Chapter 3

Partial differential equations

"... partial differential equations are the basis of all physical theorems."
Bernhard Riemann (1826-1866)

Partial Differential Equations (PDEs)

Many natural, human or biological, chemical, mechanical, economical or financial systems and processes can be described at a macroscopic level by a set of partial differential equations governing averaged quantities such as density, temperature, concentration, velocity, etc. Most models based on PDEs used in practice have been introduced in the XIXth century and involved the first and second partial derivatives only. Nonetheless, PDE theory is not restricted to the analysis of equations of two independent variables and interesting equations are often nonlinear. For these reasons, and some others, understanding generalized solutions of differential equations is fundamental as well as to devise a proper notion of generalized or weak solution.

Contents

In this chapter, we will review the main categories of partial differential equations and the principal theoretical and numerical results concerning the solving of PDEs. We refer the reader to the notes about metric and Banach spaces and about ordinary differential equations in the Appendix to follow the concepts discussed hereafter.

3.1 Partial differential equations

Let us recall that a differential equation is an equation for an unknown function of several independent variables (and of functions of these variables) that relates the value of the function and of its derivatives of different orders. An *ordinary differential equation* (ODE) is a differential equation in which the functions that appear in the equation depend on a single independent variable. A *partial differential equation* is a differential equation in which the unknown function $F : \Omega \to \mathbb{R}$ is a function of multiple independent variables and of their partial derivatives.

Notice that like ordinary derivatives, a partial derivative is defined as a limit. More precisely, given $\Omega \subset \mathbb{R}^d$ an open subset and a function $F : \Omega \to \mathbb{R}$, the partial derivative of F at $x = (x_1, \ldots, x_d) \in \Omega$ with respect to the variable x_i is:

$$\frac{\partial}{\partial x_i} F(x) = \lim_{\Delta x \to 0} \frac{F(x_1, \ldots, x_i + \Delta x, \ldots, x_d) - F(x)}{\Delta x}\,.$$

The function F is totally differentiable (*i.e.* is a C^1 function) if all its partial derivatives exist in a neighborhood of x and are continous.

Using the standard notation adopted in this classbook, we can write a typical PDE symbolically as follows. Let Ω denote an open subset of $\mathbb{R}^d$. Given $F : \mathbb{R}^{d^k} \times \mathbb{R}^{d^{k-1}} \times \cdots \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ and an integer $k \geq 1$:

Definition 3.1 *An expression of the form:*

$$F(D^k u(x), D^{k-1} u(x), \ldots, Du(x), u(x), x) = 0\,, \quad (x \in \Omega) \tag{3.1}$$

where $u : \Omega \to \mathbb{R}$ is the unknown, is called a k-th order partial differential equation.

Solving a PDE means finding all functions u verifying Equation (3.1), possibly among those functions satisfying additional boundary conditions on some part of the domain boundary $\partial\Omega$. In the absence of finding the solutions, it is necessary to deduce the existence and other properties of the solutions.

Definition 3.2 *(i) The partial differential equation (3.1) is called* linear *if it has the form:*

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = f(x)\,,$$

for given functions a_α ($|\alpha| \leq k$) and f. Moreover, this linear equation is homogeneous *if $f \equiv 0$.*

(ii) it is called semilinear *if it as the form:*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0\,,$$

(iii) it is quasilinear *if it has the form:*

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u, \ldots, Du, u, x) D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, x) = 0\,,$$

(iv) the equation is fully nonlinear *if it depends nonlinearly upon the highest order derivatives.*

By extension, a *system* of partial differential equations is a set of several PDE for several unknown functions. In general, the system involves the same number m of scalar equations as unknowns $(u^1, \ldots, u^m)$, although sometimes this system can have fewer (*underdetermined*) or more (*overdetermined*) equations than unknowns.

3.1.1 Typical PDEs

As there is no general theory known for solving all partial differential equations and given the variety of phenomena modeled by such equations, research focuses on particular PDEs that are important for theory or applications. Following is a list of partial differential equations commonly found in mathematical applications. The objective of the enumeration is to illustrate the different categories of equations that are studied by mathematicians; here, all variables are dimensionless, all constants have been set to one.

a. Linear equations.

1. Laplace's equations: $\Delta u = 0$
2. Helmholtz's equation (involves eigenvalues): $-\Delta u = \lambda u$
3. First-order linear transport equation: $u_t + c\, u_x = 0$
4. Heat or diffusion equation: $u_t - \Delta u = 0$
5. Schrödinger's equation: $i\, u_t + \Delta u = 0$
6. Wave equation: $u_{tt} - c^2\, \Delta u = 0$
7. Telegraph equation: $u_{tt} + d\, u_t - u_{xx} = 0$

b. Nonlinear equations.

1. Eikonal equation: $|Du| = 1$
2. Nonlinear Poisson equation: $-\Delta u = f(u)$
3. Burgers' equation: $u_t + u\, u_x = 0$
4. Minimal surface equation: $\operatorname{div}\left(\dfrac{Du}{(1+|Du|^2)^{1/2}}\right) = 0$
5. Monge-Ampère equation: $\det(D^2 u) = f$
6. Korteweg-deVries equation (KdV): $u_t + u\, u_x + u_{xxx} = 0$
7. Reaction-diffusion equation: $u_t - \Delta u = f(u)$

c. System of partial differential equations.

1. Evolution equation of linear elasticity: $u_{tt} - \mu\Delta u - (\lambda + \mu)D(\operatorname{div} u) = 0$
2. System of conservation laws: $u_t + \operatorname{div} F(u) = 0$
3. Maxwell's equations in vaccum: $\begin{cases} \operatorname{curl} E = -B_t \\ \operatorname{curl} B = \mu_0 \varepsilon_0 E_t \\ \operatorname{div} B = \operatorname{div} E = 0 \end{cases}$
4. Reaction-diffusion system: $u_t - \Delta u = f(u)$
5. Euler's equations for incompressible, inviscid fluid: $\begin{cases} u_t + u \cdot Du = -Dp \\ \operatorname{div} u = 0 \end{cases}$
6. Navier-Stokes equations for incompressible viscous fluid: $\begin{cases} u_t + u \cdot Du - \Delta u = -Dp \\ \operatorname{div} u = 0 \end{cases}$

3.1.2 Classification of PDE

In the previous examples, we have considered different types of equations that can be classified as follows. Usually, second-order partial differential equations or PDE systems are either elliptic, parabolic or hyperbolic. To summarize, elliptic equations are associated to a special state of a system, in principle corresponding to the minimum of the energy. Parabolic problems describe evolutionary phenomena that lead to a steady state described by an elliptic equation. And hyperbolic equations modeled the transport of some physical quantity, such as fluids or waves. In this text, we restrict ourselves to linear problems because they do not require the knowledge of nonlinear analysis.

General form of PDE

In the general expression of a partial differential equation (3.1), the highest order k of the derivatives is its *degree*. A general form of a scalar linear second-order equation in d different variables $x = (x_1, \dots, x_d)^t$ is:

$$C : D^2 u + b \cdot Du + au = f \qquad \text{in } \Omega\,, \tag{3.2}$$

or a similar, and perhaps more conventional, form:

$$-\sum_{i,j=1}^{d} c_{ij} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + au = f \qquad \text{in } \Omega\,, \tag{3.3}$$

where, $\forall x \in \Omega$, $a(x) \in \mathbb{R}$, $b(x) \in \mathbb{R}^d$, $C(x) \in \mathbb{R}^{d\times d}$ are the *coefficients* of the equation, with the notation $A : B = \sum_{i,j=1}^{d} a_{ij} b_{ij}$ denoting the contracted product. If all the coefficients are independent of the space variable x, the equation is with constant coefficients. In general, for all derivatives to exist (in the classical sense[1]), the solution and the coefficients have to satisfy *regularity* requirements: $u \in C^2(\Omega)$, $c_{ij} \in C^1(\Omega)$, $b_i \in C^1(\Omega)$, $a \in C^0(\Omega)$ and $f \in C^0(\Omega)$ We will see later that these requirements can be slightly reduced when the equation is formulated in a *weak sense*. Notice also that additional conditions need to be prescribed in order to ensure the *existence* and the *uniqueness* of the solution.

Remark 3.1 *For any twice continuous function u, it is possible to symmetrize the coefficients of the matrix $C = (c_{ij})$ by writing:*

$$\tilde{c}_{ij} = \frac{1}{2}(c_{ij} + c_{ji})\,,$$

and modifying the other coefficients accordingly to preserve the original form of the equation.

Classification of PDE

Definition 3.3 *Consider a* second-order *partial differential equation of the form (3.2) with a symmetric coefficient matrix $C(x)$; then the equation is said to be*

1. elliptic *at $x \in \Omega$ if $C(x)$ is positive definite,* i.e., *if for all $v \neq 0 \in \mathbb{R}^d$, $v^t\, Cv > 0$.*
2. parabolic *at $x \in \Omega$ if $C(x)$ is positive semidefinite ($v^t\, Cv \geq 0$, for all $v \in \mathbb{R}^d$) and not positive definite and the rank of $(C(x), b(x), a(x)$ is equal to d.*
3. hyperbolic *at $x \in \Omega$ if $C(x)$ has one negative and $n-1$ positive eigenvalues.*

[1] The classical sense is the only one we know at this stage of the discussion.

By extension, an equation will be called *elliptic*, *parabolic* or *hyperbolic* in the open set Ω if it is elliptic, parabolic or hyperbolic everywhere in Ω.

Remark 3.2 *This classification implies that the equation is not invariant if it is multiplied by -1. In this respect, the equation $-\Delta u = f$ is elliptic in $\mathbb{R}^d$ (since its coefficient matrix is positive definite). However, the equation $\Delta u = -f$ has an unknown type (as its coefficient matrix is negative definite). To overcome this problem, it is common to multiply the equation by (-1) so that the previous definition applies to this equation.*
However, the character of the equation is not altered by a change of variables.

Remark 3.3 *The terminology elliptic, parabolic and hyperbolic has also a geometric interpretation involving planar conics. Let consider a linear second-order partial differential equation with constant coefficients in $\mathbb{R}^2$ of the general form:*

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + fu = g \qquad \text{in } \Omega\,.$$

The equation is said to be elliptic if $b^2 - 4ac < 0$, parabolic if $b^2 - 4ac = 0$ and hyperbolic if $b^2 - 4ac > 0$.

Remark 3.4 *In practice, we make a distinction between* time-dependent *and* time-independent *PDEs. Often, elliptic equations are time-independent equations.*

Notation. Elliptic partial differential equations can be written in a more concise form:

$$Lu = f\,,$$

where L denotes the second-order elliptic differential operator defined according to equation (3.3) as:

$$Lu = -\sum_{i,j=1}^{d} c_{ij}\frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + au\,. \tag{3.4}$$

As many PDE are commonly used in physics, one of the independent variables represents the time t. For example, given an elliptic differential operator L, the operator form of a parabolic equation is:

$$\frac{\partial u}{\partial t} + Lu = f\,;$$

and a second-order hyperbolic equation is then:

$$\frac{\partial^2 u}{\partial t^2} + Lu = f\,.$$

Example 3.1 (An elliptic PDE: the potential equation of electrostatics) *Let the function $\rho \in C(\overline{\Omega})$ represent the electric charge density in some open bounded set $\Omega \subset \mathbb{R}^d$. Suppose the permitivity ϵ_0 of the medium is constant, the definition of the electrostatic potential provides a relationship between the potential ϕ and the charge density ρ, as a Poisson equation:*

$$\Delta\phi = -\frac{\rho}{\epsilon_0}\,,$$

Notice that this equation do not yield a well-posed problem, as it may have several solutions, since for any solution ϕ, the function $\phi + C$ (C being an arbitrary constant) is also a solution. Every elliptic equation has to be endowed with appropriate boundary conditions to yield a well-posed problem (see Section 3.2.2).

3.1.3 Boundary conditions

In general, in the mathematical models, additional information on the boundary of the domain (according to the time or the spatial dimension) are supplied with the PDE. These information are known as initial or final conditions (with respect to the time dimension) and as boundary conditions (with respect to the space dimension). Notice here the fundamental difference between ordinary differential equations (ODE) and partial differential equations. The general solution of an ODE contains constants (values) while the general solution of a PDE contains arbitrary functions. As we will see hereafter, the equation is not sufficient to ensure the unicity of the solution u. Additional information is needed on the boundary $\partial\Omega$ or on a portion Γ of the boundary of Ω. Such data is called a *boundary condition*. Hence, the partial differential equation together with a set of additional restraints is called a *boundary value* problem.

If the boundary condition gives a value to the domain: $\forall x \in \Gamma$, $u(x)$ is fixed, then it is a *Dirichlet* boundary condition. If the boundary condition gives a value to the normal derivative of the problem: $\forall x \in \Gamma$, $\dfrac{\partial u}{\partial \nu}(x) = (\nabla u \cdot \nu)(x)$ is fixed (where ν is the outward normal to Γ), then it is a *Neumann* boundary condition. *Mixed* boundary conditions indicate that different boundary conditions are used on different parts of the domain boundary. Robin (Newton) boundary conditions is another frequently used type of boundary conditions. It involves a (linear) combination of function values (Dirichlet) and normal derivatives (Neumann).

3.2 Analysis of partial differential equations

For most partial differential equations, no exact solution is known and, in some cases, it is not even clear whether a *unique* solution exists. For this reason, numerical methods have been developped, in combination with the analysis of simple cases, to provide accurate approximations of the solution in concrete scientific problems. This is indeed not an easy task and even defining precisely what is the "solution" of the equation may reveal tricky as it depends largely on the problem at hand.

3.2.1 A simple example

Let consider the case of an ideal elastic string stretched horizontally and fixed to supports at each end. We denote $u(x)$ the stretching (*i.e.,* the vertical displacement) of the string at point x. Suppose the left (resp. right) wall corresponds to the abscissa $x = 0$ (resp. $x = 1$), x representing the abscissa along the line supporting the string. This gives two Dirichlet boundary conditions: $u(0) = 0$ and $u(1) = 0$. When the string is subjected to a linear load $f : [0, 1] \to \mathbb{R}$, then for all $x \in [0, 1]$ and for Δx sufficiently small, the small portion of elastic string $[x, x + \Delta x]$ is subjected to the weight $f(x)\Delta x$. Mechanics laws give the equation of the displacement u: $-u''(x) = f(x)$, for all $x \in [0, 1]$. Hence, we face a Poisson problem: find u on [0,1] such that for a given function f:

$$(P)\begin{cases} -u''(x) = f(x) & \forall x \in [0,1] \\ u(0) = 0 \\ u(1) = 0 \end{cases}$$

Remark 3.5 *(i) The equation in problem (P) is a second-order equation,*

(ii) The problem (P) is intrinsically different from the following initial value problem:

$$(I)\begin{cases} -u''(x) = f(x) & \forall x \in [0,1] \\ u(0) = \alpha \\ u'(0) = \beta \end{cases}$$

In both problems there are two boundary conditions. It is a necessary condition for a second-order equation to have a solution.

(iii) The Cauchy-Lipschitz theorem cannot be applied directly to problem (P).

(iv) At the difference of ODE, a PDE problem may have no solution (see exercise 2).

In the following, we will show that problem (P) has a unique solution and that the solution enjoys some properties:

- if $f(x) \geq 0$ then $u(x) \geq 0$ (downwards vertical displacement),
- if for all x $f(x) > 0$, then $u(x) > 0$ for all $x \in]0,1[$.

These two results illustrate two aspects of the *maximum principle* (see Section 6.3.6).

3.2.2 Hadamard concept of well-posed problem

The notion of well-posedness, due to J. Hadamard (1865-1963), is related to the requirements that can be expected from solving a partial differential equation [Hadamard, 1902].

Definition 3.4 (Hadamard's well-posedness) *A given problem for a partial differential equation is said to be* well-posed *if:*

(1) a solution exists,

(2) the solution is unique,

(3) the solution depends continuously on the given data (in some reasonable topology).

Otherwise it is ill-posed.

The requirements of existence and uniqueness is natural and obvious. The third condition disregards problems having unstable solutions. If a problem is well-posed, obtaining a numerical approximation of the exact solution seems possible as long as the data to the problem are approximated suitably. Some problems are ill-posed because of their nature, despite the initial and boundary conditions are correctly defined. In this text, we shall consider the terms *well-posed* and *stable* as synonyms.

The notion of continuous dependence on the data indicates that the solution should not have to change much if the data are slightly perturbated. This requirement is important because in applications, the boundary data are usually obtained throug measurements and may be noisy (given only up to certain error margins) and small measurement errors should not affect dramatically the solution. Suppose our problem is of the general form: find a solution u such that

$$F(u, f) = 0\,. \tag{3.5}$$

Now, if we denote by δf a small perturbation on the data and by δu the modification in the solution that occurred because of this perturbation, such that

$$F(u + \delta u, f + \delta f) = 0\,, \tag{3.6}$$

then,

$$\forall \eta > 0\,, \exists K(\eta, f) \; : \; \|\delta f\| < \eta \Rightarrow \|\delta u\| \leq K(\eta, f)\|\delta f\|\,.$$

This concept of perturbation allows to introduce the following definition.

Definition 3.5 *Following [Quarteroni* et al.*, 2000], we consider the* relative condition number *associated with problem $F(u,f)=0$:*

$$K(f) = \sup_{\delta f \in U} \frac{\|\delta u\| \, \|f\|}{\|\delta f\| \, \|u\|}\,, \tag{3.7}$$

where U is a neighborhood of the origin and represents the set of admissible perturbations on the data for which the problem $F(u+\delta u, f+\delta f)=0$ is still relevant. Whenever $f=0$ or $u=0$, we consider the absolute condition number*, given by*

$$K_{abs}(f) = \sup_{\delta f \in U} \frac{\|\delta u\|}{\|\delta f\|}\,. \tag{3.8}$$

The problem (3.5) is called *ill-conditioned* when $K(f)$ is "big" for any admissible given function f (the meaning of big may change depending on the problem considered).

Remark 3.6 *Notice that the notion of well-posedness of a problem and its property of being well-conditioned is independent from the numerical method used to solve it.*

Even in the case where the condition number may be infinite, the problem may be well-posed. Usually, it can be reformulated into an equivalent problem with a finite condition number.

If the problem (3.5) has a unique solution, then there exists a *resolvent* mapping G between the sets of the data and the solutions, such that

$$u = G(f)\,, \quad \text{i.e.,} \quad F(G(f), f) = 0\,.$$

Hence, problem (3.6) yields $u + \delta u = G(f + \delta f)$. If the mapping G is differentiable in f, $G : \mathbb{R}^n \to \mathbb{R}^m$, $G'(f)$ will denote the Jacobian matrix of G evaluated at f, we can write

$$G(f+\delta f) = G(f) + G'(f)\delta f + o(\|\delta f\|)$$

using a Taylor's expansion, for $\delta d \to 0$ and a suitable norm $\|\cdot\|$ for δf. Neglecting high order terms, we find that

$$K(f) \simeq \|G'(f)\| \, \frac{\|f\|}{\|G(f)\|}\,, \qquad K_{abs}(f) \simeq \|G('f)\|\,,$$

the symbol $\|\cdot\|$ denoting the matrix norm associated with the vector norm. This estimate is very useful in pratice in the analysis of such problems.

The concept of well-posed problem is obviously of great importance is practical applications. Indeed, if a problem is well-posed, it is reasonable to attempt to compute an accurate approximation of the exact solution (here unique) as long as the data are suitably approximated. However, we will see now on an example that incorrect boundary conditions may jeopardize the well-posedness nature of a problem.

Example 3.2 (Ill-posed problem due to wrong boundary conditions) *(cf. [Šolin, 2005]). Consider the interval $\Omega : [-a,a], a > 0$ and the inviscid first-order hyperbolic Burgers' equation:*

$$u_t(t,x) + u(t,x)u_x(t,x) = 0\,,$$

with the initial condition

$$u(0,x) = u_0(x) = x\,, \qquad x \in \Omega\,,$$

where u_0 is a function continuous in $[-a,a]$ such that $u_0(\pm a) = \pm a$, and the boundary condition

$$u(t, \pm a) = \pm a\,, \qquad t > 0\,.$$

Every function $u(t,x)$ satisfying the Burgers' equation and the initial condition is constant along the characteristic lines

$$x_{x_0}(t) = x_0(t+1)\,, \qquad x_0 \in \Omega\,.$$

Hence, it follows that the solution to this problem with initial conditions cannot be constant in time at the endpoints of Ω. This problem has no solution.

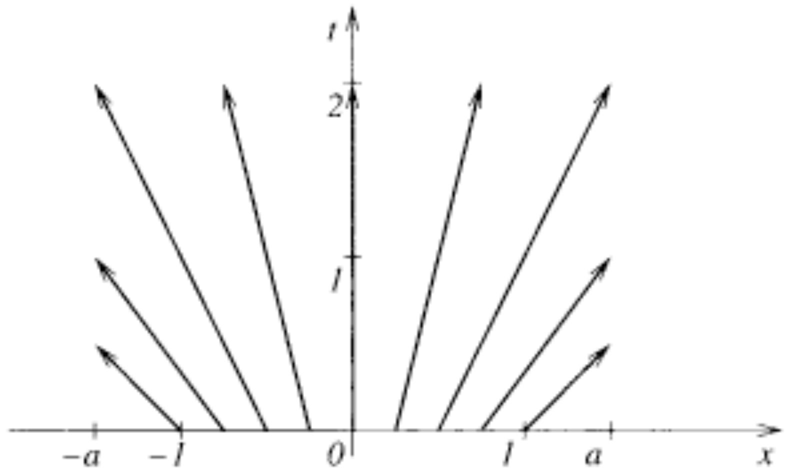

Figure 3.1: *Isolines of the solution $u(t,x)$ of Burgers' equation (reprinted from [Šolin, 2005]).*

Other problems are ill-posed because of their very nature, despite their initial and boundary conditions. We refer the reader to [Šolin, 2005] for the discussion of such example.

3.2.3 Weak solution, regularity

The notion of well-posed problem does not provide a clear definition of what the related unique solution should be. In particular, it does not indicate whether the solution u must be analytic or at least infinitely differentiable. If this may be desirable, it is perhaps too much asking for a solution. Maybe it would be wiser to ask for a solution of a PDE of order k to be at least k times continuously differentiable. In this way, all derivatives in the equation will exist and be continuous. Such a solution is called a *classical* solution of the PDE.

Although certain equations can be solved in the classical sense, many others cannot. In general, conservation laws do not admit classical solutions, but are well-posed. Indeed, such equations govern fluid dynamics problems in which shock waves develop and propagate throughout the computational domain. A shock wave being a curve of discontinuity of the solution, from the physical point of view it seems better to allow for solutions that are not continuously differentiable or continuous. In this context, we shall look for a different type of solutions called *generalized* or *weak* solutions.

For this reason, it seems reasonable to dissociate the notions of *existence* and of *regularity*. It is possible to define a notion of *weak solution* for a given PDE, having in mind that such a solution is not too regular, and then to try to establish its existence, uniqueness and continuous dependence on the data. It remains to clarify the notion of *regularity* of weak solutions. We will see in Chapter finiteelemchap that the existence of weak solutions is based on simple estimates whereas the regularity of these solutions is often related to more complex estimates. A strong requirement for the existence of a solution would be to ask for an explicit formula in terms of the boundary values. This is obviously possible in a few peculiar cases.

Hence, we attempt for instance to prove the existence by deriving a contradiction from the assumption of nonexistence. However, an existence theorem is often not useful for computing an approximation of the solution if nonconstructive techniques are employed. To overcome this problem, we can ask for a constructive method for the existence requirement, that can be used to compute approximations. It turns out that it is wiser to deal with the problems separately, first find an existence theorem in an abstract mathematical framework and then, use this result to develop a constructive and stable numerical scheme to compute approximations of the solution.

3.2.4 General existence and uniqueness

Before giving some examples of basic elliptic, parabolic and hyperbolic equations, it seems interesting to address the issue of existence and uniqueness of a solution for general equations. Since this paragraph will require some basic knowledge of functional analysis about Hilbert and Sobolev spaces and about distributions (cf. Chapters 1, 2), the reader may skip it in the first reading and continue directly with Section 3.3.

In this section, we will consider Hilbert spaces V and W and an equation of the general form

$$Lu = f\,, \tag{3.9}$$

where $L : U \to V$ is a linear operator and $f \in V$. We recall that the *null space* $N(L)$ of the linear operator L is the set

$$N(L) = \{u \in U : L(u) = 0\}\,,$$

and the *range* $R(L)$ of the operator is defined as

$$R(L) = \{v \in V : \exists u \in U \text{ such that } \; L(u) = v\}\,.$$

The linear operator is an *injection* if $N(L) = \{0\}$, a *surjection* if $R(L) = W$ and a *bijection* if it is both an injection and a surjection.

The existence of a solution to the equation (3.9) for any right-hand side function f is equivalent to the condition $R(L) = V$, while the uniqueness of the solution is equivalent to the condition $N(L) = \{0\}$.

Definition 3.6 (Closed operator) *Given two Banach spaces U, V, an operator $L = U \to V$ is said to be* closed *if for any sequence $(v_n)_{1\leq n\leq\infty} \subset U$, $v_n \to v$ and $L(v_n) \to w$ imply that $v \in U$ and $w = Lv$.*

Definition 3.7 (Monotonicity) *Let V be a Hilbert space and $L \in \mathcal{L}(V, V')$. The operator L is said to be* monotone *if*

$$\langle Lv, v\rangle \geq 0 \qquad \text{for all } v \in V\,,$$

it is strictly monotone *if*

$$\langle Lv, v\rangle > 0 \qquad \text{for all } v \neq 0 \in V\,,$$

it is strongly monotone *if there exists a constant $C > 0$ such that*

$$\langle Lv, v\rangle \geq C\|v\|^2 \qquad \text{for all } v \in V\,,$$

For every $u \in V$, the element $Lu \in V'$ is a linear form. The symbol $\langle Lu, v\rangle$ is called duality pairing *, it means the application of Lu to $v \in V$.*

Theorem 3.1 (Existence) *Let U, V be Hilbert spaces and $L : U \to V$ be a bounded linear operator. Then $R(L) = V$ if and only if $R(L)$ is closed and if $R(L)^{\perp} = \{0\}$.*

Theorem 3.2 (Existence and uniqueness) *Let U, V be Hilbert spaces and $L : U \to V$ be a closed linear operator. Suppose that there exists a constant $C > 0$ such that*

$$\|Lv\|_V \geq C\|v\|_U \,, \qquad \text{for all } v \in U \qquad \text{(coercivity estimate)}. \tag{3.10}$$

If $R(L)^\perp = \{0\}$, then the operator equation $Lu = f$ has a unique solution.

Lemma 3.1 *Let V be a Hilbert space and $L \in \mathcal{L}(V, V')$ be a continuous strongly monotone linear operator. Then, there exists a constant $C > 0$ such that L satisfies the stability estimate (3.10).*

Indeed, the strong monotonicity condition implies

$$C\|v\|_V^2 \leq \langle Lv, v\rangle \leq \|Lv\|_{V'} \|v\|_V \;\Rightarrow\; C\,\|v\|_V \leq \|Lv\|_{V'} \,.$$

Theorem 3.3 (Existence and uniqueness (strong monotonicity)) *Let V be a Hilbert space, $f \in V'$ and $L \in \mathcal{L}(V, V')$ a strongly monotone linear operator; Then, for every $f \in V'$, the operator equation $Lu = f$ has a unique solution $u \in V$.*

3.3 Fundamental examples

In this section, we will introduce four linear partial differential equations, considered as fundamental examples, for which explicit analytical solutions are known.

3.3.1 Laplace and Poisson equations

Two of the most important of partial differential equations are Laplace's equation $\Delta u(x) = 0$ and Poisson's equation $-\Delta u(x) = f(x)$, for $x \in U$. The unknow is function $u : \overline{U} \subset \mathbb{R}^d \to \mathbb{R}$ and the function $f : U \to \mathbb{R}$ is given as well as the domain U.

These equations arise in a large number of physical problems. Typically, the unknown function u denotes the density of some quantity (e.g. a concentration) in equilibrium. Then, if $V \subset U$ is any smooth region, the flux of u through the boundary ∂V is given by

$$\int_{\partial V} F \cdot \nu \, ds = 0 \,,$$

where F represents the flux density and ν the unit outer normal field. According to the Green theorem, we have

$$\int_V \operatorname{div} F \, dx = \int_{\partial V} F \cdot \nu \, ds = 0 \,.$$

from which we deduce that

$$\operatorname{div} F = 0 \,, \qquad \text{in } U \,.$$

In many applications, it is reasonable to consider that the flux is proportional to the gradient Du in the opposite direction:

$$F = -cDu \,, \qquad (c > 0) \,,$$

thus leading to the Laplace's equation

$$\operatorname{div}(Du) = \Delta u = 0 \,.$$

3.3.2 Transport equation

3.3.3 Heat equation

We consider the non homogeneous *heat equation*:

$$u_t - \Delta u = f\,.$$

3.3.4 Wave equation

3.4 Second-order elliptic problems

This section is devoted to the discussion of linear second-order elliptic problems. We will introduce important notions like the weak formulation of a model problem, the Lax-Milgram lemma as the basic tool for proving the existence and uniqueness of a solution and maximum principles for elliptic problems used to prove their well-posedness.

3.4.1 Weak formulation

Suppose an open bounded set $\Omega \subset \mathbb{R}^d$ with Lipschitz-continuous boundary and let consider a general linear second-order equation:

$$-\sum_{i,j=1}^{d} c_{ij}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{d} b_i\frac{\partial u}{\partial x_i} + au = f\,, \tag{3.11}$$

where the coefficients and the right-hand side term satisfy the regularity assumptions expressed previously. Recal that this equation is elliptic if the symmetric coefficient matrix $C = (c_{ij})_{i,j=1,\dots,d}$ is positive definite everywhere in Ω. This problem is fairly general and describe, for example, the transverse deflection of a cable (c is the axial tension and f the transversal load, cf. Section 3.2.1).

Now, for the sake of simplicity, let consider first a model equation in the *divergence form*:

$$-\nabla\cdot(c(x)\nabla u(x)) + a(x)u(x) = f(x) \qquad \text{in } \Omega\,, \tag{3.12}$$

obtained from the previous general equation by assuming $c_{ij}(x) = c(x)\delta_{ij}$ and $b(x) = 0$ in Ω.

In order to establish existence and uniqueness results, we need to introduce two additional assumptions: $c(x) \geq C_{min} > 0$ and $a(x) \geq 0$ in Ω. We consider first homogeneous Dirichlet boundary conditions $u(x) = 0$ on the boundary $\partial\Omega$. Using the notations introduced hereabove, we can consider the boundary-value problem:

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{3.13}$$

where L denotes a second-order partial differential operator of the form (3.4).

Definition 3.8 *The linear operator L is (uniformly)* elliptic *if there exists a constant $\alpha > 0$ such that, for each point $x \in \Omega$, the symmetric matrix $C(x) = (c_{ij}(x))$ is positive definite has smallest eigenvalue greater than or equal to α.*

Classical solution

Suppose $f \in C(\overline{\Omega})$. As we know, a *classical solution* of the problem (3.12) endowed with homogeneous Dirichlet conditions on $\partial\Omega$ is a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying the equation (3.12) everywhre in Ω and the boundary condition on the boundary. However, there is no guarantee that this problem can be solved, stronger regularity of the function f is indeed required. Therefore, we will abandon the search for smooth classical solutions but still looking to satisfy the conditions for the well-posedness of the problem. To this end, we need to investigate a broader class of functions for solutions.

Weak formulation

To reduce the regularity constraints, we introduce a *weak formulation* of the problem at hand. Assuming that the solution u is sufficiently smooth, we multiply the equation (3.12) by a smooth test function $v \in C_c^\infty(\Omega)$ and we integrate over Ω. Thus we obtain, after integrating by parts (Green's formula) in the first term of the left hand side:

$$\int_\Omega \sum_{ij=1}^{d} c_{ij} u_{x_i} v_{x_j} + auv\, dx = \int_\Omega fv\, dx\,, \tag{3.14}$$

or similarly:

$$-\int_\Omega c\nabla u \cdot \nabla v\, dx + \int_\Omega auv\, dx = \int_\Omega fv\, dx\,. \tag{3.15}$$

Notice that no boundary term appears since $v = 0$ on $\partial\Omega$. The aim is to find the largest possible function spaces for u, v where all integrals are finite. Actually, the same identity holds with the smooth functions u, v are weakened to

$$u, v \in H_0^1(\Omega)\,, \qquad f \in L^2(\Omega)\,,$$

where $H_0^1(\Omega)$ is the Sobolev space $W_0^{1,2}(\Omega)$ (cf. Chapter 2). Similarly, the constraints on the coefficients can be reduced to

$$a, c \in L^\infty(\Omega)\,.$$

Finally, the *weak form* of the problem 3.12 is stated as follows: Given $f \in L^2(\Omega)$, find a function $u \in H_0^1(\Omega)$ such that

$$-\int_\Omega c\nabla u \cdot \nabla v\, dx + \int_\Omega auv\, dx = \int_\Omega fv\, dx\,. \qquad \forall v \in H_0^1(\Omega)\,.$$

It is obvious that the classical solution to the problem 3.12 solves also the weak formulation of the problem. Conversely, if the weak solution is sufficiently smooth, *i.e.* $u \in C^2(\Omega) \cap C(\overline{\Omega})$, it also solves the classical formulation.

Linear forms

Let $V = H_0^1(\Omega)$. We define

(i) the *bilinear form* $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ associated with the divergence form elliptic operator L defined by (3.9) is:

$$a(u, v) := \int_\Omega (c\nabla u \cdot \nabla v + auv)\, dx\,.$$

(ii) the *linear form* $l \in V^*$:

$$l(v) := \langle l, v \rangle = \int_\Omega f v \, dx \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Omega)$.

Then, the weak formulation of the problem (3.13) reads: Find a function $u \in V$ such that

$$a(u, v) = l(v) \qquad \forall v \in V \,.$$

3.4.2 Existence of weak solutions

The Lax-Milgram theorem is the basic and most important result to prove the existence and uniqueness of solution to elliptic problems.

We assume that H is a real Hilbert space, with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$.

Theorem 3.4 (Lax-Milgram) *Let H be a Hilbert space, $a : H \times H \to \mathbb{R}$ a bilinear mapping, for which there exist constant $M, \alpha > 0$ such that*

$$|a(u, v)| \;\leq\; M\|u\|\|v\| \,, \qquad u, v \in H$$

and

$$\alpha\|u\|^2 \;\leq\; a(u, u) \,, \qquad u \in H \qquad (\alpha\text{-ellipticity}) \,,$$

and $l \in H \to \mathbb{R}$ be a bounded linear functional on H. Then, there exists a unique solution to the problem

$$a(u, v) = l(v) \,, \qquad \forall v \in H \,. \tag{3.16}$$

Furthermore, we have

$$\|u\|_H \;\leq\; \frac{1}{\alpha}\|l\| \,.$$

Proof. For each element $u \in H$, the mapping $\varphi_u : H \to \mathbb{R}$, $v \mapsto \varphi_u(v) = a(u, v)$ is a bounded linear functional on H. Moreover, $\|\varphi_u\| \leq M$. The Riesz representation Theorem (**??**) asserts the existence of a unique element $w \in H$ such that

$$a(u, v) = \langle w, v \rangle \,, \qquad v \in H$$

and we have $\|w\| = \|\varphi_u\| \leq M$. Let us write $w = \mathcal{A}u$ whenever the previous equation holds, so that

$$a(u, v) = \langle \mathcal{A}u, v \rangle \,, \qquad \forall v \in H \,.$$

We claim that $\mathcal{A} : H \to H$ is a bounded linear operator. Next, we observe from the Riesz representation Theorem that there exists a unique $f \in H$ such that

$$\langle f, v \rangle = l(v) \,, \qquad \forall v \in H \,.$$

It follows that (3.16) is equivalent to

$$\langle \mathcal{A}u, v \rangle = \langle f, v \rangle \,, \qquad \forall v \in H \,,$$

hence $\mathcal{A}u = f$. The existence and uniqueness are obtained by establishing that the linear operator $\mathcal{A}$ is a bijection (one-to-one).

(i) $\mathcal{A}$ is an injection, *i.e.* $Ker\mathcal{A} = \{0\}$. Actually,

$$v \in Ker\mathcal{A} \Leftrightarrow \mathcal{A}v = 0 \Rightarrow \langle \mathcal{A}v, v \rangle = 0 \,,$$

but $\langle \mathcal{A}v, v \rangle = a(v, v) \geq \alpha\|v\|^2$, thus $v = 0$ and consequently $Ker\mathcal{A} = \{0\}$.

(ii) $\mathcal{A}$ is surjection, *i.e.* $\mathcal{I}m\mathcal{A} = R(\mathcal{A}) = H$. We need to show that $R(\mathcal{A})$ is closed and dense. Let $(u_n) \subset R(\mathcal{A})$ be a Cauchy sequence such that $\|u_n - u_m\| \to 0$ when $n, m \to \infty$. Then, there exists $x_n \in \mathcal{A}$ such that $u_n = \mathcal{A}x_n$. Next, we show that (x_n) is a Cauchy sequence.

$$\alpha\|x_n - x_m\|^2 \leq \langle \mathcal{A}(x_n -_m), x_n - x_m\rangle = \langle u_n - u_m, x_n - x_m\rangle \leq \|u_n - u_m\|\|x_n - x_m\| .$$

Hence,

$$\|x_n - x_m\| \leq \frac{1}{\alpha}\|u_n - u_m\| \to 0 \qquad n, m \to \infty .$$

$(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists $x \in H$ such that x_n converges to x. By continuity, $u_n = \mathcal{A}x_n \to u\mathcal{A}x$ and thus (u_n) converges in $R(\mathcal{A})$. Hence, $R(\mathcal{A})$ is closed. To prove that $R(\mathcal{A})^\perp = \{0\}$, take an arbitrary $u \in R(\mathcal{A})^\perp$. Then, for any $v \in H$, $\langle \mathcal{A}v, u\rangle = 0$. Putting $v = u$ we obtain

$$\alpha\|u\|^2 \leq \langle \mathcal{A}u, u\rangle = 0 \Rightarrow u = 0 .$$

Finally, considering $u = v$ in equation (3.16) yields

$$\alpha\|u\|^2 \leq a(u, u) = l(u) \leq \|l\|\|u\| .$$

The result follows. □

Remark 3.7 *If the bilinear form $a(\cdot, \cdot)$ is symmetric, then the unique solution $u \in H$ of equation(3.16) corresponds to the unique representant of the linear form $l \in H^*$ with respect to the inner product $\langle\cdot, \cdot\rangle = a(\cdot, \cdot)$. In other words, the Lax-Milgram theorem can be seen as a peculiar case of the Riesz representation theorem.*

Proposition 3.1 *Consider a continuous bilinear form $a(\cdot, \cdot)$, α-elliptic and suppose a is symmetric. Then, the solution u of problem (3.16) is the unique solution of the following minimization problem:*

$$J(u) = \inf_{v\in H} J(v) , \tag{3.17}$$

where $J(v) = \frac{1}{2}a(v, v) - l(v)$, for all $v \in H$.

Remark 3.8 *When the bilinear form $a(\cdot, \cdot)$ is symmetric and considering approximations of this problem in finite dimension, this result allows to use descent algorithms to compute an approximation of the solution.*

Example 3.3 *Regarding problem (3.13), we have*

$$J(v) = \frac{1}{2}\int_0^1 cv'^2 + av^2 , \qquad \forall v \in H_0^1([0, 1])$$

and the solution u is such that

$$J(u) = \frac{1}{2}\int_0^1 cu'^2 + cu^2 \leq J(v) , \qquad \forall v \in H_0^1([0, 1]) .$$

The existence and uniqueness of solution to the problem (3.13) can be obtained using the Lax-Migram theorem under the following assumptions:

Lemma 3.2 *Assume that $c(x) \geq C_{min} > 0$ and $a(x) \geq 0$ almost everywhere in Ω. Then the weak form (3.13) has a unique solution $u \in H_0^1(\Omega)$.*

3.4.3 Applications of Lax-Milgram theorem

Suppose $b \in C^1([0,1])$ and $c, f \in C([0,1])$ and consider the problem: Find a function $u : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} -u'' + bu' + au &= f\,, \qquad \text{on } [0,1] \\ u(0) = u(1) &= 0 \end{cases} \tag{3.18}$$

A *weak solution* to this problem is any function $u \in H_0^1([0,1])$ such that

$$\int_0^1 u'v' + bu'v + auv = \int_0^1 fv\,, \qquad \forall v \in H_0^1([0,1])\,.$$

This weak formulation verify the hypothesis of the Lax-Milgram theorem. Here,

$$\begin{aligned} a(u,v) &= \int_0^1 u'v' + bu'v + auv\,, \qquad \forall u, v \in H_0^1([0,1]) \\ l(v) &= \int_0^1 fv\,, \qquad \forall v \in H_0^1([0,1])\,. \end{aligned} \tag{3.19}$$

It is easy to verify that $a(\cdot,\cdot)$ is a continuous bilinear form on $H_0^1([0,1])$ and that $l(\cdot)$ is a continuous functional on $H_0^1([0,1])$. However, the ellipticity of the bilinear form is obtained under specific assumptions.

Proposition 3.2 *Suppose that for all $x \in [0,1]$, we have $a(x) - \frac{1}{2}b'(x) \geq 0$. Then, the bilinear form $a(\cdot,\cdot)$ is elliptic (*i.e.* α-elliptic with $\alpha = 1$).*

The Lax-Milgram theorem leads to the following result.

Theorem 3.5 *If $a(x) - \frac{1}{2}b'(x) \geq 0$ for all $x \in [0,1]$, then the problem (3.18) has a unique weak solution $u \in H_0^1([0,1])$.*

3.4.4 Non-homogeneous Dirichlet conditions

3.4.5 Neumann boundary conditions

Consider the problem model (3.13) endowed with Neumann boundary conditions of the form

$$\frac{\partial u}{\partial \nu} = g \qquad \text{on } \partial\Omega \tag{3.20}$$

where $g \in C(\partial\Omega)$ and ν denotes the outer normal vector to $\partial\Omega$. Suppose the following assumption on $a(x)$ holds:

$$a(x) \geq C_{min} > 0 \qquad \text{in } \Omega\,.$$

The weak formulation of this problem is obtained as previously at the difference that the boundary integrals do not vanish as they did in the homogeneous Dirichlet case. Noticing that $\partial u/\partial \nu = \nabla u \cdot \nu$, we obtain the weak formulation: Given $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$, find $u \in H^1(\Omega)$ such that

$$\int_\Omega (c\nabla u \cdot \nabla v + auv)\,dx = \int_\Omega fv\,dx + \int_{\partial\Omega} cgv\,dS\,, \qquad \forall v \in H^1(\Omega)\,. \tag{3.21}$$

The reader will identify easily the bilinear form $a(\cdot,\cdot)$ and the linear form $l(\cdot)$ in the previous formulation. Notice that the bilinear form $a(\cdot,\cdot)$ corresponds to the same formula as in the case of Dirichlet boundary conditions, only the space changed. The Lax-Milgram theorem yields the existence of a unique solution to this problem, the assumption on $a(x)$ is used to prove the α-ellipticity of the bilinear form $a(cdot,\cdot)$.

Remark 3.9 *Dirichlet boundary conditions are sometimes called* essential *boundary conditions since they influence the weak formulation and determine the function space in which the solution exists. Neumann boundary conditions are naturally incorporated in the boundary integrals and are thus called* natural.

Another type of boundary condition is often used, that involves a combination of function values and normal derivatives. They are called Newton or Robin boundary conditions. For instance, consider the model problem:

$$\begin{aligned} -\nabla\cdot(c\nabla u) + au &= f \qquad \text{in } \Omega \\ c_1 u + c_2 \frac{\partial u}{\partial \nu} &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{3.22}$$

where $f \in C(\Omega)$, $g \in C(\partial\Omega)$ and $c_1, c_2 \in C(\partial\Omega)$ are such that $c_1 c_2 > 0$ and $0 < \epsilon < |c_2|$ on $\partial\Omega$. For such problem, we obtain the following weak formulation: Given $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $a, c \in L^\infty(\Omega)$, find $u \in H^1(\Omega)$ such that:

$$\int_\Omega c\nabla u\cdot\nabla v + auv\,dx + \int_{\partial\Omega} \frac{cc_1}{c_2} uv\,dS = \int_{\partial\Omega} fv\,dx + \int_{\partial\Omega} \frac{cg}{c_2} v\,dS \qquad \forall v \in H^1(\Omega)\,. \tag{3.23}$$

Since the bilinear form $a(\cdot,\cdot)$ is bounded and α-elliptic, the Lax-Milgram theorem yields a unique solution to this problem.

3.4.6 Maximum principles

Maximum principles are important in elliptic problems and their discrete counterparts allow to compute nonnegative quantities in application problems. Here, we like to study linear elliptic operators of the form

$$Lu(x) = -\sum_{i,j=1}^{d} c_{ij}(x) \frac{\partial^2 u}{\partial x^2}\,, \tag{3.24}$$

where $\Omega \subset \mathbb{R}^d$ is an open bounded set and $c_{ij} \in C(\Omega)$.

Theorem 3.6 (Maximum principle) *Consider a symmetric elliptic operator of the form (3.24) and let $u \in C^2(\Omega) \cap C(\Omega)$ be the solution of the equation $lu = f$, where $f \in C(\Omega)$ and $f \le 0$ in Ω. Then, the maximum of u in $\overline{\Omega}$ is attained on the boundary $\partial\Omega$:*

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u\,.$$

Furthermore, if the maximum is attained at an interior point of Ω, then the function u is constant.

Proof. (i) Let first assume that we have $f < 0$ in Ω and yet there exists a point $x_0 \in \Omega$ such that

$$u(x_0) = \sup x \in \Omega u(x) > \sup_{x\in\partial\Omega} u(x)\,.$$

Since the matrix $C(x_0) = (c_{ij}(x_0))$ is symmetric and positive definite, there exists an orthogonal matrix $O = (o_{ij})$ so that

$$\Lambda = O^{-1}\,C(x_0)\,O\,,$$

where $\Lambda = diag(\lambda_1(x_0), \dots, \lambda_d(x_0))$. Introducing the change of coordinates $\xi = \xi(x) = Ox$, we have

$$\begin{aligned} 0 > f(x_0) &= (Lu)(x_0) \\ &= -\sum_{i,j=1}^{d} (O^{-1}\,C(x_0)\,O)_{ij} \frac{\partial^2 u}{\partial\xi_i\partial\xi_j}(x_0) \\ &= -\sum_{i,j=1}^{d} \lambda_i(x_0) \frac{\partial^2 u}{\partial\xi_i^2}(x_0)\,, \end{aligned} \tag{3.25}$$

which is in contradiction with $\lambda_i(x_0) > 0$ for all $1 \leq i \leq d$, and $x_0 \in \Omega \backslash \partial\Omega$ is a maximum point of u, *i.e.* $\dfrac{\partial^2 u}{\partial \xi_i^2}(x_0) \leq 0$ for all $1 \leq i \leq d$.

(ii) we proceed similarly for the weaker assumption $f \leq 0$ in Ω. □

There follow several results from the maximum principle as straightforward consequences of the theorem 3.6.

Corollary 3.1 (Minimum principle) *Consider an elliptic operator L of the form (3.24). If $Lu = f \geq 0$ in Ω, then u attains its minimum on the boundary $\partial\Omega$:*

$$\min_{\overline{\Omega}} = \min_{\partial\Omega} u\,.$$

Corollary 3.2 (Comparison principle) *Consider an elliptic operator L of the form (3.24). Suppose that functions $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ solve the equations $Lu = f_u$ and $Lv = f_v$ respectively, and*

$$f_u \leq f_v \quad \text{in } \Omega \qquad u \leq v\,, \quad \text{on } \partial\Omega\,.$$

Then, $u \leq v$ in Ω.

Corollary 3.3 (Continuous dependence on boundary data) *Consider an elliptic operator L of the form (3.24). Suppose that u_1 and u_2 solve the equation $Lu = f$ with different Dirichlet boundary conditions. Then,*

$$\sup_{x\in\Omega} |u_1(x) - u_2(x)| = \sup_{x\in\partial\Omega} |u_1(x) - u_2(x)|\,.$$

Corollary 3.4 (Continuous dependence on the RHS) *Consider a uniformly elliptic operator L of the form (3.24). Then, there exists a constant C only depending on the set Ω and the uniform ellipticity constant α, such that:*

$$|u(x)| \leq \sup_{y\in\partial\Omega} |u(y)| + C \sup_{y\in\Omega} |f(y)|\,, \qquad \forall x \in \Omega\,.$$