
- *the estimate* (39) *for the nonconforming term.*

Then

$$\frac{\mu}{\nu}(1-\beta)\sqrt{1-\gamma^2}\,|\!|\!| u-u_h |\!|\!|_{\mathcal{H}} \leqslant |\!|\!| \hat{e}_h |\!|\!|_{\mathcal{H}} \leqslant \frac{(\nu+\delta)}{\mu} |\!|\!| u-u_h |\!|\!|_{\mathcal{H}}. \tag{42}$$

Proof. The proof of the left hand inequality in (42) is exactly the same as in the proof of Theorem 1. The right hand inequality also follows the established pattern. Let $w \in \mathcal{W}_h$. Using (37), (40), (4) and (39) we have

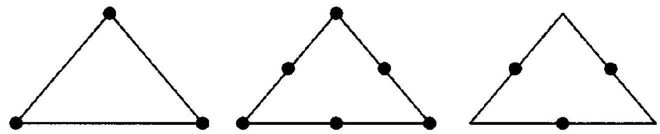

Fig. 1. Piecewise polynomial spaces $\mathcal{M}_h$ (left), $\overline{\mathcal{M}}_h$ (middle), and $\overline{\mathcal{V}}_h$ (right).

$$\mu \|\hat{e}_h\|_{\mathcal{H}} \leqslant \sup_{\|w\|_{\mathcal{K}} \leqslant 1} B(\hat{e}_h, w) = \sup_{\|w\|_{\mathcal{K}} \leqslant 1} B(u - u_h, w) - G(u - u_h, w) \leqslant (\nu + \delta)\|u - u_h\|_{\mathcal{H}},$$

proving the right-hand inequality in (42). □

5. Examples

In this section, we present two simple examples.

5.1. Example 1

We consider the solution of the Poisson equation

$$\begin{aligned} -\Delta u &= f \quad \forall x \in \Omega \subset \mathbb{R}^2, \\ u &= 0 \quad \forall x \subset \partial\Omega. \end{aligned} \tag{43}$$

For simplicity, we assume Ω is polygonal. In this case, $\mathcal{H} \equiv \mathcal{K} \equiv \mathcal{H}_0^1(\Omega)$ are the usual Sobolev spaces, and

$$\begin{aligned} B(u, v) &= \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x, \\ f(v) &= \int_\Omega f v \,\mathrm{d}x, \\ \|u\|_{\mathcal{H}}^2 &= \|u\|_{\mathcal{K}}^2 = \|u\|^2 = B(u, u). \end{aligned} \tag{44}$$

The finite element spaces $\mathcal{M}_h \equiv \mathcal{N}_h$ will consist of continuous piecewise linear polynomials on a shape regular triangulation $\mathcal{T}_h$ of Ω. The spaces $\overline{\mathcal{M}}_h \equiv \overline{\mathcal{N}}_h$ will consist of continuous piecewise quadratic polynomials. The basis for $\mathcal{M}_h$ is the standard nodal basis for piecewise linear finite elements. The basis for $\overline{\mathcal{M}}_h$ is the hierarchical basis consisting of the nodal basis for $\mathcal{M}_h$ and the so-called "bump functions", the piecewise quadratic nodal basis functions associated with edge midpoints. The bump functions form a basis for $\overline{\mathcal{V}}_h \equiv \overline{\mathcal{W}}_h$. This is illustrated symbolically in Fig. 1.

For this setting it is easy to check that $\mu = \nu = 1$. For smooth enough solutions, one can anticipate that $\beta = \mathrm{O}(h)$. The strengthened Cauchy inequality for this case has been analyzed completely by Maitre and Musy [20]. It is also known that the stiffness matrix for the bump functions is comparable to its diagonal [9], so that both classes of a posteriori error estimators analyzed in Section 4 can be applied.

For the case of nonconforming error estimates the spaces $\mathcal{V}_h \equiv \mathcal{W}_h$ are spaces of discontinuous quadratic bump functions. There are three basis functions per element, so that one must solve a small 3×3 linear system for the error estimate in each element. The bilinear form $G(\cdot,\cdot)$ can be taken as

$$G(u,v) = \sum_e \int_e \{\nabla u \cdot \boldsymbol{n}\}_A v_J \, \mathrm{d}x,$$

where e is an internal edge in the triangulation $\mathcal{T}_h$, $\{\nabla u \cdot \boldsymbol{n}\}_A$ is the average of the normal derivative for the two elements sharing edge e ($\boldsymbol{n}$ is chosen arbitrarily from the two possibilities), and v_J is the jump in v on e (with sign chosen consistently with $\boldsymbol{n}$). Computing the bound (39) is fairly technical, involving the use of trace and inverse inequalities as well as some additional regularity for the solution u (since line integrals of $\partial u/\partial n$ appear). See [12,23] for some details.

5.2. Example 2

As our second example, we consider a simple elliptic system, the Stokes equations

$$\begin{aligned} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \forall x \in \Omega \subset \mathbb{R}^2, \\ \nabla \cdot \boldsymbol{u} = 0 \quad & \forall x \in \Omega, \\ \boldsymbol{u} = 0 \quad & \forall x \in \partial\Omega. \end{aligned} \tag{45}$$

Now $\boldsymbol{u} = (u_1, u_2)^{\mathrm{T}}$ is a vector velocity field, and p is the pressure. The pressure is determined only up to an additive constant.

For this case $\mathcal{H} \equiv \mathcal{K} \equiv \mathcal{H}_0^1(\Omega) \times \mathcal{H}_0^1(\Omega) \times \mathcal{L}^2(\Omega)$, and

$$\begin{aligned} B(\{\boldsymbol{u},p\},\{\boldsymbol{v},q\}) &= \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} - p\nabla\cdot\boldsymbol{v} - q\nabla\cdot\boldsymbol{u}\,\mathrm{d}x, \\ \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} &= \sum_{i=1}^{2} \nabla u_i\cdot\nabla v_i, \\ f(\{\boldsymbol{v},q\}) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,\mathrm{d}x, \\ (\{\boldsymbol{u},p\},\{\boldsymbol{v},q\})_{\mathcal{H}} &= \int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v} + pq\,\mathrm{d}x, \\ \|\boldsymbol{u},p\|_{\mathcal{H}}^2 = \|\boldsymbol{u},p\|_{\mathcal{K}}^2 &= \|\boldsymbol{u},p\|^2 = (\{\boldsymbol{u},p\},\{\boldsymbol{v},q\})_{\mathcal{H}}. \end{aligned} \tag{46}$$

We will compute a finite element approximation using the mini-element discretization of Arnold et al. [3]. The triangulation $\mathcal{T}_h$ will be as in the first example. The space $\mathcal{M}_h \equiv \mathcal{N}_h$ is the usual mini-element space. The velocity components are approximated using continuous piecewise linear polynomials satisfying the Dirichlet boundary conditions, plus the cubic *bubble* functions associated with the barycenter of each element (see Fig. 2). The pressure is approximated by a continuous piecewise linear polynomial. The pressure can be made unique by requiring it to have average value zero. This requirement can be easily imposed as part of the solution process and does not affect the computational basis, which is just the span of the usual nodal basis functions. The space $\overline{\mathcal{M}}_h \equiv \overline{\mathcal{N}}_h$ is

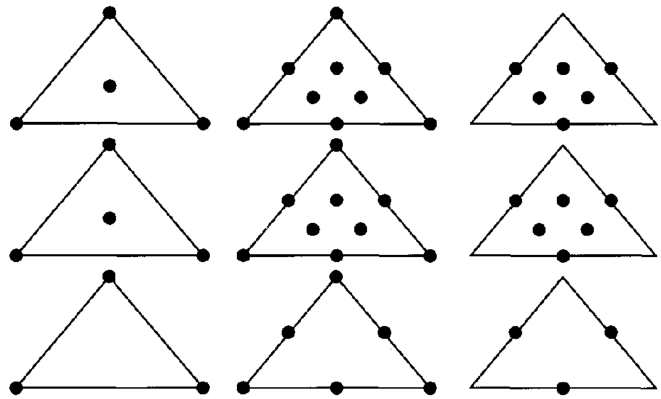

Fig. 2. Mini-element spaces $\mathcal{M}_h$ (left), $\overline{\mathcal{M}}_h$ (middle) and $\overline{\mathcal{V}}_h$ (right) for the two components of the velocity (rows one and two) and the pressure (row three).

the second member of the family of mini-element spaces [3]. Each velocity component is approximated using continuous piecewise quadratic polynomials plus the quartic bubble functions. The pressure is continuous piecewise quadratic.

This pair of mini-element spaces is nested. The cubic bubble for a given triangle can be expressed as a simple linear combination of the three quartic bubbles for the same triangle. Normally, equations for the bubble functions are statically condensed from the system of linear equations to be solved, so that the unknowns that are actually computed correspond to the degrees of freedom associated with the linear and quadratic basis functions only. Thus, we define $\mathcal{M}_h = \mathcal{M}_h^\dagger \oplus \mathcal{B}_h$, where $\mathcal{M}_h^\dagger$ consists of just the piecewise linear functions and $\mathcal{B}_h$ of the cubic bubbles functions. Similarly, we have $\overline{\mathcal{M}}_h = \overline{\mathcal{M}}_h^\dagger \oplus \overline{\mathcal{B}}_h$, where $\overline{\mathcal{M}}_h^\dagger$ consists of the piecewise quadratic polynomials and $\overline{\mathcal{B}}_h$ of the quartic bubble functions. Note that $\mathcal{B}_h \subset \overline{\mathcal{B}}_h$. Now we have the hierarchical decomposition $\overline{\mathcal{M}}_h = \mathcal{M}_h^\dagger \oplus \overline{\mathcal{V}}_h^\dagger \oplus \overline{\mathcal{B}}_h$, where $\overline{\mathcal{V}}_h^\dagger$ is the space of quadratic bump functions as in the other examples. We will take $\overline{\mathcal{V}}_h \equiv \overline{\mathcal{W}}_h = \overline{\mathcal{V}}_h^\dagger \oplus \overline{\mathcal{B}}_h$.

We now verify the hypotheses of Theorem 1. The continuity condition (3) is straightforward to check. The inf–sup condition (4) for the spaces $\mathcal{H}$, $\mathcal{M}_h$ and $\overline{\mathcal{M}}_h$ are standard results [3]. There are slight technical hurdles connected with nonuniqueness of the pressure, which can be made unique, for example, by requiring an average value of zero. The solutions $\{\boldsymbol{u}_h, p_h\}$ and $\{\overline{\boldsymbol{u}}_h, \overline{p}_h\}$ satisfy the saturation assumption with $\beta = O(h)$, provided $\{\boldsymbol{u}, p\}$ is sufficiently smooth.

To prove the inf–sup condition for $\overline{\mathcal{V}}_h$, one can use a variation of the argument used in [3, Section 2] for the mini-element spaces themselves; one lets the space $\overline{\mathcal{M}}_h$ play the role analogous to $\mathcal{H}$ and $\overline{\mathcal{V}}_h$ play the role analogous to $\mathcal{M}_h$. The argument simplifies somewhat because both spaces are finite-dimensional, and one can use strengthened Cauchy inequalities to bound the norm of the interpolation operator.

A slightly tricky technical point in the analysis concerns the strengthened Cauchy inequality (14). Because $\mathcal{B}_h \subset \overline{\mathcal{B}}_h$, we must use the hierarchical decomposition $\overline{\mathcal{M}}_h = \mathcal{M}_h^\dagger \oplus \overline{\mathcal{V}}_h$. Let $\{\boldsymbol{v}, q\} \subset \mathcal{M}_h^\dagger$, $\{\boldsymbol{w}, r\} \in \overline{\mathcal{V}}_h$. Then the relevant strengthened Cauchy inequality is

$$\left|B\left(\{\boldsymbol{v}, q\}, \{\boldsymbol{w}, r\}\right)\right| \leqslant \gamma \, |\!|\!| \boldsymbol{v}, q |\!|\!| \, |\!|\!| \boldsymbol{w}, r |\!|\!|,$$

which is established in the usual fashion. One can check that the argument used in the proof of Theorem 1 is affected in only a trivial way by this modification. Some a posteriori error estimators for the mini-element formulation of the Stokes equations are given in [13,24].

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