THEORETICAL AND NUMERICAL ISSUES OF INCOMPRESSIBLE FLUID FLOWS

Chapter 3: The Navier-Stokes Model

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University of Tehran, 2018
A CLASSICAL EXAMPLE

- Von Karman vortex street: flow around a cylinder

Figure 1: Von Karman vortex street at various Reynolds numbers.
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- Von Karman vortex street: flow around a cylinder

Figure 2: Re \( \ll 1 \) creeping flow (streamlines of velocity).
A CLASSICAL EXAMPLE

- **Von Karman vortex street**: flow around a cylinder

![Von Karman vortex street](image)

Figure 3: $Re \sim 10$ steady laminar flow (streamlines of velocity).
A CLASSICAL EXAMPLE

- Von Karman vortex street: flow around a cylinder

Figure 4: Re ~ 100 Karman vortex street (streamlines of velocity).
A CLASSICAL EXAMPLE

- Von Karman vortex street: flow around a cylinder

Figure 5: $\text{Re} \sim 1e5$ laminar flow with largely unsteady and turbulent wake (Streamlines of turbulent kinetic energy).
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- Von Karman vortex street: flow around a cylinder

Figure 6: Re $\sim 1e6$ steady fully turbulent flow field (streamlines of velocity).
Section 3.1

Mathematical and numerical analysis
The Navier-Stokes model

- Suppose the fluid is confined in a domain $\Omega_t \subset \mathbb{R}^d$ (i.e. an open and connected region in two or three dimensions of space), for $t \in [0, T]$.

- We recall that the isothermal flow of Newtonian viscous fluids of constant density is modeled by the following Navier-Stokes equations:

\[
\begin{aligned}
\left\{ \begin{array}{l}
\text{Given } f, \text{ find } u \text{ such that } \\
\rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) - \mu \Delta u + \nabla p = \rho f, \\
\text{div } u = 0,
\end{array} \right. \\
\text{in } \Omega_t \times (0, T),
\end{aligned}
\]

where $u$ represents the velocity $[m/s]$, $p$ the pressure $[Pa = n/m^2]$, $\mu$ is the dynamic viscosity $[kg/(m \cdot s)]$, $\rho$ is the density $[kg/m^3]$ and $f$ denotes a density of volume forces per mass unit $[N/m^3]$.

- The non-linear term $(u \cdot \nabla)u$ corresponds to the convective acceleration, while the diffusion term $\Delta u$ models the viscous effects.
The Navier-Stokes model

By dividing both sides of the first equation by the constant density $\rho$, we obtain the alternate formulation

$$\begin{align*}
\begin{cases}
\text{Given } f, \text{ find } u \text{ such that } \\
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \frac{1}{\rho} \nabla p = f, \\
\text{div } u = 0,
\end{cases}
\end{align*}$$

in $\Omega_t \times (0, T)$,

where $\nu = \mu / \rho$ is the kinematic viscosity coefficient $[m^2/s]$.

Introducing the Reynolds number $Re = UL/\nu = \rho UL/\mu$, where $L$ and $U$ are characteristic of the length and velocity scales of the flow, and dividing both sides of (2) by $U^2/L$, allows us to rewrite the equations in a dimensionless form:

$$\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \frac{1}{Re} \Delta u + \nabla p = f, \\
\text{div } u = 0,
\end{cases}
\end{align*}$$

in $\Omega_t \times (0, T)$.

where the introduced the change of variables

$$x = x/L, \quad t = t(U/L), \quad u = u/U, \quad p = p/\rho U^2, \quad f = f(L/U^2).$$
solving the Navier-Stokes equations is a nontrivial task for the following reasons:

- the momentum equation is nonlinear
- the incompressibility condition $\text{div} \ u = 0$
- solving the Navier-Stokes equation amounts to solve a system of $(d + 1$ if $\Omega \subset \mathbb{R}^d$) PDEs coupled through the nonlinear term $(u \cdot \nabla)u$, the incompressibility condition $\text{div} \ u = 0$ and sometimes through the viscous term and the boundary condition;

- a time discretization by operator splitting will partly overcome these difficulties, in particular to decouple the difficulties associated to the nonlinearity with those associated to the incompressibility condition.

- the numerical solution of time dependent PDEs by operator-splitting methods has been largely studied by researchers, but the specialists of ODEs have so far shown little interest in splitting methods.
Initial and boundary conditions

The Navier-Stokes model

- As such, problems (1) or (3) are not well-posed since these equations may have an infinite number of solutions.
- It is mandatory to introduce further conditions, namely
  (a) an initial condition (i.e. at time \( t = 0 \)):

\[
u(x, 0) = u_0(x), \quad \text{in } \Omega
\]

where \( u_0 \) is a given divergence-free vector field and

(b) a Dirichlet boundary condition:

\[
u(x, t) = g(x, t), \quad \text{on } \partial \Omega \times (0, T),
\]

where \( g(x, t) \) is a given function defined over \( \partial \Omega \times (0, T) \).

If \( \Omega \) is bounded, then we have, thanks to the divergence theorem and the continuity condition, the compatibility condition:

\[
\int_{\partial \Omega} g(x, t) \cdot n \, ds = \int_{\Omega} \text{div} \, u(x, t) \, dx = 0,
\]

where \( n \) is the outward unit normal to the domain boundary \( \partial \Omega \).
Other boundary conditions, for example

(c) mixed boundary conditions:

\[ u = g_0, \quad \text{on } \Gamma_D \times (0, T) \]
\[ \sigma n = g_1, \quad \text{on } \Gamma_N \times (0, T) \]  \hspace{1cm} (7)

\( \Gamma_D \) and \( \Gamma_N \) are such that \( \bar{\Gamma}_D \cap \bar{\Gamma}_N = \emptyset \), \( \Gamma_D \cup \Gamma_N = \partial \Omega \), \( g_0 \) and \( g_1 \) are two given functions of \( (x, t) \).

\( \sigma \) is the stress tensor: \( \sigma = 2\mu D(u) - p|_d \)

and \( D(u) \) is the deformation rate tensor: \( 2D(u) = \nabla u + (\nabla u)^t \).

(d) In applications, other mixed conditions are often preferred:

\[ u = g_0, \quad \text{on } \Gamma_D \times (0, T) \]
\[ \mu \frac{\partial u}{\partial n} - pn = g_1, \quad \text{on } \Gamma_N \times (0, T) \]  \hspace{1cm} (8)

Notice that in most applications, the function \( g_1 \) is set to zero in the Neumann boundary condition.
We return to the Navier-Stokes equations in primitive variables:

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \frac{1}{\rho} \nabla p &= f, & \text{in } \Omega_t \times (0, T), \\
\text{div } u &= 0, & \text{in } \Omega_t \times (0, T), \\
u &= u_0, & \text{with } \text{div } u_0 = 0,
\end{align*}
\]

endowed with the following mixed boundary conditions:

\[
\begin{align*}
u &= g_0, & \text{on } \Gamma_D \times (0, T) & \quad \sigma n = g_1, & \text{on } \Gamma_N \times (0, T)
\end{align*}
\]

where we considered \( \sigma = 2\nu D(u) - \frac{p}{\rho} \text{Id} \).

The mathematical theory for the analysis of the Navier-Stokes model and the estimation of the approximation error in finite element method rely on functional analysis and Sobolev spaces.

In particular, the finite element approximation requires to recast the Navier-Stokes equations into a weak form.
Weak form of the problem

• We introduce the Hilbert functional space

\[ V = \{ v \in H^1_0(\Omega_t) \; ; \; v = 0 \; \text{on} \; \Gamma_D \} , \]

which is a Hilbert space for the scalar product and the norm

\[ (u, v)_{H^1(\Omega)} = \int_{\Omega} u(x)v(x) \, dx + \int_{\Omega} \nabla u(x) : \nabla v(x) \, dx \]
\[ \|u\|_{H^1(\Omega)} = \int_{\Omega} |u(x)|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx \]

• Space \( V \) will coincide with \( H^1_0(\Omega_t) \) if the Dirichlet boundary \( \Gamma_D \) coincide exactly with \( \partial\Omega_t \). In this case, the prescribed Dirichlet function \( g_0 \) must be compatible with the divergence-free constraint, \textit{i.e.}

\[ \int_{\partial\Omega_t} g_0 \cdot n \, ds = \int_{\Omega_t} \text{div} \, u \, dx = 0 . \]
Weak form of the problem

• By multiplying the first equation by a test function \( v \in V \) and integrating over \( \Omega_t \), we obtain for almost every \( t \in (0, T) \):

\[
\int_{\Omega_t} \frac{\partial u}{\partial t} \cdot v \, dx + \int_{\Omega_t} ((u \cdot \nabla) u) \cdot v \, dx - \int_{\Omega_t} \nu \Delta u \cdot v \, dx + \frac{1}{\rho} \int_{\Omega_t} \nabla p \cdot v \, dx = \int_{\Omega_t} f \cdot v \, dx.
\]

• Green’s formula allows to rewrite the partial terms as follows

\[
- \int_{\Omega_t} \nu \Delta u \cdot v \, dx = \int_{\Omega_t} \nu \nabla u : \nabla v \, dx - \int_{\partial\Omega_t} \nu \frac{\partial u}{\partial n} \, ds
\]

\[
\int_{\Omega_t} \nabla p \cdot v \, dx = - \int_{\Omega_t} p \text{div} \, v \, dx + \int_{\partial\Omega_t} p v \cdot n \, ds
\]

• Then, taking into account the mixed boundary conditions (10), we have for almost every time \( t \) in \( (0, T) \)

\[
\int_{\Omega_t} \frac{\partial u}{\partial t} \cdot v \, dx + \int_{\Omega_t} ((u \cdot \nabla) u) \cdot v \, dx + \int_{\Omega_t} \nu \nabla u : \nabla v \, dx - \frac{1}{\rho} \int_{\Omega_t} p \text{div} \, v \, dx
\]

\[
= \int_{\Omega_t} f \cdot v \, dx + \int_{\Gamma_N} g_1 \cdot v \, ds, \quad \forall v \in V \ (11)
\]

completed with the Dirichlet boundary condition (10) on \( \Gamma_D \times (0, T) \).
As expected, the Neumann boundary condition \( \sigma n = g_1 \) on \( \Gamma_N \times (0, T) \) is naturally satisfied by the formulation, which is often called the variational formulation of the momentum equation.

Introducing the deformation rate tensor \( D(u) \) yields the following formulation, for almost any \( t \in (0, T) \):

\[
\int_{\Omega_t} \frac{\partial u}{\partial t} \cdot v \, dx + \int_{\Omega_t} ((u \cdot \nabla) u) \cdot v \, dx + 2\nu \int_{\Omega_t} D(u) : D(v) \, dx - \frac{1}{\rho} \int_{\Omega_t} p \, \text{div} \, v \, dx = \int_{\Omega_t} f \cdot v \, dx + \int_{\Gamma_N} g_1 \cdot v \, ds, \quad \forall \, v \in V \tag{12}
\]

completed with the Dirichlet boundary condition (10) on \( \Gamma_D \times (0, T) \). Likewise, the Neumann boundary condition is satisfied automatically.

Similarly for the continuity equation, we first multiply the equation by a test function \( q \) in a space \( Q \) and then we integrate it over \( \Omega_t \), to obtain the equation:

\[
\int_{\Omega_t} q \, \text{div} \, u \, dx = 0, \quad \forall \, q \in Q \tag{13}
\]
Existence of solution to the problem

**Remark 1.** (i) The main difference between this weak formulation and the Stokes weak formulation is related to the convective term. The latter is identified with a trilinear form $c : H^1(\Omega_t)^3 \rightarrow \mathbb{R}$ that will be defined later.

(ii) All integrals involved in the bilinear and the trilinear forms are finite.

- The mathematical analysis of incompressible Navier-Stokes equations for viscous flows has been a topic of interest for almost a century since they were introduced by Navier in 1822 and Stokes in 1845.

- The seminal work of Leray proved the existence of solutions when the flow region is the full space $\mathbb{R}^n$.

- An early worthwhile result is due to J.L. Lions established that if the flow domain is two-dimensional then the unsteady Navier-Stokes equations lead a unique solution.

- In three dimensions of space, there is no theoretical result about the uniqueness (and regularity) of a solution to the time-dependent Navier-Stokes equations modelling the flow of incompressible viscous fluids.
The steady Navier-Stokes problem

- We first consider the steady Navier-Stokes equations that describe the motion of an homogeneous incompressible Newtonian fluid.
- The motion is here independent of time.
- The problem is posed in a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz continuous boundary $\partial \Omega$ and reads as follows:

\[
\begin{aligned}
&\text{Given } f, \text{ find } u \text{ such that } \\
&- \nu \Delta u + (u \cdot \nabla)u + \frac{1}{\rho} \Delta p = f, \quad \text{in } \Omega \\
&\text{div } u = 0, \quad \text{in } \Omega
\end{aligned}
\]  

(14)

In addition, we consider homogeneous Dirichlet boundary conditions,

\[
u = 0, \quad \text{on } \partial \Omega,
\]

(15)

corresponding to the case where the fluid is confined in the domain with fixed boundary.

- This problem close to the Stokes problem, except for the presence of the non-linear convective term $(u \cdot \nabla)u$ that makes it much more difficult to solve.
Mathematical results

• Let consider the functional spaces $V = H^1_0(\Omega)$ and $Q = L^2_0(\Omega)$.

• The variational formulation is almost identical to the formulation (11), except for the integral of the time derivative:
Given $f \in L^2(\Omega)$, find $(u, p)$ such that for all $v \in V$

\[
\int_{\Omega} ((u \cdot \nabla) u) \cdot v \, dx + \int_{\Omega} \nu \nabla u \cdot \nabla v \, dx - \frac{1}{\rho} \int_{\Omega} p \, \text{div} \, v \, dx = \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_N} g_1 \cdot v \, ds
\]  

(16)

• The weak formulation of the continuity equation is strictly identical to equation (13):

\[
\int_{\Omega} q \, \text{div} \, u \, dx = 0, \quad \forall q \in Q.
\]  

(17)

• The formulation (16) can be conveniently recast using bilinear and trilinear forms.
Mathematical results

- We introduce the bilinear forms \( a(\cdot, \cdot) : (u, v) \in V \times V \to a(u, v) \in \mathbb{R} \) and \( b(\cdot, \cdot) : (u, q) \in V \times Q \to b(u, q) \in \mathbb{R} \), (same as the Stokes problem):

\[
 a(u, v) = \int_{\Omega} \nu \nabla u : \nabla v \, dx = \int_{\Omega} \nu \sum_{i=1}^{d} \nabla u_i : \nabla v_i \, dx ,
\]

\[
 b(u, q) = -\int_{\Omega} \text{div} \, u \, q \, dx .
\]

and the trilinear form \( c(\cdot; \cdot, \cdot) : (u, w, v) \in V \times V \times V \to c(w; u, v) \in \mathbb{R} \) associated with the non-linear term and defined as:

\[
 c(w; u, v) = \int_{\Omega} ((w \cdot \nabla)u) \cdot v \, dx = \sum_{i,j=1}^{d} \int_{\Omega} w_j \frac{\partial u_i}{\partial x_j} v_i \, dx .
\]

- We have already stated that the bilinear forms \( a(\cdot, \cdot) \) and \( b(\cdot, \cdot) \) are continuous.

- The next two propositions give useful properties of the trilinear form \( c(\cdot; \cdot, \cdot) \).
Mathematical results

- We have the following results:
  
  **Proposition 1.** The trilinear form $c(\cdot; \cdot, \cdot)$ is continuous on $(H^1(\Omega))^d$.
  
- For any given $u \in V$, the map $v \mapsto c(u, u, v)$ is linear and continuous on $V$.
  
- **Proposition 2.** Let $u, v, w$ in $H^1(\Omega)$ and let assume $\text{div} w = 0$ and $w \cdot n|_{\partial\Omega} = 0$. Then, the trilinear form $c : (H^1(\Omega))^3 \rightarrow \mathbb{R}$ satisfies the following properties, which are equivalent:

  \begin{align*}
  c(w; v, v) &= 0, \quad (21) \\
  c(w; u, v) + c(w; v, u) &= 0. \quad (22)
  \end{align*}

- Assume the constant density $\rho = 1$. We can recast the problem (16): given $f \in L^2(\Omega)$, find $u \in V, p \in Q$ such that

  \begin{align*}
  \begin{cases}
  a(u, v) + c(u; u, v) + b(v, p) = (f, v), & \text{for all } v \in V \\
  b(u, q) = 0, & \text{for all } q \in Q.
  \end{cases} \quad (23)
  \end{align*}

- A pair of function $(u \in V, p \in Q)$ satisfying these equations, given $f \in L^2(\Omega)$, is a weak solution of the stationary Navier-Stokes problem.
The nonlinear term

different forms of the nonlinear term

• \((u \cdot \nabla)u\) : convective form
• \(\text{div}(uu^t)\) : divergence form
• \(\text{div}(uu^t) + \frac{1}{2} \nabla (u^t u)\) : divergence form with modified pressure
• \((\nabla \times u) \times u\) : rotational form (with modified pressure)

the equivalence of these forms can be established.

• from \((u \cdot \nabla)u\) one can obtain:

\[
(u \cdot u)u + \nabla p = \text{div}(uu^t) + \frac{1}{2} \nabla (u^t u) - \frac{1}{2} \nabla (u^t u) + \nabla p
\]

\[
= \text{div}(uu^t) + \frac{1}{2} (u^t u) + \nabla p_{mod} \quad \text{with} \quad p_{mod} = p - \frac{1}{2} (u^t u);
\]

• the convective form and the divergence form are trivially equivalent since one gets:

\[
\text{div}(uu^t) = (u \cdot \nabla u) + (\text{div} u)u = (u \cdot \nabla)u.
\]
**Properties of the convective term**

**Lemma 1 (basic properties).**

- Let $v$ be weakly differentiable, then it is
  \[(u \cdot \nabla)v = (\nabla v)u\]

- Let $u$ be weakly differentiable, then it holds
  \[(u \cdot \nabla)u = D(u)u + \frac{1}{2}(\text{div } u) \times u\]

- The variational form of the convective term is trilinear, i.e., it is linear in each argument;

- Let $u, v, w \in H^1(\Omega)$, then
  \[((u \cdot \nabla)v, w) = (\text{div}(vu^t), w) - ((\text{div } u)v, w),\]
  \[((u \cdot \nabla)v, w) = (u, \nabla(v \cdot w)) - ((u \cdot \nabla)w, v),\]
  \[((u \cdot \nabla)v, w) = ((\nabla v)^t w, u)\]
  \[((u \cdot \nabla)v, w) = \int_{\partial \Omega} (u \cdot n)(v \cdot w) \, ds - (\text{div } u, v \cdot w) - ((u \cdot \nabla)w, v),\]
  where $n$ is the outward unit normal to $\partial \Omega$.

- Let $u, v \in H^1(\Omega)$, $w \in H_{\text{div}}$ then
  \[(((\nabla \times u) \times v, w) = ((v \cdot \nabla)u, w) - ((w \cdot \nabla)u, v).\]
**Lemma 2 (Estimates of the Convective Term).** Let \( u, v, w \in H^1(\Omega) \), where \( \Omega \subset \mathbb{R}^d \) is a bounded domain with Lipschitz boundary, then there is a \( C \in \mathbb{R} \) such that

\[
((u \cdot \nabla)v, w) \leq C \|u\|_{H^1(\Omega)} \|\nabla v\|_{L^2(\Omega)} \|w\|_{H^1(\Omega)},
\]

and from the divergence form of the convective term

\[
((u \cdot \nabla)v, w) + \frac{1}{2}((\text{div } u)v, w) \leq C \|u\|_{H^1(\Omega)} \|\nabla v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)},
\]

If \( w \in V_{\text{div}} \) then it is for the rotational form

\[
((\nabla \times u) \times v, w) \leq C \|u\|_{L^2(\Omega)} \|\nabla v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}.
\]

**Lemma 3 (Estimate of the Convective Term).** Let \( u \in L^2(\Omega) \), \( v \in W^{1,\infty}(\Omega) \) and \( w \in H^1(\Omega) \) then

\[
((u \cdot \nabla)v, w) \leq \|u\|_{L^2(\Omega)} \|\nabla v\|_{L^\infty(\Omega)} \|w\|_{L^2(\Omega)}.
\]
Mathematical results

Let consider the following divergence-free functional subspaces:

\[ V_{\text{div}} = \{ v \in H^1(\Omega) ; \ \text{div} \ v = 0 \}, \quad \text{and} \quad V_{\text{div}}^0 = \{ v \in V_{\text{div}} ; v = 0 \ \text{on} \ \Gamma_D \}, \]

Problem (23) has an equivalent form:

\[
\begin{cases}
\text{find } (u, p) \in V_{\text{div}}^0 \times L^2_0(\Omega) \text{ such that } \\
a(u, v) + c(u; u, v) + b(v, p) = (f, v), \quad \text{for all } v \in H^1_0(\Omega).
\end{cases}
\]

(24)

If \((u, p)\) is a solution to problem (23), then \(u\) is a solution to problem (24). The converse is also true as stated next.

Lemma 4. Let \(\Omega\) be a bounded domain of \(\mathbb{R}^d\) with a Lipschitz-continuous boundary \(\partial \Omega\). Given \(f \in L^2(\Omega)\), there exists at least one pair \((u, p) \in V_{\text{div}} \times L^2_0(\Omega)\) which satisfies problem (24) or equivalently problem (23).
Mathematical results

- Let consider the norm of the trilinear form \( (c; \cdot, \cdot) \) of \( V^3 \) and recall the norm of \( V' \):

\[
\sup_{u,v,w \in V} \frac{c(w; u, v)}{|u|_{H^1(\Omega)}|v|_{H^1(\Omega)}|w|_{H^1(\Omega)}} = C, \quad \|f\|_{V'} = \sup_{v \in V} \frac{(f, v)}{|v|_{H^1(\Omega)}}.
\]

- The next result gives the uniqueness of a weak solution \((u, p)\) to the steady problem (24).

\[\text{Theorem 1. Under the hypothesis of the previous lemma, and assuming that} \]
\[
\frac{C}{v^2} \|f\|_{V'} \leq 1,
\]
\[\text{then, problem (24) has a unique solution} \ (u, p) \ \text{in} \ V \times L^2_0(\Omega).\]
From the numerical point of view, we have to consider two types of boundary conditions:

(i) **artificial conditions**: a virtual boundary is considered, e.g. by introducing a wall to account for the symmetry of the flow and reduce the size of half;

(ii) **physical conditions**: a boundary exists in the domain and must be treated accordingly, e.g. no-slip condition at wall.

The **homogeneous Dirichlet boundary condition** $u = 0$, (no-slip), is the correct and natural condition for a viscous fluid contained in a rigid domain. Indeed, the viscous effect constraints the fluid particles to adhere to the wall.

When Dirichlet conditions $u = g$ are specified on the whole boundary $\partial \Omega$, the pressure solution is known up to a hydrostatic constant only and then the function $g$ has to satisfy a **compatibility condition**:

$$\int_{\partial \Omega} g \cdot n \, ds = 0.$$
Boundary conditions

- For specific applications or in different situations, more appropriate boundary conditions need to be worked out.

- The problem of well-posed boundary conditions is thus a key issue in many fields.
  - Slip boundary condition: the influence of the wall rugosity on the slip behavior of the fluid is still a matter of debate. In case of impermeable boundary, there is a classical condition for the normal velocity component:
    \[ u \cdot n = 0, \quad \text{on } \partial \Omega \]
  
  - For the tangential behavior however, the situation is less clear. A complete slip boundary condition has been proposed:
    \[ u \cdot n = [D(u) \cdot n]_{\tan} = 0, \quad \text{on } \partial \Omega, \]

which means that the fluid is confined in the domain but the fluid particle can slip along the boundary.
Additional boundary conditions have been proposed:

- **Neumann non-friction boundary conditions**: used to prescribe a force per unit area that involves the normal component of the stress tensor $\sigma$:

  $$\sigma n = (2\mu D(u) - p)n = g_1, \text{ on } \Gamma_N,$$

  where $\Gamma_N$ is a subset of $\partial\Omega$.

  When $g_1 = 0$, this condition is usually used as a free outflow condition on a fictitious boundary, e.g. symmetry wall, or at the interfaces, e.g. free surfaces or between two fluids. In this case, the part of the boundary $\Gamma_N$ is called a free outflow. In other situations, $g_1 = -p_en$, where $p_e$ denotes the external pressure.

- Alternate boundary conditions and formulations for the viscous term $-\nu \Delta u$: when the coefficient $\nu$ is constant, the viscous term can be rewritten in different equivalent forms:

  $$\nu \Delta u = \text{div}(\nu (\nabla u + (\nabla u)^t))$$

  $$= -\nu \text{ curl(curl } u) = \nu (\nabla (\text{div } u) - \text{curl(curl } u)).$$

  (25)

  (26)
• Suppose $\Gamma_n$ and $\Gamma_\tau$ are two subsets of $\partial \Omega$, such that $\hat{\Gamma}_n \cap \hat{\Gamma}_\tau = \emptyset$, $\Gamma_n \cup \Gamma_\tau = \partial \Omega$.

• We want to prescribe the normal and the tangent velocities on $\Gamma_n$ and $\Gamma_\tau$, respectively:

\[
\begin{align*}
u \cdot n &= g_n, \quad \text{on } \Gamma_n \quad \text{and} \quad n \times (u \times n) = g_\tau, \quad \text{on } \Gamma_\tau,
\end{align*}
\]

where $n$ denotes the outward normal vector to the domain boundary, $u \cdot n$ is the component of $u$ normal to the boundary and $n \times u \times n = u - (u \cdot n)n$ represents the projection of $u$ onto the tangent (plane) to the boundary.

• We introduce the functional spaces:

\[
\begin{align*}
V_g &= \{v \in H^1(\Omega); \quad v \cdot n = g_n \text{ on } \Gamma_n, \quad n \times v \times n = g_\tau \text{ on } \Gamma_\tau\} \\
V_0 &= \{v \in H^1(\Omega); \quad v \cdot n = 0 \text{ on } \Gamma_n, \quad v \times n = 0 \text{ on } \Gamma_\tau\} \\
\end{align*}
\]

and we consider here $Q = L^2_0(\Omega)$ if $\Gamma_n = \partial \Omega$ or $Q = L^2(\Omega)$ otherwise.
The weak formulation of problem (14) reads:

\[
\begin{cases}
\text{find } (u, p) \in V_g \times Q \text{ such that} \\
\quad a(u, v) + c(u; u, v) + b(v, p) = (f, v) + d(v), \quad \text{for all } v \in V_0 . \\
\quad b(u, q) = 0, \quad \text{for all } q \in Q
\end{cases}
\]

Here, we have introduced in the right-hand side a linear functional \( d(\cdot) \) defined as:

\[
d(v) = \int_{\partial \Omega \backslash \Gamma_n} r \cdot n \, ds + \int_{\partial \Omega \backslash \Gamma_\tau} s \cdot v \times n \, ds,
\]

where the functions \( r \) and \( s \) have a physical meaning.

Taking for \( a(\cdot, \cdot) \) the form (18), the natural boundary conditions become

\[
\begin{align*}
-p + \nu n \cdot \nabla u \cdot n &= r, \quad \text{on } \partial \Omega \backslash \Gamma_n, \\
\nu n \cdot \nabla u \times n &= s, \quad \text{on } \partial \Omega \backslash \Gamma_\tau
\end{align*}
\]
• As such, these conditions have no physical meaning, and the boundary conditions (27) must be imposed on all boundaries.

• For more practical conditions, e.g. for free-surface problems or artificial outflow boundaries, one may prefer dealing with the formulation (25) for the viscous term associated with the bilinear form:

\[
a(u, v) = \frac{1}{2} \int_{\Omega} \nu D(u) : D(v) \, dx,
\]

where

\[
2D(u) = \nabla u + (\nabla u)^t.
\]

And the natural boundary conditions write:

\[
-p + 2\nu n \cdot D(u) \cdot n = r, \quad \text{on } \partial \Omega \setminus \Gamma_n, \tag{31}
\]

\[
2\nu n \cdot D(u) \times n = s, \quad \text{on } \partial \Omega \setminus \Gamma_\tau \tag{32}
\]

where \(r\) and \(s\) are then the normal and tangent stresses, respectively.
Non-linear iterative procedures

Model: the main difference with Stokes equations is the presence of nonlinear terms in the (stationary) Navier-Stokes equations.

- we need iterative methods to solve incompressible flows; the literature contains a variety of such methods which are all based on a linearization of the equations.
- given an initial guess, \( u_0 \in H^1(\Omega) \), a sequence of iterates \( \{u^k\}_{k=1,\ldots,n} \in H^1(\Omega) \) is computed which is expected to converge towards the solution of the weak formulation.
- the fixed-point (Picard) linearization of the stationary Navier-Stokes equations leads to a sequence of so-called Oseen problems, i.e., linear problems (endowed with boundary conditions) of the form

\[
\begin{aligned}
- \nu \Delta u + (w \cdot \nabla) u + cu + \nabla p &= f, & \text{in } \Omega, \\
\text{div } u &= 0, & \text{in } \Omega \\
u &= 0, & \text{on } \partial \Omega
\end{aligned}
\]

where \( w \in V \) is the approximation of the solution from the previous Picard iteration.
Weak solution of Oseen problem

- The weak form of Oseen problem reads as follows:
  given \( w \in V \), find \( (u, p) \in V \times Q \) such that
  \[
  \nu(\nabla u, \nabla v) + ((w \cdot \nabla)u + cu, v) - (\text{div } v, p) = (f, v) \quad \forall v \in V
  \]
  \[
  -(\text{div } u, q) = 0 \quad \forall q \in Q
  \]

\[
(34)
\]

**Theorem 2 (Existence and Uniqueness of a Weak Solution of the Oseen Equations).**
Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d, d = 2, 3 \), with a Lipschitz continuous boundary \( \partial \Omega \); and let the conditions on the data of the Oseen problem be fulfilled (\( w \) in \( L^{d+1}(\Omega) \)), \( u, v \in L^q(\Omega) \). Then, there exists a unique solution \( u, p \in H^1_0(\Omega) \times L^2_0(\Omega) \) of (34).

**Lemma 5 (Stability of the Solution).** Under the conditions of the theorem, the solution of the Oseen problem (34) depends continuously on the data of the problem:

\[
\frac{\nu}{2} \| \nabla u \|^2_{L^2(\Omega)} + \| c^{1/2} u \|^2_{L^2(\Omega)} \leq \frac{1}{2\nu} \| f \|^2_{H^{-1}(\Omega)} \| \nabla u \|
\]

If \( f \in L^2(\Omega) \) and \( c(x) \geq c_0 > 0 \), then we have also the stability bound

\[
\nu \| \nabla u \|^2_{L^2(\Omega)} + \frac{1}{2} \| c^{1/2} u \|^2_{L^2(\Omega)} \leq \frac{1}{2c_0} \| f \|^2_{L^2(\Omega)} \| \nabla u \|
\]
Oseen iteration

- the weak formulation of the Oseen problem is:
  given \( w \in V \), find \((u, p) \in V \times Q\) such that
  \[
  \begin{align*}
  a(u, v) + c(w, u, v) + b(v, p) &= (f, v), \quad \text{for all } v \in V \\
  b(u, q) &= 0 \quad \text{for all } q \in Q.
  \end{align*}
  \]

- the Oseen-iteration is a secant modulus method, where the equations are linearized by freezing the nonlinearity: the first term in \( c(\cdot, u, v) \) is evaluated at the last iterate;

**Algorithm 1 (Oseen iteration).**

for \( k = 0, 1, 2, \ldots \)

given \( u^k \in V, p^k \in Q \), solve \((u^{k+1}, p^{k+1})\)
\[
\begin{align*}
  a(u^{k+1}, v) + c(u^k, u^{k+1}, v) + b(v, p^{k+1}) &= (f, v), \quad \text{for all } v \in V \quad (35) \\
  b(u^{k+1}, q) &= 0 \quad \text{for all } q \in Q.
\end{align*}
\]

- At each iteration step \( k \), one linear system (35) has now to be solved.
Oseen iteration

- note that the previous result does not depend on the size of the data;
- however, if the data are larger the numerical solution of the Oseen equation is difficult (similar to the Navier–Stokes equations with large Reynolds number) and it is not guaranteed that the nonlinear iteration converges.

- the convergence of the Oseen–Iteration can be shown for small data using the Banach fixed point theorem:

  **Theorem 3.** If \( \nu^2 \geq C \| f \|_{L^2(\Omega)} \), the iterate \( u^k \) obtained from the Oseen method converge to the unique solution \( (u, \cdot) \in V \times Q \) of the Navier-Stokes problem.

- in general, the rate of convergence of Oseen iteration is linear.
Newton's method seems appealing because of its theoretical potential to achieve quadratic convergence rate.

Consider the abstract problem of finding the root $\tilde{x}$ of a general nonlinear function $f$, $f(x) = 0$.

Suppose $x_k$ is an approximate solution of the root $\tilde{x}$.

Then, by a Taylor expansion around an initial guess $x_k$ we write

$$0 = f(\tilde{x}) = f(x_k) + f'(x_k)(\tilde{x} - x_k) + o(|\tilde{x} - x_k|)$$

$$\approx f(x_k) + f'(x_k)(\tilde{x} - x_k),$$

and thus

$$\tilde{x} \approx x_k - [f'(x_k)]^{-1} f(x_k).$$

Newton's method for solving $f(x) = 0$ writes:

$$x_{k+1} = x_k - [f'(x_k)]^{-1} f(x_k), \quad k = 0, 1, 2, \ldots$$
Newton's method

1. likewise, let define the function for the Navier-Stokes model:

   \[ f(u, \nabla u) = (u \cdot \nabla) u . \]

2. in this case, a Taylor expansion of \( f(\cdot, \nabla \cdot) \) around \( u_k, \nabla u_k \) reads

   \[ f_{k+1}(u, \nabla u) \approx f_k(u_k, \nabla u_k) + (u_{k+1} - u_k) \cdot \frac{\partial f_k}{\partial u} + \nabla(u_{k+1} - u_k) \cdot \frac{\partial f_k}{\partial \nabla u} . \]

3. neglecting the quadratic terms and substituting the definition of \( f(\cdot, \nabla \cdot) \) yields

   \begin{align*}
   u_{k+1} \cdot \nabla u_{k+1} & \approx u_k \cdot \nabla u_k + \delta_k \cdot \nabla u_k + \nabla \delta_k \cdot u_k \\
   & = u_{k+1} \cdot \nabla u_k + u_k \cdot \nabla u_{k+1} - u_k \cdot \nabla u_k ,
   \end{align*}

   which forms the kernel of the classical Newton linearization of the nonlinear term.

4. The convergence rate is \textbf{quadratic}; if \( \|u - u_0\| \leq \delta \)

   \[ \|u_{k+1} - u\| \leq M\|u_k - u\|^2 , \quad \text{and} \quad \|u_k - u\| \leq \frac{(M\delta)^{2k}}{M} , \]
Newton’s method applied to the weak formulation of the stationary Navier-Stokes problem and using identity \((37)\) takes the following form:

\[
\begin{align*}
\text{Algorithm 2 (Newton-method).} \\
\text{for } k = 0, 1, 2, \ldots \\
1. \text{given } u^k \in V, \text{ solve } \delta u^k \in V \\
\quad a(\delta u^k, v) + c(u^k, \delta u^k, v) + c(\delta u^k, u^k, v) = \\
\quad -\left(a(u^k, v) + c(u^k, u^k, v) - (f^k, v)\right) \\
2. \text{update } u^{k+1} = u^k + \delta u^k ;
\end{align*}
\]

- an important aspect for Newton-type methods is the starting value, since one only obtains local convergence in general.

- one possibility is to use some steps of an Oseen-iteration first, and to use the result as a starting value for a Newton method.
Section 3.2
Discretization procedures
We return to the unsteady Navier-Stokes equations:

\[
\begin{cases}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nu \Delta u + \frac{1}{\rho} \nabla p = f, & \text{in } \Omega_t \times (0, T), \\
div u = 0, & \text{in } \Omega_t \times (0, T), \\
u = u_0, & \text{with } div u_0 = 0,
\end{cases}
\]

endowed with the conditions (to be well-posed):

\[
u = g, \quad \text{on } \partial \Omega \times (0, T) \quad \text{with } \int_{\partial \Omega} g(t) \cdot n \, ds = 0, \quad \text{on } (0, T) \quad (40)
\]

Then, we have for almost every time \( t \) in \((0, T)\)

\[
\int_{\Omega_t} \frac{\partial u}{\partial t} \cdot v \, dx + \int_{\Omega_t} ((u \cdot \nabla) u) \cdot v \, dx + \int_{\Omega_t} \nu \nabla u : \nabla v \, dx - \frac{1}{\rho} \int_{\Omega_t} p \, \text{div} \, v \, dx
\]

\[
= \int_{\Omega_t} f \cdot v \, dx, \quad \forall \, v \in V \quad (41)
\]

completed with the Dirichlet boundary condition on \( \partial \Omega \times (0, T) \).
Temporal discretizations

- discretization in time:

  $\theta$-schemes use the following strategy for the full discretization and linearization:

  1. semi-discretization of (41) in time. The semi-discretization in time leads in each discrete time to a nonlinear system of equations of saddle point type.
  2. variational formulation and linearization. The nonlinear system of equations is reformulated as variational problem and the nonlinear variational problem is linearized.
  3. discretization of the linear systems in space. The linear system of equations arising in each step of the iteration for solving the nonlinear problem is discretized by a finite element discretization using, e.g., an inf-sup stable pair of finite element spaces.

- this approach, which applies first the discretization in time and then in space, is also called method of Rothe or horizontal method of lines. The other way, discretizing first in space to get an ordinary differential equation and then in time, is called vertical) method of lines.
one step $\theta$-schemes for the Navier-Stokes equations

- let $\theta \in [0, 1]$ and consider the time step from $t^n$ to $t^{n+1}$.
- one step $\theta$-schemes for the Navier-Stokes equations are of the form:

$$
\frac{u^{n+1}}{\Delta t^{n+1}} + \theta (-\nu \Delta u^{n+1} + (u^{n+1} \cdot \nabla) u^{n+1}) + \nabla p^{n+1} = u^n \frac{\Delta t^n}{\Delta t^{n+1}} - (1 - \theta) (-\nu \Delta u^n + (u^n \cdot \nabla) u^n) + (1 - \theta) f^n + \theta f^{n+1}
$$

and $\text{div} \, u^{n+1} = 0$

- explicit and implicit Euler scheme, Crank–Nicolson scheme:
  three well known one-step $\theta$-schemes are obtained by an appropriate choice of:
  - $\theta = 0$: forward or explicit Euler scheme,
  - $\theta = 1/2$: Crank–Nicolson scheme,
  - $\theta = 1$: backward or implicit Euler scheme.
one step $\theta$-schemes for the Navier-Stokes equations

- forward or explicit Euler scheme: the appearance of the viscous term leads to a stiff ODE with respect to time. From the numerical analysis of ODEs, it is known that explicit schemes have to be used with very small time steps for stiff problems to obtain stable simulations. For the Navier–Stokes equations, the time step has to be usually so small that the simulations become very inefficient, hence not recommended for the discretization.

- backward or implicit Euler scheme: this first order scheme is quite popular. However, the use of the backward Euler scheme in combination with higher order discretizations in space might lead to rather inaccurate results, compared with the results computed with higher order temporal discretizations.

- Crank–Nicolson scheme: is a second order scheme whose use is very popular.
Temporal discretizations concerning the incompressibility constraint

- the temporal derivative is approximated in $(42)$ at $t^n + \theta t^{n+1}$.
  Thus, the velocity is approximated at this time and the pressure acts as a Lagrangian multiplier for this velocity.

- for instance, in the case of the Crank-Nicolson scheme, the computed solution approximates $(u^{n+1/2}, p^{n+1/2})$.

- if one is interested in a good approximation of the pressure at time $t^{n+1}$, one has to modify the pressure term in the Crank–Nicolson scheme to
  $$\frac{1}{2} (p^{n+1} + p^n).$$

- however, this scheme requires in the first step the pressure at the initial time.
We perform the space discretization of the NS system as previously, to find \((u_h, p_h)\) in \(V_h \times Q_h\) for all \(t \in (0, T)\) such that:

\[
\begin{align*}
\int_{\Omega_t} \frac{\partial u_h}{\partial t} \cdot v_h \, dx + \int_{\Omega_t} ( (u_h \cdot \nabla) u_h ) \cdot v_h \, dx + \int_{\Omega_t} \nu \nabla u_h : \nabla v_h \, dx \\
- \frac{1}{\rho} \int_{\Omega_t} p_h \text{div} \, v_h \, dx &= \int_{\Omega_t} f_h \cdot v_h \, dx, \quad \forall v_h \in V_{0h} \\
\int_{\Omega_t} \text{div} \, u_h q_h \, dx &= 0 \quad \forall q_h \in Q_h \\
u_h &= g_h(t) \quad \text{on} \, \partial \Omega_t \\
u_h(0) &= u_{0h}
\end{align*}
\]

. The finite element spaces \(V_h\) and \(Q_h\) are classically

\[
V_{0h} = V_h \cap \left( H^1_0(\Omega_t) \right)^2 = \{ v_h \in V_h ; \ v_h = 0 \, \text{on} \, \partial \Omega_t \},
\]

. The function \(f_h, u_{0h}\) and \(g_h\) are convenient approximations of \(f, u_0\) and \(g\) and \(g_h\) must verify

\[
\int_{\partial \Omega_t} g_h(t) \cdot n \, ds = 0 \quad \text{on} \, (0, T).
\]
We introduce a $\theta$-scheme (Glowinski, 1985) to discretize an initial value problem:

$$\frac{\partial u}{\partial t} + A(u, t) = 0, \quad \text{with } u(0) = u_0.$$

- Let $\theta$ be a number in the open interval $(0, 1/2)$,

- the scheme applied to the solution of this problem, when $A = A_1 + A_2$, solves $u^{n+\theta}$, $u^{n+1-\theta}$ and $u^{n+1}$

- it writes:

\[
\begin{align*}
& u^0 = u_0 \\
& \frac{u^{n+\theta} - u^n}{\theta \delta t} + A_1(u^{n+\theta}, (n + \theta)\delta t) + A_2(u^n, n\delta t) = 0 \\
& \frac{u^{n+1-\theta} - u^{n+\theta}}{(1 - 2\theta)\delta t} + A_1(u^{n+\theta}, (n + \theta)\delta t) + A_2(u^{n+1-\theta}, (n + 1 - \theta)\delta t) = 0 \\
& \frac{u^{n+1} - u^{n+1-\theta}}{\delta t} + A_1(u^{n+1}, (n + 1)\delta t) + A_2(u^{n+1-\theta}, (n + 1 - \theta)\delta t) = 0
\end{align*}
\]
The full discretization of unsteady Navier-Stokes equations by a $\theta$-scheme leads to the following systems:

- initializations and parameter settings:
  \[
  \begin{aligned}
  u^0_h &= u_{0h} \\
  \theta &= 1 - 1/\sqrt{2}, \\
  \alpha &= (1 - 2\theta)/(1 - \theta), \\
  \beta &= \theta/(1 - \theta),
  \end{aligned}
  \]

- and then, $u^n_h$ being known,

- compute first $(u^{n+\theta}_h, p^{n+\theta}_h) \in V_h \times Q_h$ by solving the elliptic systems, for $n \geq 0$:
  \[
  \begin{aligned}
  \forall v_h \in V_{0h}, \\
  \int_{\Omega_t} \frac{u^{n+\theta}_h - u^n_h}{\theta \delta t} \cdot v_h \, dx + \alpha \nu \int_{\Omega_t} \nabla u^{n+\theta}_h : \nabla v_h \, dx - \int_{\Omega_t} p^{n+\theta}_h \text{div} \, v_h \, dx \\
  = \int_{\Omega_t} f^{n+\theta}_h \cdot v_h \, dx - \beta \nu \int_{\Omega_t} \nabla u^n_h : \nabla v_h \, dx - \int_{\Omega_t} (u^n_h \cdot \nabla) u^n_h \cdot v_h \, dx \\
  \int_{\Omega_t} \text{div} \, u^{n+\theta}_h \, q_h = 0, \quad \forall q_h \in Q_h \\
  u^{n+\theta}_h = g^{n+\theta}_h \quad \text{on } \partial \Omega_t
  \end{aligned}
  \]
compute $u_h^{n+1-\theta} \in V_h$, for all $v_h \in V_{0h}$

$$
\begin{align*}
\int_{\Omega_t} \frac{u_h^{n+1-\theta} - u_h^{n+\theta}}{(1 - 2\theta)\delta t} \cdot v_h \, dx &+ \beta \nu \int_{\Omega_t} \nabla u_h^{n+1-\theta} : \nabla v_h \, dx \\
&\quad + \int_{\Omega_t} (u_h^{n+1-\theta} \cdot \nabla) u_h^{n+1-\theta} \cdot v_h \, dx \\
&\quad = \int_{\Omega_t} f_h^{n+\theta} \cdot v_h \, dx - \alpha \nu \int_{\Omega_t} \nabla u_h^{n+\theta} : \nabla v_h \, dx - \int_{\Omega_t} p_h^{n+\theta} \text{div} v_h \, dx
\end{align*}
$$

\begin{align*}
u_h^{n+1-\theta} &= g_h^{n+1-\theta} \quad \text{on } \partial \Omega_t,
\end{align*}
and finally compute \( (u_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h \)

\[
\begin{align*}
\int_{\Omega_t} \frac{u_h^{n+1} - u_h^{n+1-\theta}}{(\theta)\delta t} \cdot v_h \, dx + \alpha v \int_{\Omega_t} \nabla u_h^{n+1} : \nabla v_h \, dx - \int_{\Omega_t} p_h^{n+1} \text{div} \, v_h \, dx & \\
= \int_{\Omega_t} f_h^{n+1} \cdot v_h \, dx - \beta v \int_{\Omega_t} \nabla u_h^{n+1-\theta} : \nabla v_h \, dx & \\
- \int_{\Omega_t} (u_h^{n+1-\theta} \cdot \nabla)u_h^{n+1-\theta} \cdot v_h \, dx & \\
\int_{\Omega_t} \text{div} \, u_h^{n+1} q_h \, dx = 0 \quad \forall q_h \in Q_h & \\
u_h^{n+1} = g_h^{n+1} \quad \text{on} \quad \partial \Omega_t
\end{align*}
\]
Discretization of the total derivative

- Until now, the term $(u \cdot \nabla)u$ was treated as a general nonlinearity;
- However, in Navier-Stokes equations, the term

\[
\rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right)
\]

models the transport of the momentum $\rho u$ by the velocity field $u$.

- The idea of combining FE approximations and methods of characteristics for numerical simulations of incompressible viscous fluid flows is quite old (1970’s): J.P. Benque, O. Pironneau, Gresho, … but still considered a bit "exotic".
- Despite the simplicity of the principle that underlies the methods of characteristics, their practical implementation is delicate and not fully understood;
- "Any method for approximating hyperbolic equations sacrifices a good deal if it takes no account of the method of characteristics" (Morton, 1992), "Numerical schemes that follow characteristics backward in time and then interpolate at their feet have a history stretching back in the very early days of computational fluid dynamics" (Courant et al., 1952).
We introduce the **method of characteristics** to solve a transport problem:

\[
\begin{cases}
\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi = 0 \quad \text{in } \Omega \times (0, T) \\
\varphi(0) = \varphi_0 \\
\varphi = g \quad \text{on } \Gamma_- \times (0, T)
\end{cases}
\]

with

\[
\frac{\partial u}{\partial t} = 0 \quad \text{div } u = 0 \quad \frac{\partial g}{\partial t} = 0
\]

and

\[\Gamma_- = \{ x \in \partial \Omega, \ u(x) \cdot n(x) < 0 \} .\]

Let consider \((x_*, t_*) \in \Omega \times (0, T)\); we associate the solution of the system

\[
\begin{cases}
\frac{dX}{dt} = u(x) \\
X(t_*) = x_*
\end{cases}
\]

the solution of this system is \(X(\cdot; x_*, t_*)\).
The curve $C_{x_*, t_*}$ described by the points $\{X(t; x_*, t_*), t\}$ as $t$ varies is called the characteristic curve associated to the transport equation (43) and to $\{x_*, t_*\}$.

Suppose $\varphi$ is solution of (43) and restrict it to the curve $C_{x_*, t_*}$, we have then

$$\frac{d}{dt}\varphi(X(t; x_*, t_*), t) = \left(\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi\right)(X(t; x_*, t_*), t) = 0.$$ 

This shows that $\varphi$ is constant along the curve $C_{x_*, t_*}$.

It follows that if $\varphi$ solve (43), we have, for $\tau > 0$ sufficiently small, the relation

$$\varphi(x, t) = \varphi(X(t - \tau; x, t), t - \tau).$$
• The backward method of characteristics can be used to solve Navier-Stokes problem:
  - initialization
    \[ u^0 = u_0 \]
  - and for \( n \geq 0 \):
    \[
    \begin{cases}
      \frac{u^{n+1} - u^n}{\delta t} - \nu \Delta u^{n+1} + \nabla p^{n+1} = f^{n+1} & \text{in } \Omega_t \\
      \text{div } u^{n+1} = 0, & \text{in } \Omega_t \\
      u^{n+1} = g^{n+1} & \text{on } \partial \Omega_t;
    \end{cases}
    \]
    \tag{44}
  - where \( u^n \) is obtained by solving the system:
    \[
    \begin{cases}
      \frac{dX}{dt} = u^n(X) & \text{on } (0, \delta t) \\
      X(\delta t) = x
    \end{cases}
    \tag{45}
  
  we denote by \( X^n(\cdot, x) \) the solution of the problem (45).
  - We take \( u^n(x) = u^n(X^n(0, x)) \)
Section 3.3
Numerical examples
classical test case: simulations for Reynolds number ranging from 400 up to 10000.

- problem involves a primary vortex at the cavity center and vortices in the corners,
- number of vortices increases with $Re$ and position of the center of primary vortex moves toward the center,
- 4 meshes have been used: a regular triangulation (2,461 nodes, 5,000 elements); a uniform triangulation (2,143 nodes, 4,136 elements); a refined uniform triangulation (8,421 nodes, 16,544 elements); and a regular triangulation (10,201 nodes, 20,000 elements);
- results in good accordance with experimental results (Ghia et al.).
we consider a two-dimensional channel flow over a downstream-facing step.

it consists in an upstream channel of height \( h \) followed by a suddenly enlarged channel of height \( H = 2h \).

the channel lengths of the portions situated upstream and downstream of the backstep are \( l_0H \) and \( l_1H \), respectively.

at the upstream end of the channel there is a fully developed laminar flow defined by the steady axial velocity

\[
u(y) = 24u_0y(0.5 - y), \quad \text{for } x = -l_0 \quad \text{and } y \in [0, 0.5]\]

where \( u_0 \) represents the mean axial flow velocity.
in the unsteady flow case, a portion of the lower wall of length $Hl$, situated just behind the downstream-facing step, is assumed to execute transverse oscillations defined by the following lower wall equation:

$$H_y = \begin{cases} Hg(x, t) - h_s & \text{for } x \in [0, l] \\ -h_s & \text{for } x > l \end{cases}$$

where $g(x, t)$ is a sinusoidal-shape mode defined as

$$g(x, t) = e(t) \sin(\pi x / l) \quad \text{where } e(t) = A \cos(\omega t)$$
in the unsteady flow case, a portion of the lower wall of length $H_l$, situated just behind the downstream-facing step, is assumed to execute transverse oscillations.

reprinted from [Mateescu D., J. Fluids and Structures, 2001]