Chapter 5: Error estimates and mesh adaptation

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Section 5.1
Error estimates
context: computation of a solution $u_h$ for a model problem using a triangulation $\mathcal{T}_h$.

- the analysis of this solution allows to decide whether $u_h$ is an accurate approximation of $u$ or not. In such case, we deduce from $u_h$ a sizing information that allows to modify the triangulation $\mathcal{T}_h$ such as to compute a more accurate solution $u_h$.

- such process is called an adaptative scheme.

- it assumes that the error estimate can be performed only based on known quantities: the diameter $h$, the numerical solution $u_h$, the right-hand side $f$...

- this estimate is called a posteriori estimate, to make a distinction with a priori estimates like:

$$
\|u - u_h\|_{H^1(\Omega)} \leq c h^k \|u\|_{H^{k+1}(\Omega)},
$$

which are quite limited in practice since $\|u\|_{H^{k+1}(\Omega)}$ is unknown.
Motivations:

- we consider the model problem posed in $\Omega$ of the form:

$$\text{find } u \in W \text{ such that } a(u, v) = f(v), \text{ for all } v \in V,$$

where $V$ and $W$ are Hilbert spaces, $f \in V'$ and $a \in \mathcal{L}(W \times V, \mathbb{R})$.

- we suppose also that $a(\cdot, \cdot)$ satisfies the hypothesis of the Nečas (or Lax-Milgram) theorem, and that the problem is well-posed.

- for now, we consider given a triangulation $\mathcal{T}_h$ of the domain and the approximation spaces $W_h$ and $V_h$. The approximate problem is then:

$$\text{find } u_h \in W_h \text{ such that } a(u_h, v_h) = f(v_h), \text{ for all } v_h \in V_h,$$

which is also considered to be well-posed.
• we introduce the definition

**Definition 1.** A function $e(h, u_h, f)$ is called an a posteriori error if it provides an upper bound on the approximation error: $\|u - u_h\|_W \leq e(h, u_h, f)$. Furthermore, if $e(h, u_h, f)$ is such that

$$e(h, u_h, f) = \left( \sum_{K \in T_h} e_K(u_h, f)^2 \right)^{1/2},$$

then $e_K(u_h, f)$ is called an error indicator.

• it remains to connect the error indicator and the mesh adaptation.

• intuitively, if the value $e_T(u_h, f)$ is large (resp. small), we can locally (**i.e.,** in the neighborhood of $T$) refine (resp. coarsen) the triangulation.

• the aim is thus to **equidistribute** the error on $T_h$ so as to achieve a maximal accuracy for a given number of degrees of freedom (nodes).
one main concern about the error estimate: is it **optimal**?

the optimality of the estimate is a guarantee that the evaluation $e_K(u_h, f)$ is not too pessimistic and does not lead to over-refine the triangulation. It is thus interesting to introduce estimates of the form:

$$c_1 e_K(u_h, f) \leq \|u - u_h\|_{W,K} \leq c_2 e_K(u_h, f), \quad \text{for all } K \in \mathcal{T}_h.$$  

for the norm $\|u - u_h\|_W$ taken as $(\sum_{K \in \mathcal{T}_h} \|u - u_h\|^2_{W,T})^{1/2}$.

this type of inequality means that the error indicator $e_K(u_h, f)$ is **equivalent** to the local error $\|u - u_h\|_{W,T}$.

the **objective** is thus to obtain:

- "anisotropic" bounds on the derivatives, so as to prescribe the **size**, the **shape** and the **orientation** of the simplices of $\mathcal{T}_h$.
- estimates in $L^1, L^2$ norms or for the $H^1$ seminorm.
A PRIORI ERROR ESTIMATE

**Theorem 1.** Let $M$ be the constant associated with the continuity of the bilinear form $a$: $a(u, v) \leq M\|u\|\|v\|$ and $\alpha$ be the constant associated with the hypothesis of ellipticity (coercivity), we have:

$$\|u - u_h\| \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|.$$ 

If $a(\cdot, \cdot)$ is symmetric, $a(u - u_h, v_h) = 0$ for all $v_h \in V_h$ means that $u_h$ is the projection of $u$ in $V_h$ according to the inner product $a$. Thus, we have:

$$\|u - u_h\| \leq \sqrt{\frac{M}{\alpha}} \inf_{v_h \in V_h} \|u - v_h\|.$$ 

The result follows:

**Theorem 2.** For Lagrange $P^k$ finite elements, and considering a sufficiently smooth solution $u \in H^{k+1}(\Omega)$, we have the error bound:

$$\|u - u_h\|_{H^1(\Omega)} \leq C h^k |u|_{H^{k+1}(\Omega)}$$

where $h$ is the maximal diameter of the simplices in $T_h$ and $C$ is a constant independent of $h$. 
A priori error estimate

We look at the one dimensional problem. Let consider a discretisation of $I = [a, b]$ in $N$ subintervals $K_i = [x_{i-1}, x_i]$ and suppose $V_h$ is the space of continuous piecewise affine functions. We rely on the classical bound:

$$\|u - u_h\|_{H^1(I)} \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(I)}, \quad \forall v_h \in V_h.$$ 

and then we deduce the following results:

**Theorem 3.** Let denote $h = \max_i |x_i - x_{i-1}|$ and for every $x \in [a, b]$ we have:

$$|u(x) - \pi_{T_h} u(x)| \leq \frac{h^2}{8} \max_{x \in [a,b]} |u''(x)|$$

$$|u'(x) - \pi_{T_h} u(x)| \leq \frac{h}{2} \max_{x \in [a,b]} |u''(x)|$$

$$\|u(x) - \pi_{T_h} u(x)\|_{H^1(I)} \leq Ch \max_{x \in [a,b]} |u''(x)|$$

$$\|u(x) - u_h(x)\|_{H^1(I)} \leq Ch \max_{x \in [a,b]} |u''(x)|$$

By noticing that: $\|u - \pi_{T_h} u\|_{H^1(\Omega)} = \left( \sum_K \|u - \pi_{T_h} u\|_{H^1(\Omega)}^2 \right)^{\frac{1}{2}}$. 
**Remark 1.** The previous bounds are only available if the forms $a(\cdot, \cdot)$ and $l(\cdot)$ are employed in the exact and in the approximated problems (hence, in the exact evaluation of the integrals).

**Proof:** For each $[x_{i-1}, x_i]$, we introduce the two basis functions of $V_h$ associated with the points $x_{i-1}$ and $x_i$, respectively. Let us recall that their restriction in $[x_{i-1}, x_i]$ are written as:

$$w_{i-1}(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}, \quad w_i(x) = \frac{x - x_{i-1}}{x_i - x_{i-1}}$$  \hspace{1cm} (1)

We have the following relations for all $x$ in $[x_{i-1}, x_i]$:

$$w_{i-1}(x) + w_i(x) = 1 \quad x_{i-1}w_{i-1}(x) + x_iw_i(x) = x$$

and

$$\pi_{T_h} u(x) = u(x_{i-1})w_{i-1}(x) + u(x_i)w_i(x)$$

and thus

$$(\pi_{T_h} u)'(x) = u(x_{i-1})w_{i-1}'(x) + u(x_i)w_i'(x).$$

Now, we use the second order Taylor expansions as follows:

$$u(x_{i-1}) = u(x) + (x_{i-1} - x)u'(x) + \frac{(x_{i-1} - x)^2}{2}u''(\xi_i)$$

$$u(x_i) = u(x) + (x_i - x)u'(x) + \frac{(x_i - x)^2}{2}u''(\eta_i)$$
and we obtain:

$$\pi_T u(x) = u(x) + \frac{1}{2}[(x_{i-1} - x)^2 w_{i-1}(x) u''(\xi_i) + (x_i - x)^2 w_i(x) u''(\eta_i)]$$

and

$$(\pi_T u)'(x) = u'(x) + \frac{1}{2}[(x_{i-1} - x)^2 w_{i-1}(x) u''(\xi_i) + (x_i - x)^2 w_i'(x) u''(\eta_i)]$$

On the one hand, we introduce a bound on:

$$\frac{1}{2}[(x_{i-1} - x)^2 w_{i-1}(x) u''(\xi_i) + (x_i - x)^2 w_i(x) u''(\eta_i)] \leq \frac{h^2}{8}$$

and then

$$\frac{1}{2}[(x_{i-1} - x)^2 w_{i-1}(x) + (x_i - x)^2 w_i(x)] \max_{x \in [a,b]} |u''(x)|$$

and similarly for the derivatives:

$$\frac{1}{2}[(x_{i-1} - x)^2 w_{i-1}'(x) u''(\xi_i) + (x_i - x)^2 w_i'(x) u''(\eta_i)] \leq \frac{h^2}{8}$$

$$\frac{1}{2}[(x_{i-1} - x)^2 |w_{i-1}'(x)| + (x_i - x)^2 |w_i'(x)|] \max_{x \in [a,b]} |u''(x)|$$
A priori error estimate

**Remark 2.** In each segment $K_i$ of length $h_i$, we obtain the bounds on the local error:

$$\|u - \pi_T h u\|_{L^2(K_i)} \leq C \frac{h_i^2}{8} \max_{x \in K_i} |u''(x)|$$

$$\|u - \pi_T h u\|_{H^1(K_i)} \leq C h_i \max_{x \in K_i} |u''(x)|$$

And we deduce the choice of the sizes $h_i$ to equidistribute the errors:

- in $L^2$ norm: $h_i \propto 1/\sqrt{\max_{x \in K_i} |u''(x)|}$

- in $H^1$ norm: $h_i \propto 1/\max_{x \in K_i} |u''(x)|$

It can be shown that each choice minimizes the global errors $L^2$ and $H^1$, respectively.
A priori error estimate

• hence, for the $L^2$ error, by using for example:

\[
\pi T_h u(x) = u(x) + \frac{1}{2}[(x_{i-1} - x)^2 w_{i-1}(x) u''(\xi_i) + (x_i - x)^2 w_i(x) u''(\eta_i)],
\]

• we have then

\[
\|u - \pi T_h u\|_{L^2(\Omega)} \leq \left[ \frac{1}{4} \sum_{i=1}^{i=N} \left( \int_{x_{i-1}}^{x_i} [(x_{i-1} - x)^2 w_{i-1}(x) + (x_i - x)^2 w_i(x)]^2 \, dx \right) \max_{[x_{i-1},x_i]} |u''|^2 \right]^{1/2}
\]

• an explicit computation yields to write

\[
[(x_{i-1} - x)^2 w_{i-1}(x) + (x_i - x)^2 w_i(x)] = (x_i - x)(x - x_{i-1})
\]

• and we obtain, by posing $h_i = x_i - x_{i-1}$:

\[
\|u - \pi T_h u\|_{L^2(\Omega)} \leq \left[ \frac{1}{120} \sum_{i=1}^{i=N} h_i^5 \max_{[x_{i-1},x_i]} |u''|^2 \right]^{1/2}.
\]
A priori error estimate

- the minimum is obtained by computing the gradient of this error with respect to the $x_i$
- this yields the following relation:

$$h_i \sqrt{\max_{[x_{i-1}, x_i]} |u''|} = h_{i+1} \sqrt{\max_{[x_i, x_{i+1}]} |u''|} \quad \forall i = 1, \ldots, N - 1$$

Figure 1: Optimal point placement for an integration (Matlab) and to equidistribute the interpolation error in $L^2$ norm.
a few comments:

**Remark 3.** Suppose $u$ if of class $C^2$, a polynomial of degree 2. Then, $u''$ is constant. In this case, we observe that the maximum of the interpolation error is achieved at the midpoint of the interval.

Furthermore, the derivative of $u$ is an affine function. The derivative of its interpolate, $(\pi_{T_i} u)'$, is a constant function which is equal to the value of $u'$ at the midpoint of the interval.

In summary, at the midpoint of the interval $[x_{i-1}, x_i]$, we have both a maximum of the interpolation error on $u$ and no interpolation error on $u'$.

This explains why it is desirable to estimate the derivatives of the solution at the midpoints of the intervals.
Two dimensional error estimate

- consider a polygonal planar domain $\Omega$ covered by a triangulation $\mathcal{T}_h$. Consider the space of continuous piecewise affine functions $V_h$ (affines on the triangles $K_i$). Again, we introduce the bound:

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{M}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \quad \forall v_h \in V_h$$

(2)

- similarly, we take for $v_h$ the interpolate $\pi_{\mathcal{T}_h} u$ of $u$ in the space of finite elements $V_h$. We bound the approximation error $\|u - u_h\|$ by a constant multiplied by the interpolation error $\|u - \pi_{\mathcal{T}_h} u\|$.

- denote by $h$ the diameter and by $\theta_0$ the smallest vertex angle among each triangle of $\mathcal{T}_h$; The Hessian matrix $D^2v$ of a $C^2$ continuous function $v$ is then:

$$D^2v(x, y) = \begin{pmatrix}
\frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial x \partial y} \\
\frac{\partial^2 v}{\partial x \partial y} & \frac{\partial^2 v}{\partial y^2}
\end{pmatrix}$$

(3)

and $|D^2v(x, y)|$ will then denote its spectral norm.
we have the following result:

**Theorem 4.** For all \(x, y \in \Omega\):

\[
\|u(x, y) - \pi_{T_h} u(x, y)\| \leq \frac{h^2}{2} \sup_{x, y \in \Omega} |D^2 v(x, y)|
\]

\[
\|\nabla u(x, y) - \nabla \pi_{T_h} u(x, y)\| \leq 3 \frac{h}{\sin(\theta_0)} \sup_{x, y \in \Omega} |D^2 v(x, y)|
\]

hence the interpolation error is bounded:

\[
\|u - \pi_{T_h} u\|_{H^1(\Omega)} \leq C h \sup_{x, y \in \Omega} |D^2 v(x, y)|
\]

and the approximation error is bounded:

\[
\|u - u_h\|_{H^1(\Omega)} \leq C h \sup_{x, y \in \Omega} |D^2 v(x, y)|
\]
**Remark 4.** The previous result raises several comments and remarks:

1. The dominant term in the error bound is related to the gradients. It will be minimal if \( \frac{h}{\sin(\theta_0)} \) is minimal; this yields to consider the equilateral triangle as the optimal shape. It is also the largest triangle inscribed in a circle of given radius.

2. Suppose \( u \) is a polynomial of degree 2, then \( D^2u \) is a matrix with constant coefficients.
   - In the case of isotropic and homogeneous problem, the Hessian matrix \( D^2u \) is a scalar matrix.
   - In each triangle, the maximum of the interpolation error is attained at the center of the circumscribed circle.
   - The gradient of \( u \) is an affine function and the gradient of its interpolate \( \nabla \pi_{T_h}u \) is a constant function equal to the value of \( \nabla u \) at the center of gravity of the triangle.
   - At the circumcenter of the equilateral triangle, we have a maximum for the interpolation error on \( u \) and the interpolation error on \( \nabla u \) vanishes.
3. For non-isotropic problems, the previous analysis is incomplete. We have to consider the eigenvalues and eigenvectors of the Hessian matrix in each triangle and to find the largest triangle inscribed in an ellipse as the optimal triangle.

- hence, we go back to the expressions:

\[
\pi_{T_h} u(x, y) = u(x, y) + \frac{1}{2} \sum_{i=1}^{3} \lambda_i(x, y) \overrightarrow{MA_i} \cdot D^2u(\xi, \eta) \overrightarrow{MA_i}
\]

\[
\nabla \pi_{T_h} u(x, y) = \nabla u(x, y) + \frac{1}{2} \sum_{i=1}^{3} \nabla \lambda_i(x, y) \cdot \overrightarrow{MA_i} \cdot D^2u(\xi, \eta) \overrightarrow{MA_i}.
\]

- however, the Hessian matrices diagonalized in their eigenbasis are no longer scalar matrices. We have to consider taking the absolute values of the eigenvalues.
we define a new inner product and then a new metric for which the isolines of error become concentric ellipses. Let us denote

$$D^2 = \begin{pmatrix} |d_1| & 0 \\ 0 & |d_2| \end{pmatrix}$$

the symmetric and positive definite matrix associated with the inner product in this metric. We still have the bounds:

$$|u(x, y) - \pi_{\mathcal{T}} u(x, y)| \leq \frac{1}{2} \sum_{i=1}^{i=3} \lambda_i(x, y) \overrightarrow{MA_i} . D^2 \overrightarrow{MA_i}$$

$$|\nabla u(x, y) - \nabla \pi_{\mathcal{T}} u(x, y)| \leq \frac{1}{2} \sum_{i=1}^{i=3} |\nabla \lambda_i(x, y)| \overrightarrow{MA_i} . D^2 \overrightarrow{MA_i}$$

**Remark 5.** Again, we observe that the error on $u$ vanishes along the ellipse circumscribed to the triangle and is maximal at its center. Regarding the error on $\nabla u$, it is the opposite.
Figure 2: Isolines of the error on the equilateral triangle: isotropic case (left), anisotropic case with adapted (middle) and non-adapted triangle (right).

**Remark 6.** We can conclude that the choice of the optimal shape for the elements is related to the error on the gradients. The interpolation error on $u$ is only related to the size. Regarding the approximation error, it depends on both the size and the shape of the elements, as it is known by the $H^1$ norm.
Section 5.2
Residual based estimates
• we consider a model problem, involving a second order scalar PDE. Let $\Omega$ be a polyhedral domain in $\mathbb{R}^d$ and consider the problem posed for $f$ in $L^2(\Omega)$:

$$\text{find } u \in H^1_0(\Omega) \text{ solving}$$

$$a(u, v) = \int_{\Omega} f v, \quad \forall v \in H^1_0(\Omega).$$

where $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v$ is elliptic.

• let $\mathcal{T}_h$ denotes a family of affine triangulations of $\Omega$ composed of Lagrange finite element Lagrange $P^k$. We denote $V_h \subset H^1_0(\Omega)$ the approximation space and the approximate problem is then:

$$\text{find } u_h \in V_h \text{ solving}$$

$$a(u_h, v_h) = \int_{\Omega} f v_h, \quad \forall v_h \in V_h.$$

• we recall here the a priori error estimate:

$$\|u - u_h\|_{H^1(\Omega)} \leq c_1 \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)} \leq c_2 \left( \sum_{K \in \mathcal{T}_h} h_{K}^{2k} \|u\|^2_{H^{k+1}(K)} \right)^{1/2}. $$
to be slightly more general, we do not suppose here the ellipticity of the form but only its stability of $a(\cdot, \cdot)$ according to Nečas theorem:

$$\inf_{u \in H^1_0(\Omega)} \sup_{v \in H^1_0(\Omega)} \frac{a(u, v)}{\|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}} \geq \alpha. \tag{4}$$

we can deduce:

$$\alpha \|u - u_h\|_{H^1(\Omega)} \leq \sup_{v \in H^1_0(\Omega)} \frac{a(u - u_h, v)}{\|v\|_{H^1(\Omega)}} \leq \sup_{v \in H^1_0(\Omega)} \frac{\langle \Delta(u - u_h), v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}}{\|v\|_{H^1(\Omega)}} \leq \|f + \Delta u_h\|_{H^{-1}(\Omega)},$$

that is now a true \textit{a posteriori} estimate.

**Proposition 1.** We have, for any $h$ (discretization parameter):  

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{1}{\alpha} \|f + \Delta u_h\|_{H^{-1}(\Omega)}. \tag{5}$$
Residual based error estimates

**Remark 7.** The previous result appeals two comments:

1. **the error estimate** \( (5) \) **is indeed an a posteriori estimate, but in practice, the norm** \( \| \cdot \|_{H^{-1}} \) **is quite tedious to evaluate.**

2. The intuitive idea of integrating by parts is interesting since it leads to put the unknown \( u \) in the right-hand side. It is possible to avoid the norm \( \| \cdot \|_{H^{-1}} \) by performing the integration on each element \( K \) of \( T_h \).

**Theorem 5.** Suppose the family \( (T_h) \) is regular. Then it exists \( c > 0 \) such that for any \( h \)

\[
\| u - u_h \|_{H^1(\Omega)} \leq c \left( \sum_{K \in T_h} e_K(u_h, f)^2 \right)^{1/2},
\]

where we introduced the **error estimate**

\[
e_K(u_h, f) = h_K \| f + \Delta u_h \|_{L^2(T)} + \frac{1}{2} \sum_{e \in E_K} h_e^2 \| [\partial_n u_h] \|_{L^2(I)},
\]

with \( e \) a face of \( K \), \( E_K \) the set of (non boundary) faces of \( K \), \( h_e \) the diameter of \( e \) and \( [\partial_n u_h] \) the jump of the normal derivative of \( u_h \) across \( e \).
Residual based error estimates

- a family of triangulations \((\mathcal{T}_h)_h\) is regular if there exists a constant \(\sigma_0\) such that

\[
\forall h, \forall K \in \mathcal{T}_h, \quad \sigma_K = \frac{h_K}{\rho_K} \leq \sigma_0 .
\]

- before writing the proof, we need to introduce Clément’s operator.

**Proposition 2.** Let \((\mathcal{T}_h)_h\) be a regular family of affine triangulations on \(\Omega\). Let \(V_h\) be an approximation space built with \(\mathcal{T}_h\) supposed \(H^1\)-conforming. Let \(K\) be a simplex. We denote \(V(K)\) the set of all simplices of \(\mathcal{T}_h\) that share a common intersection with \(K\). Let \(e\) be a face of \(K\) and \(V(e)\) bet the set of all simplices having an intersection with \(e\). There exists an operator \(C_h : H^1(\Omega) \to V_h\) and a constant \(c > 0\) such that:

\[
\forall h, \forall K \in \mathcal{T}_h, \quad \begin{cases} 
\|v - C_h v\|_{L^1(K)} \leq ch_K \|v\|_{H^1(V(K))} \\
\|v - C_h v\|_{L^1(e)} \leq ch_e^{1/2} \|v\|_{H^1(V(e))}
\end{cases}
\]
Proof: We notice that $a(u - u_h, v_h) = 0$ for all $v_h \in V_h$, and thus the stability inequality allows to write

$$\|u - u_h\|_{H^1(\Omega)} \leq \frac{1}{\alpha} \sup_{v \in H^1_0(\Omega)} \frac{a(u - u_h, v - v_h)}{\|v\|_{H^1(\Omega)}}.$$ 

On the right-hand side, we develop the upper part:

$$a(u - u_h, v - v_h) = \int_{\Omega} (-\Delta u)(v - v_h) - \nabla u_h \cdot \nabla (v - v_h)$$

$$= \sum_{K \in T_h} \left[ \int_K (f + \Delta u_h)(v - v_h) - \sum_{e \in \partial T_e} \int_e (\partial_n u_h)(v - v_h) \right].$$

Since $v - v_h$ vanishes on $\partial \Omega$, the sum on $e$ involves only the common faces between two simplices. Since $v - v_h$ is continuous across each interface $e$, it comes

$$a(u - u_h, v - v_h) \leq \sum_{K \in T_h} \left( \|f + \Delta u_h\|_{L^2(K)} \|v - v_h\|_{L^2(K)} + \sum_{e \in E_K} \frac{1}{2} \|[\partial_n u_h]\|_{L^2(e)} \|v - v_h\|_{L^2(e)} \right)$$

$$\leq \sum_{K \in T_h} e_T(u_h, f) \max_{e \in E_K} \left( h_K^{-1} \|v - v_h\|_{L^2(K)}, \max_{e \in E_K} (h_e^{-1/2} \|v - v_h\|_{L^2(e)}) \right)$$
We chose $v_h = C_h v$ and thus we have

$$a(u - u_h, v - C_h v) \leq c \sum_{K \in \mathcal{T}_h} e_K(u_h, f) \eta_K,$$

where $c$ is the interpolation constant associated with $C_h$ and $

\eta_K = \max(\|v\|_{H^1(V(K))}, \max_{e \in E_K}(\|v\|_{H^1(V(e)))}).$

We introduce the two integers

$$M = \max_{K \in \mathcal{T}_h} \text{card}\{K' \in \mathcal{T}_h; K \in V(K')\}, \quad N = \max_{e \in E_h} \text{card}\{e \in E_h; K \in V(e)\},$$

where $E_h$ denotes the set of internal (non boundary) faces of $\mathcal{T}_h.$ The numbers $M$ and $N$ depend only on the smoothness of $\mathcal{T}_h$ and are independent of the granularity $h.$ It appears clearly that

$$\sum_{K \in \mathcal{T}_h} \eta_K^2 \leq \max(M, N) \|v\|_{H^1(\Omega)}^2,$$

and thus we have:

$$\frac{a(u - u_h, v - C_h v)}{\|v\|_{H^1(\Omega)}} \leq c \left(\sum_{K \in \mathcal{T}_h} e_K(u_h, f)^2\right)^{1/2}.$$

that can be combined with the bound on $\|u - u_h\|_{H^1(\Omega)}$ deduced from the stability of $a(\cdot, \cdot)$ to conclude.
Remark 8. This result leads to several comments:

1. $e_K(u_h, f)$ is called a residual based error estimate; indeed, the value $f + \Delta u_h$ is the residual of the equation $f = -\Delta u$.

2. If $V_h$ is build with $\mathbb{P}^1$ Lagrange finite elements, $\Delta u_h|_K = 0$ and $\partial_n u_h|_e$ is constant along $e$, which leads to the result:

$$e_K(u_h, f) = h_K \| f \|_{L^2(K)} + \frac{1}{2} \sum_{e \in E_K} h_e^{1/2} \text{mes}(e)^{1/2} \| \partial_n u_h \|_e.$$

3. The coercivity of the bilinear form $a(\cdot, \cdot)$ is not really involved, but we relied on the stability inequality (4). Hence, it this analysis can be generalized to non-coercive problems, provided that they fulfill the hypothesis of of Nečas theorem.
• the approach can be generalized to the following type of problem:

\[
\mathcal{L}u = -\nabla \cdot (\sigma \cdot \nabla u) + (\beta \cdot \nabla)u + \mu u,
\]

supplied with mixed Dirichlet-Neumann boundary conditions.

• suppose \( \partial \Omega = \Gamma_D \cup \Gamma_N \), \( f \in L^2(\Omega) \) and \( g \in L^2(\Omega) \). We consider the problem

find \( u \in V \) solution to

\[
a(u, v) = \int_{\Omega} fv + \int_{\Gamma_N} gv, \quad \forall v \in V.
\]

where \( V = \{ v \in H^1(\Omega); \; v|_{\Gamma_D} = 0 \} \) and \( a(u, v) = \int_{\Omega} \nabla u \cdot \sigma \cdot \nabla v + v(\beta \cdot \nabla u) + \mu uv \).

• suppose the problem is well posed. Its solution satisfies

\[
\begin{cases}
\mathcal{L}u = f & \text{a.e. in } \Omega \\
u = 0 & \text{a.e. on } \Gamma_D, \\
n \cdot \sigma \nabla u = g & \text{a.e. on } \Gamma_N.
\end{cases}
\]
Residual based error estimates

consider a regular family $\mathcal{T}_h$ of affine triangulations, a Lagrange $\mathbb{P}^k$ finite element, and $V_h \subset V$ the corresponding approximation space. The discrete problem reads:

find $u_h \in V_h$ solution to

$$a(u_h, v_h) = \int_{\Omega} fv_h + \int_{\Gamma_N} gv_h, \quad \forall v_h \in V_h.$$ 

**Proposition 3.** under these assumptions, we derive the following error indicator:

$$e_K(u_h, f, g) = h_K \| f - \mathcal{L} u_h \|_{L^2(K)}$$

$$+ \frac{1}{2} \sum_{e \in E_K} h_e^{1/2} \| [n \cdot \sigma \cdot \nabla u_h] \|_{L^2(e)} + \sum_{e \in N_K} h_e^{1/2} \| g - n \cdot \sigma \cdot \nabla u_h \|_{L^2(e)}$$

where $N_K$ is the set of faces of $K$ located on $\Gamma_N.$
Section 5.3

Geometric estimates
**Geometric Error estimates**

The analysis is here very similar to the study of parameterized surfaces. We have the result:

**Theorem 6.** Let $T_h$ be a triangulation of the domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) and $u$ be a $C^2$ function on $\Omega$. Let $V_h$ the space of continuous functions on $\Omega$ whose restriction to every simplex of $T_h$ is a $P_1$ function, and denote by $\pi_T : C(\Omega) \to V_T$ the usual $P_1$ finite elements interpolation operator. Then for every simplex $K \in T_h$

$$\|u - \pi_T u\|_{L^\infty(K)} \leq \frac{1}{2} \left( \frac{d}{d+1} \right)^2 \max_{x \in K} \max_{y,z \in K} \langle |\mathcal{H}(u)|(x)yz, yz \rangle$$

(6)

where, for a symmetric matrix $S \in S_d(\mathbb{R})$, which admits the following diagonal shape in orthonormal basis $S = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_d \end{pmatrix}^t P$, we denote

$$|S| := P \begin{pmatrix} |\lambda_1| & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & |\lambda_d| \end{pmatrix}^t P.$$
This result can be rewritten under the following (more convenient) form:

\[
\| u - \pi_{\mathcal{T}_h} u \|_{L^\infty(K)} \leq \frac{1}{2} \left( \frac{d}{d + 1} \right)^2 \max_{x \in K} \max_{e \in E_K} \langle |\mathcal{H}(u)|(x)e, e \rangle
\]

However, the right-hand side is still tedious to evaluate.

Hence, we are searching for a metric tensor \( \widetilde{M}_K \) on \( K \in \mathcal{T}_h \) such that:

\[
\max_{x \in K} \langle e, |\mathcal{H}(u)|(x)e, e \rangle \leq \langle \widetilde{M}_K \ e, e \rangle,
\]

and such that the region defined by \( \{ \langle v, \widetilde{M}_K \ v \rangle / \forall v \subset K \} \) has a minimal surface. This yields the result:

\[
\| u - \pi_{\mathcal{T}_h} u \|_{\infty,K} \leq \frac{1}{2} \left( \frac{d}{d + 1} \right)^2 \max_{e \in E_K} \langle e, \widetilde{M}_K \ e \rangle.
\]
Geometric Error estimates

- a few comments on the result:

**Remark 9.** 1. The relation shows that the interpolation error is locally proportional to the squared value of the diameter $h$ of a simplex $K$, with respect to the metric $\widetilde{M}_K$.
2. Hence, a control on the edge lengths for all $K \in \mathcal{T}_h$ allows to bound the interpolation error on $\mathcal{T}_h$.
3. The aim is then to equidistribute the error to control the accuracy of the solution.

- let $\epsilon$ be the tolerance on the error on each mesh element.

- all edges in the mesh must be such that:

$$\epsilon = c\langle e, \widetilde{M}_K e \rangle, \quad \text{for all } e \in E_K,$$

- by introducing the metric tensor $\mathcal{M}_T = \frac{c}{\epsilon} \widetilde{M}_T$, edges must satisfy:

$$\langle e, \mathcal{M}_T e \rangle = 1 \Rightarrow \text{length} \mathcal{M}_T (e) = 1.$$
The metric notion allows to define an equivalence class among triangulations: 2 unit triangulations are equivalent.

The triangulation becomes then an unknown of the problem, given $\mathcal{M}$:

- The density $d = \prod_{i=1}^{n} \frac{1}{h_i}$ where $h_i$ is the local size,

- The complexity of a triangulation is given by:

$$C(\mathcal{M}) = \int_{x \in \Omega} d(x) dx = \int_{x \in \Omega} \prod_{i=1}^{n} \frac{1}{h_i}(x) dx,$$

for given $N = Card(\mathcal{T}_h)$, we search for $\mathcal{M}$ minimizing the $L^p$ norm of the error, which yields:

$$\min_{\mathcal{M}} \int_{\Omega} (e_{\mathcal{M}}(x))^p dx = \min_{h_i} \int_{\Omega} \left( \sum_{i=1}^{n} h_i^2 \left| \frac{\partial^2 u}{\partial \alpha_i^2}(x) \right| \right)^p dx,$$

under the constraint: $\int_{\Omega} d(x) dx = N$. 
Mesh adaptation for level sets

- We consider a bounded domain \( \Omega \subset \mathbb{R}^d \), implicitly defined by a function \( \phi : \mathbb{R}^d \to \mathbb{R} \):

\[
\Omega = \{ x \in \mathbb{R}^d, \phi(x) < 0 \} ; \partial \Omega = \{ x \in \mathbb{R}^d, \phi(x) = 0 \} ; \overline{c \Omega} = \{ x \in \mathbb{R}^d, \phi(x) > 0 \}
\]

- We want to adapt the mesh \( \mathcal{T}_h \) so that the 0 level set of \( \pi_{\mathcal{T}_h} \phi \) obtained from \( \phi \) by \( \mathbb{P}^1 \) FE interpolation, \( \partial \Omega_{\mathcal{T}_h} \) is as close as possible to the 0 level set of \( \phi \) in terms of the Hausdorff distance.

**Definition 2.** Let \( K_1, K_2 \) two compacts subsets of \( \mathbb{R}^d \). For all \( x \in \mathbb{R}^d \), denote \( d(x, K_1) = \inf_{y \in K_1} d(x, y) \) the Euclidean distance from \( x \) to \( K_1 \). We define

\[
\rho(K_1, K_2) := \sup_{\in K_1} d(x, K_2)
\]

and the Hausdorff distance between \( K_1 \) and \( K_2 \) as the quantity

\[
d^H(K_1, K_2) := \max(\rho(K_1, K_2), \rho(K_2, K_1))
\]
the following lemma is useful to measure the distance to $\Omega$ relying on any implicitly defining function $\phi$

**Lemma 1.** Let $\phi$ a $C^1$ continuous function on $\mathbb{R}^d$. Assume there exists a tubular neighborhood $V = \{x \in \mathbb{R}^d, |\phi(x)| < \alpha\}$ for some $\alpha > 0$, such that $\phi$ does not exhibit any critical point in $V$. Then, $\partial \Omega$ is a submanifold of $\mathbb{R}^d$, and $\Omega$ is a bounded subdomain of $\mathbb{R}^d$ with $C^1$ boundary. For any point $x \in V$, we have the estimate:

$$d(x, \partial \Omega) \leq \frac{\sup_{z \in V} |\nabla \phi(z)|}{\inf_{z \in V} |\nabla \phi(z)|^2 |\phi(x)|}$$

(7)

formula (7) expresses the idea that the closer $\phi$ is to the signed distance function to $\Omega$, the more reliable the evaluation of $\phi$ as an estimate of the Euclidean distance to $\partial \Omega$. 

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we can derive a formal estimate of the Hausdorff distance between $\partial \Omega$ and $\partial \Omega_{T_h}$:

$$d^H(\partial \Omega, \partial \Omega_{T_h}) \leq c \sup_{y \in \mathbb{R}^d} |\nabla \phi(y)| \cdot \max_{K \in T_h} \max_{x \in K} \max_{y,z \in K} \langle |\mathcal{H}(\phi)(x)| yz, yz \rangle .$$

very similar to a previous estimate (6)

this results has a nice geometric interpretation when $\phi = d_\Omega$ (signed distance function to $\Omega$. In that case, the second fundamental form reads, for any $x \in \partial \Omega$:

$$\forall \xi \in T_x \partial \Omega, \quad \Pi_x(\xi, \xi) = \langle \mathcal{H}d_\Omega(x) \xi, \xi \rangle .$$

hence, the eigenvalues of $\mathcal{H}d_\Omega(x)$ are the two principal curvatures of the surface $\partial \Omega$, associated with the two principal directions at point $x$. 
the above estimate means that so as to get the best reconstruction of $\partial \Omega$, we have to align the circumscribed ellipsoids of the simplices of $\mathcal{T}_h$ with the curvature of this surface.

Figure 3: piecewise affine reconstruction of $\partial \Omega_{\mathcal{T}_h}$ of $\partial \Omega$ with a regular background mesh $\mathcal{T}_h$ (left) and an adapted anisotropic background mesh $\mathcal{T}_h$ (right).
Mesh Adaptation for the Advection Equation

- For all $t \in [0, T]$, consider a bounded regular domain with smooth boundary:
  \[ \Omega(t) := \{ x \in \mathbb{R}^d, \phi(t, x) < 0 \}, \quad \partial \Omega(t) := \{ x \in \mathbb{R}^d, \phi(t, x) = 0 \} \]

- For $n = 0, \ldots, N$, consider $\Omega^n$ and $\partial \Omega^n$ the piecewise affine reconstructions of $\Omega(t^n)$ and $\partial \Omega(t^n)$ obtained as
  \[ \Omega^n := \{ x \in \mathbb{R}^d, \phi^n(x) < 0 \}, \quad \partial \Omega^n := \{ x \in \mathbb{R}^d, \phi^n(x) = 0 \} \]

- A formal use of the previous lemma yields:
  \[
  \rho(\partial \Omega(T), \partial \Omega^N) \leq \sup_{x \in \partial \Omega(T)} \frac{\sup_{K \in T_h} |\nabla \phi^N_K|}{\inf_{K \in T_h} |\nabla \phi^N_K|^2} |\phi^N(x)|
  \]
  \[
  = \sup_{x \in \partial \Omega(T)} \frac{\sup_{K \in T_h} |\nabla \phi^N_K|}{\inf_{K \in T_h} |\nabla \phi^N_K|^2} |\phi^N(x) - \phi(T, x)|
  \]
  \[
  \leq \frac{\sup_{K \in T_h} |\nabla \phi^N_K|}{\inf_{K \in T_h} |\nabla \phi^N_K|^2} \| \phi^N - \phi(T, \cdot) \|_{L^\infty(\mathbb{R}^d)}
  \]
slightly modified version of the Zalesak slotted disk.
Here, \( \Omega = [-1, 1] \times [-1, 1] \), the disk of radius 0.2, centered at \((0, 0.5)\), slot width is 0.04 and the maximum width of the lower bridge is 0.2. The constant angular velocity is set to 1.

Figure 4: Zalesak slotted disk test: zoom on the initial mesh (left), the final mesh (center) and the mesh at iteration 9. The location of the interface is shown using a green or red line. The time step is here \( \Delta t = \pi/8 \), much larger than the time step prescribed by the restrictive CFL condition: \( \Delta t = 5 \times 10^{-4} \) for an element size \( h_{\text{min}} = 10^{-3} \).
We introduce two $L^1$ error measures:

\[
E^n = \frac{|M^n - M^0|}{M^0}, \quad E^{Z,n} = \sum K \frac{|S^n(K) - S^0(K)|}{M^0},
\]

where $M^n = \sum_{K \in T^n_h} S^n(K)$. The initial mass $M^0$ is 0.117640, the final mass $M^{16}$ is 0.116981 and the errors are $E^{16} = 0.00559$ and $E^{Z,16} = 0.00597$. 