THEORETICAL AND NUMERICAL ISSUES OF INCOMPRESSIBLE FLUID FLOWS

Chapter 1: Fluid Mechanics

Instructor: Pascal Frey

Sorbonne Université, CNRS
4, place Jussieu, Paris

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The laws of nature are drawn from experience, but to express them one needs a special language . . .

H. Poincaré, in *Analysis and Physics*. 
Section 1.1

Notations, vectors, tensors
### Differential Operators

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<td>Partial derivative (in the distributional sense) of $u$ with respect to $x_i$</td>
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<td>$</td>
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Vectors in fluid dynamics

- **Velocity gradient**

\[ \nabla \mathbf{u} = (\nabla u_x, \nabla u_y, \nabla u_z) = \begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{pmatrix} \]

The trace of \( \nabla \mathbf{u} \) is \( \text{div} \; \mathbf{u} = \nabla \cdot \mathbf{u} \)

- **Deformation rate tensor**

\[ D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t) = \frac{1}{2} \begin{pmatrix} 2\frac{\partial u_x}{\partial x} & (\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) & (\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}) \\ (\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) & 2\frac{\partial u_y}{\partial y} & (\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}) \\ (\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) & (\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}) & 2\frac{\partial u_z}{\partial z} \end{pmatrix} \]

- **Convective derivative**

\[ \mathbf{u} \cdot \nabla \mathbf{u} = u_x \frac{\partial \mathbf{u}}{\partial x} + u_y \frac{\partial \mathbf{u}}{\partial y} + u_z \frac{\partial \mathbf{u}}{\partial z} \]
Let consider $T = \{t_{ij}\} \in \mathbb{R}^{3 \times 3}$, $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$. We recall:

1. $\alpha T = \{\alpha t_{ij}\}$,

2. $T^1 + T^2 = \{t^1_{ij} + t^2_{ij}\}$

3. $u \cdot T = (u_1, u_2, u_3) \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} = \sum_{i=1}^{3} u_i \begin{pmatrix} t_{i1} \ t_{i2} \ t_{i3} \end{pmatrix}_{i-th \ row}$

4. $T \cdot u = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \sum_{j=1}^{3} \begin{pmatrix} t_{1j} \\ t_{2j} \\ t_{3j} \end{pmatrix} u_j \quad (j-th \ column)$

5. $T^1 \cdot T^2 = \begin{pmatrix} t^1_{11} & t^1_{12} & t^1_{13} \\ t^1_{21} & t^1_{22} & t^1_{23} \\ t^1_{31} & t^1_{32} & t^1_{33} \end{pmatrix} \begin{pmatrix} t^2_{11} & t^2_{12} & t^2_{13} \\ t^2_{21} & t^2_{22} & t^2_{23} \\ t^2_{31} & t^2_{32} & t^2_{33} \end{pmatrix} = \left\{ \sum_{k=1}^{3} t^1_{ik} t^2_{kj} \right\}$

6. $T^1 : T^2 = \text{tr}(T^1 \cdot (T^2)^t) = \sum_{i=1}^{3} \sum_{k=1}^{3} t^1_{ik} t^2_{ik}$. 

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Gauss-Green theorem

- Let $\Omega$ be a bounded, open subset of $\mathbb{R}^d$ and $\partial \Omega$ is $C^1$. Then, along $\partial \Omega$ is defined the outward unit normal vector field $n$.

- Let $u \in C^1(\bar{\Omega})$, the normal derivative of $u$ is defined by:
  \[
  \frac{\partial u}{\partial n} = \nabla u \cdot n = \frac{\partial u}{\partial x} \cdot n. \]

**Theorem 1 (Gauss-Green theorem).** Suppose $u \in C^1(\bar{\Omega})$; then
\[
\int_{\Omega} \frac{\partial u}{\partial x_i} \, dx = \int_{\partial \Omega} u n_i \, ds \quad (i = 1, \ldots, n).
\]

**Theorem 2 (Integration by parts formula).** Let $u, v \in C^1(\bar{\Omega})$; then
\[
\int_{\Omega} \frac{\partial u}{\partial x_i} v \, dx = - \int_{\Omega} u \frac{\partial v}{\partial x_i} \, dx + \int_{\partial \Omega} u v n_i \, ds \quad (i = 1, \ldots, n).
\]

**Theorem 3 (Green’s formulas).** Let $u, v \in C^2(\bar{\Omega})$; then
1. \[
\int_{\Omega} \Delta u \, dx = \int_{\partial \Omega} \frac{\partial u}{\partial n} \, ds;
\]
2. \[
\int_{\Omega} \nabla v \cdot \nabla u \, dx = - \int_{\Omega} u \Delta v \, dx + \int_{\partial \Omega} \frac{\partial v}{\partial n} u \, ds;
\]
3. \[
\int_{\Omega} u \Delta v - v \Delta u \, dx = \int_{\partial \Omega} \frac{\partial v}{\partial n} - \frac{\partial u}{\partial n} \, ds.
\]
Section 1.2
Conservation laws
Mathematical models

Partial Differential Equations (PDE) form the kernel of very many mathematical models and are used to describe physical, chemical, biological, economical phenomena, ...

- **Mathematical Modelling:**
  - **role:** description of the behavior of a system using a set of PDEs endowed with initial and boundary conditions;
  - **challenge:** theoretical framework for mathematical analysis;
  - **well-posedness:** existence, uniqueness (or multiplicity) of solutions and stability w/r to the data.

- **Numerical Simulation:**
  - **target:** imitating / reproducing the behavior of the complex system to analyze and possibly improve it;
  - **methods:** relies on mathematical analysis, numerical analysis, scientific computing.
Numerical simulation • has become a consequent part of the mathematical modelling process and is possibly the sole that can give access to specific solutions that cannot be solved analytically;

• the choice of the approximation method is strongly related to the model and to the nature and the behavior of the phenomenon represented as well as to the desired properties of the solutions (regularity, stability, etc.);

• the quality of the approximation method is related to mathematical properties (stability, convergence and order of convergence, error estimate) and to more practical issues (accuracy, reliability, efficiency). Both criteria are often antagonists and incompatible with one another.

• numerical analysts shall then try to obtain a good compromise between these criteria, preserving the accuracy of the solution at a reasonable computational effort.
the mathematical modelling and numerical simulation of a PDE problem is a highly multidisciplinary topic to obtain the numerical solution;

process requires advanced knowledge of theoretical and applied mathematics and of computer science.

the numerical simulation bridges the gap between theoretical analysis and experiments.

Figure 1: A continuum from modelling to simulation.
Objective: describe the motion of a fluid filling a domain $\Omega_t \subset \mathbb{R}^d, t > 0$, given $\Omega_0$.

Representations: motion of fluid flow can be modeled in two ways:

1. In the Lagrangian description, the trajectory $t \mapsto \chi(x_0, t, t_0)$ of any individual fluid particle, occupying position $x_0$ at time $t_0$, is followed during the time period. This approach is widely used in solid and particle mechanics, but yields more tedious analysis in the case of fluid flow.

2. In the Eulerian representation, the study takes place in a fixed referential, and no particular fluid particle is followed. Each point in the domain $\Omega_t \subset \mathbb{R}^d$ - whose Cartesian coordinates are denoted by $x = (x_i)_{i=1,...,d}$ - is considered over the time period, independently of which specific particle is occupying the place.
Let \( u(x, t) \) denote the velocity of the fluid particle occupying the position \( x \) in \( \Omega_t \) at time \( t \). At each instant \( t \), \( u \) is a vector field on \( \Omega_t \), called the velocity field of the fluid.

**Definition 1.**

1. The flow field is said to be **conservative** if there exists a scalar function \( \phi(x, t) \) (called a scalar potential) such that

\[
    u = \nabla \phi.
\]

2. The **vorticity** \( \omega \) of the flow is defined as:

\[
    \omega = \text{curl}(u) = \nabla \times u.
\]

The flow is said **irrotational** (or **curl-free**) if \( \omega = 0 \).

3. The flow field is said **incompressible** (or **divergence-free**) if the material density \( \rho \) is constant in time and space; this is equivalent to:

\[
    \text{div} \ u = \nabla \cdot u = 0.
\]

4. The **deformation tensor** \( \mathbb{D} = \mathbb{D}(u) \) of the flow is defined as:

\[
    \mathbb{D}(u) = \frac{1}{2}(\nabla u + \nabla^t u).
\]
**PROPOSITION 1.** Let \( u : U \rightarrow \mathbb{R}^d \) any vector field, defined over a subset \( U \subset \mathbb{R}^d \). Then the following asymptotic expansions hold:

\[
(d = 2), \quad u(x + h) = u(x) + \nabla u(x) \cdot h + \frac{1}{2} \text{curl}(u)(x)h^\perp + O(h^2).
\]

\[
(d = 3), \quad u(x + h) = u(x) + \nabla u(x) \cdot h + \frac{1}{2} \text{curl}(u)(x) \times h + O(h^2).
\]

Hence, locally in the vicinity of any point \( x \), \( u \) is submitted to:

1. an **infinitesimal deformation** \( \nabla u \), which admits \( d \) principal deformation directions: the principal directions of the symmetric matrix \( \nabla u \).

2. an **infinitesimal rotation** \( h \mapsto \frac{1}{2} \text{curl}(u)(x) \times h \).

   If \( d = 3 \), the axis of the rotation is the unit vector in the direction of \( \text{curl}(u)(x) \), and the speed of the rotation is proportional to the modulus of \( \text{curl}(u)(x) \).
Exercises: prove the following identities (\( u \) vector field, \( f \) scalar function)

- \( \text{div}(fu) = u \cdot \nabla f + f \text{div} u; \)
- \( \text{div}(\text{curl} u) = \text{div}(\nabla \times u) = 0 \) and \( \text{curl}(\nabla f) = 0; \)
- \( \text{curl}(\text{curl} u) = \nabla (\text{div} u) - \Delta u. \)

The divergence theorem relates the flow of a vector field through a surface to the behavior of the vector field in the domain.

\[ \int_{\Omega} \nabla \cdot v \, dx = \int_{\partial \Omega} v \, n \, ds, \]  (1)

where \( n \) is the outward pointing unit normal field of the boundary \( \partial \Omega \).
The motion of a fluid, filling a domain $\Omega_t$, is considered during the time $I = [t_0, t]$.

Usually, $\Omega_{t_0}$ (resp. $\Omega_t$) is called the reference configuration (resp. current configuration).

In a Lagrangian perspective, the motion is described by a family of mappings and we denote by $\chi(\cdot, t, t_0)$ the diffeomorphism:

$$x_0 \in \Omega_{t_0} \mapsto x = \chi(x_0, t, t_0) \in \Omega_t,$$

which maps the position $x_0$ of a considered particle at time $t_0$ to its position $x$ at $t$.

**Definition 2.** The trajectory of a particle of fluid is the curve $\{\chi(x_0, t, t_0)\}_{t \in I}$.

The velocity of this fluid particle at position $x$ at time $t$ is then the vector $u$, independent of the reference time $t = 0$, i.e.

$$u = u(x, t) = u(\chi(x_0, t, t_0), t) = \frac{\partial \chi}{\partial t}(x_0, t, t_0).$$  \hspace{1cm} (2)
DEFINITION 3. The acceleration $\gamma(x, t)$ of the particle at time $t$ and at $x = \chi(x_0, t, t_0)$ is:

$$
\gamma(x, t) = \gamma(\chi(x_0, t, t_0), t) = \frac{d}{dt} u(x, t) = \frac{d}{dt} u(\chi(x_0, t, t_0), t)
$$

$$
= \frac{\partial u}{\partial t}(\chi(x_0, t, t_0), t) + \sum_i \frac{\partial u}{\partial x_i}(\chi(x_0, t, t_0), t) \frac{\partial \chi_i}{\partial t}(x_0, t, t_0), \quad (3)
$$

which also reads, using (2)

$$
\gamma(\chi(x_0, t, t_0), t) = \frac{\partial u}{\partial t}(\chi(x_0, t, t_0), t) + \sum_i \frac{\partial u}{\partial x_i}(\chi(x_0, t, t_0), t) u_i(\chi(x_0, t, t_0), t)
$$

$$
= \frac{\partial u}{\partial t}(\chi(x_0, t, t_0), t) + (u \cdot \nabla) u(\chi(x_0, t, t_0), t). \quad (4)
$$
Considering the inverse mapping $\chi^{-1}$, for $x \in \Omega_t$ we deduce from the previous formula

$$\gamma(x, t) = \frac{\partial u}{\partial t} + (u \cdot \nabla) u(x, t)$$

which is known as the fundamental Eulerian formula for acceleration of fluids.

**Definition 4.** The operator

$$\frac{D}{Dt} \overset{\text{def}}{=} \frac{\partial}{\partial t} + (u \cdot \nabla)$$

is called the total derivative or the material derivative. It accounts for the derivative of a quantity along the trajectory of a particle with velocity $u$.

For an arbitrary function $f(x, t)$, the chain rule allows to write

$$\dot{f}(x, t) = \frac{d}{dt} f(x, t) \overset{\text{def}}{=} \frac{\partial f}{\partial t}(x, t) + (u(x, t) \cdot \nabla) f(x, t) = \frac{Df}{Dt}(x, t).$$

With this notation, we have $\gamma(x, t) = \dot{u}(x, t)$. 

Notion of mass

• For each time $t > 0$, we assume that there exists a positive measure $\mu_t$ carried by $\Omega_t$ called the mass distribution, which is regular with respect to the Lebesgue measure $^*dx$.

• Hence, there exists a regular function $\rho = \rho(x, t)$ (sufficiently smooth) such that:

$$d\mu_t(x) = \rho(x, t) \, dx,$$

where $\rho$ is called the mass density of the system at point $x$ at time $t$.

• For any subset $W_t \subset \Omega_t$, the mass of fluid contained in $W_t$ at time $t$ is defined by:

$$m(W_t) = \int_{W_t} d\mu_t(x) = \int_{W_t} \rho(x, t) \, dx.$$

Notice that the assumption of existence for $\rho$ is a continuum assumption that no longer holds at molecular level.

*Non mathematicians can simply consider $d\mu_t(x)$ as a notation for the measure $\rho(x, t)dx$. 
FORCES

The motion of a material system is entailed by the action of external forces of two types, represented by a regular vector measure $d\varphi_t(x)$ carried by a volume or a surface.

1. Contact (or stress) forces:
   expressed as forces across the surface $\partial W_t$ of any subsystem $W_t \subset \Omega_t$. Their description relies on Cauchy’s theory of stress tensor.

**Definition 5.** In a $d$-dimensional evolving continuum $\Omega_t$, Cauchy’s stress tensor $\sigma(x, t)$ is a $d$-dimensional tensor, defined by the property that, for any subsystem $W_t \subset \Omega_t$ with smooth enough boundary $\partial W_t$, the surface density of forces applied on $\partial W_t$ by the rest of the continuum reads:

$$s(x, t) = \sigma(x, t)n(x, t),$$

The fact that there is no momentum of the stress efforts inside a fluid in static equilibrium has the following fundamental mathematical translation.

**Theorem 5.** Cauchy’s stress tensor $\sigma$ is symmetric.
The total force exerted on the fluid inside $W_t$ by means of stress on its boundary is

$$s_{\partial W_t} = \int_{\partial W_t} \sigma(x, t) \cdot n(x, t) \, ds(x).$$

where $ds$ is a regular surface measure.

**Definition 6.** A fluid is said to be an ideal fluid when there exists a function $p(x, t)$, called the pressure, such that the stress tensor $\sigma$ is of the form:

$$\sigma(x, t) \cdot n = -p(x, t) \cdot n,$$

(5)

for any unit vector $n$.

This means that there is no tangential component in the contact forces. This definition also implies that there cannot be a rotation initiating in such fluid.
**Definition 7.** A *viscous Newtonian fluid* is defined by the form of its stress tensor \( \sigma \):

\[
\sigma = -p \text{Id} + \mathcal{L}(\mathbf{D}(u)),
\]

where \( p \) is the pressure of the fluid, \( \mathbf{D}(u) \) is the deformation tensor and \( \mathcal{L} \) is a linear mapping.

Further hypothesis (isotropic medium, invariance under a change of observer), lead to the following form of the stress tensor

\[
\sigma = -p \text{Id} + \lambda \text{div} \, u \text{Id} + 2\mu \mathbf{D}(u). \tag{6}
\]

where \( \lambda \), the *volume viscosity*, and \( \mu > 0 \), the *dynamic viscosity*, are the viscosity coefficients of the fluid (also called the Lamé coefficients).

**Viscosity** measures the resistance of a fluid under a shear stress deformation.
2. External (or body) forces:

- exert a force per unit volume on the fluid; e.g. gravity

- described by a density

\[ d\varphi_t(x) = f(x, t) \, dx, \quad \text{for } x \in \Omega_t. \]

For instance, in the classical case of \textit{gravity}, we have

\[ f(x, t) = -\rho g e_3, \]

where \( g \in \mathbb{R}^3 \) is the gravitational acceleration and \( e_3 \) is the vertical unit vector pointing upward.
Two kinds of mechanical works (energy exchange with exterior) can be distinguished, due to external or interior forces.

1. **External work:**

   **Definition 8.** Suppose a fluid in motion, with velocity field \( u(x, t) \), experiences an external volume force \( f(x, t) \); the instantaneous mechanical work at time \( t \) done by \( f \) on a portion \( W_t \) of \( \Omega_t \) is defined as:

   \[
   \int_{W_t} f(x, t) \cdot u(x, t) \, dx.
   \]

   Similarly, if external loads \( g(x, t) \) are applied on the boundary of \( \Omega_t \), the mechanical work done by \( g \) reads:

   \[
   \int_{\Omega_t} g(x, t) \cdot u(x, t) \, ds.
   \]

   When \( f \) and \( u \) have the same orientation, then \( f \cdot u \leq 0 \), and the mechanical work done by \( f \) is positive, in the sense that the action of the force "helps" the motion.
2. Internal work:

**Definition 9.** The instantaneous power received by a portion of fluid in $W_t \subset \Omega_t$ at time $t$ reads:

$$-\int_{W_t} \sigma : D(u) \, dx$$

To get an intuition, assume that the considered fluid is incompressible and viscous. As seen before, one possibility for modelling the stress-strain relation reads:

$$\sigma = -p \text{Id} + 2\mu D(u),$$

where $p$ stands for the scalar pressure field. Then, the instantaneous power received by a portion $W_t$ of fluid is:

$$-\int_{W_t} \sigma : D(u) \, dx = -\int_{W_t} p \, \text{div}(u) + 2\mu |D(u)|^2 \, dx = -2\int_{W_t} \mu |D(u)|^2 \, dx.$$  

Hence viscous forces only entail energy dissipation.
Conservation principles

Establishing the partial differential equations that describe the evolution of $\Omega_t$ according to Newtonian mechanics relies on three conservation laws:

1. **mass conservation**: mass is neither created nor destroyed;

2. **balance of momentum**: according to Newton’s second law, the rate of change of the linear momentum equals the total applied force on the considered system;

3. **conservation of energy**: energy is neither created nor destroyed.

The derivation of the three forthcoming equations of conservation follows a common general sketch: a small portion of fluid, evolving in time, is chosen, and the quantities attached to it (mass, momentum, and energy) should satisfy the aforementioned conservation principles.

Giving a precise mathematical meaning to this idea involves differentiation of integrals on moving domains.
It is often required to compute the time derivative of integrals of the form:

$$g(t) = \int_{\Omega_t} f(x, t) \, dx,$$

where $f = f(x, t)$ is a given scalar function and $\Omega_t$ is a bounded domain in $\mathbb{R}^d$, evolving with respect to a velocity field $u(x, t)$.

**Theorem 6 (Transport theorem (Liouville’s theorem)).** Assume $f = f(x, t)$ is a function of class $C^1$ for $x \in \Omega_t$ and $t \in I$, and that $u$ is of class $C^1$ with respect to $x$ and $t$. Then,

$$\frac{dg}{dt}(t) = \int_{\Omega_t} \frac{\partial f}{\partial t}(x, t) \, dx + \int_{\Omega_t} \text{div}(f \cdot u)(x, t) \, dx,$$

$$= \int_{\Omega_t} \frac{\partial f}{\partial t}(x, t) \, dx + \int_{\partial \Omega_t} f(x, t) u(x, t) \cdot n(x) \, ds(x),$$

where $\partial \Omega_t$ denotes the boundary of $\Omega_t$, $n$ is the unit outward normal vector on $\partial \Omega_t$ and $ds$ is the surface measure on $\partial \Omega_t$. 

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**Differentiation of a Volume Integral**
**Conservation of Mass**

The rate of change of mass in a subset $W_t$ is

$$\frac{d}{dt} m(W_t) = \frac{d}{dt} \int_{W_t} \rho(x,t) \, dx .$$

Using the Transport theorem with $f = \rho$ yields, for all $W_t$ in $\Omega_t$:

$$\int_{W_t} \frac{\partial \rho}{\partial t} + \text{div}(\rho u) \, dx = 0 ,$$

We obtain the so-called continuity equation:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0 \quad \text{in} \quad \Omega_t , \quad (9)$$

In case of an incompressible fluid, $\rho$ is constant and equation (9) becomes

$$\text{div} \, u = 0 \quad \text{in} \quad \Omega_t .$$
Based on this continuity equation, the following convenient form of theorem 6 is suited to deal with mass quantities:

**Corollary 1.** Let \( f = f(x, t) \) is a function of class \( C^1 \) for \( x \in \Omega_t \) and \( t \in I \), and assume that \( u \) is of class \( C^1 \) with respect to \( x \) and \( t \). Define the integral quantity \( g(t) \) as:

\[
g(t) = \int_{\Omega_t} \rho(x, t) f(x, t) \, dx.
\]

Then,

\[
\frac{dg}{dt}(t) = \int_{\Omega_t} \rho(x, t) \frac{Df}{Dt}(x, t) \, dx .
\] (10)

**Proof:**

\[
\frac{dg}{dt}(t) = \int_{\Omega_t} \frac{\partial(\rho f)}{\partial t}(x, t) + \text{div}(\rho f u)(x, t) \, dx ,
\] (11)

\[
= \int_{\Omega_t} \rho \left( \frac{\partial f}{\partial t} + u \cdot \nabla f \right)(x, t) + \left( \frac{\partial \rho}{\partial t} + \text{div}(\rho u) \right)(x, t) \, dx ,
\] (12)

+ definition 4 of the material derivative \( \frac{Df}{dt} \).
Balance of momentum:

- related to Newton’s second law \( F = m\gamma \);
- change of momentum related to the force acting on \( \Omega_t \).

Given \( W_t = \chi(W, t, t_0) \), volume of fluid moving, the linear momentum in \( W_t \) writes:

\[
M_L(W_t) = \int_{W_t} \rho(x, t) u(x, t) \, dx ,
\]

and the balance of momentum reads, thanks to (13):

\[
\frac{d}{dt} M_L(W_t) = \int_{W_t} f(x, t) \, dx + \int_{\partial W_t} \sigma(x, t) \cdot n(x, t) \, ds(x) ,
\]

where \( n = n(x, t) \) is the unit outward normal to \( \partial W_t \).
The left-hand side of previous equation rewrites (cf. corollary 1):

\[
\frac{d}{dt}M_L(W_t) = \int_{W_t} \rho(x, t) \frac{Du}{Dt}(x, t) \, dx = \int_{W_t} \rho(x, t) \dot{u}(x, t) \, dx.
\]

Now, using the divergence theorem 4, one has:

\[
\int_{\partial W_t} \sigma(x, t) n(x, t) \, ds(x) = \int_{W_t} \text{div} \, \sigma(x, t) \, dx.
\]

Since \( W_t \) is an arbitrary subset, we can gather all previous results to write the linear momentum formula:

\[
\rho \dot{u} = \text{div} \, \sigma + f. \tag{15}
\]

Interestingly, the linear momentum law can be used in the context of energy considerations.
Let $W_t \subset \Omega_t$ any evolving portion of fluid, $u$ the velocity field in the frame of reference.

**Definition 10.** The kinetic energy of $W_t$ at time $t$ is given by:

$$E_c = \frac{1}{2} \int_{\Omega_t} |u(x, t)|^2 \ d\mu_t(x) = \frac{1}{2} \int_{\Omega_t} \rho(x, t)|u(x, t)|^2 \ dx,$$

The rate of change of kinetic energy of a moving subset $W_t$ of fluid is calculated as follows

$$\frac{d}{dt} E_c = \frac{d}{dt} \left( \frac{1}{2} \int_{W_t} \rho(x, t)|u(x, t)|^2 \ dx \right)$$

$$= \frac{1}{2} \int_{W_t} \rho(x, t) \frac{D|u(x, t)|^2}{D_t} \ dx \quad \text{(corollary 1)}$$

$$= \frac{1}{2} \int_{W_t} \rho(x, t) \left( u \cdot \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) \right) (x, t) \ dx$$

in other terms,

$$\frac{d}{dt} E_c = \int_{W_t} \rho u \cdot \dot{u} \ dx. \quad (16)$$
**Theorem 7.** The instantaneous rate of change in kinetic energy $\frac{d}{dt} E_c$ in any portion of fluid $W_t \subset \Omega_t$ equals the sum of:

1. the power of internal forces within $W_t$:

$$- \int_{W_t} \sigma : \mathbb{D}(u) \, dx,$$

2. the mechanical work done by surface loads $\sigma \cdot n$ (owing to the continuity of the stress tensor in $\Omega_t$) applied on $\partial W_t$:

$$\int_{\partial W_t} (\sigma \cdot n) \cdot u \, dx,$$

3. the mechanical work done by external forces on $W_t$:

$$\int_{W_t} f \cdot u \, dx.$$

Hence, we write:

$$\int_{W_t} \rho u \cdot \dot{u} \, dx = - \int_{W_t} \sigma : \mathbb{D}(u) \, dx + \int_{\partial W_t} (\sigma \cdot n) \cdot u \, dx + \int_{W_t} f \cdot u \, dx.$$

(17)
Section 1.3

Flows models
Summary of the Equations

1. **Continuity equation**
   
   \[
   \frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0
   \]  
   conservation of mass

2. **Momentum equation**
   
   \[
   \frac{\partial (\rho u)}{\partial t} + \text{div}(\rho u \times u) = \text{div} \sigma + f
   \]  
   Newton’s second law

3. **Energy equation**
   
   \[
   \frac{\partial (\rho E_c)}{\partial t} + \text{div}(\rho E_c u) = -D(u) : \sigma + f \cdot u
   \]  
   first law of thermodynamics
The Navier-Stokes equations

- Physical considerations:
  we consider the motion of a Newtonian incompressible viscous fluid, with the following assumptions on the Cauchy stress tensor $\sigma$:
  1. $\sigma$ depends linearly on the velocity gradients $\nabla u$
     - normal stress: stretching
     - shear stress: deformation
  2. $\sigma$ is invariant under translations and rigid rotations (Galilean invariance);
  3. $\sigma$ is symmetric (consequence of the balance of angular momentum).

We introduced $\sigma$ as (viscous fluid $\mu > 0$):

$$\sigma = -p \mathrm{Id} + \lambda \text{div} \, u \, \mathrm{Id} + 2\mu \mathcal{D}(u),$$

which for an incompressible fluid ($\text{div} \, u = 0$) can be simplified:

$$\sigma = -p \mathrm{Id} + 2\mu \mathcal{D}(u).$$
Substituting $\sigma$ in the expression $\rho \dot{u} = \text{div} \sigma + f$ yields the Navier-Stokes equations for a viscous incompressible fluid filling a domain $\Omega_t$, for any $t \in [0, T]$:

$$
\begin{cases}
\rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + 2 \text{div}(\mu \mathcal{D}(u)) + f \\
\text{div} u = 0
\end{cases}
$$

(19)

where $\rho = \rho(x, t)$ denotes the density, $u = u(x, t)$ the velocity of the fluid particle at position $x$ at time $t$, $p = p(x, t)$ the pressure, and $f = f(x, t)$ the volume forces.

Given $\mu > 0$ and $\text{div} u = 0$, then $\text{div} \nabla u^t = \nabla \text{div} u$ and thus $\text{div}(\mu \mathcal{D}(u)) = \mu \Delta u$, the "classical" form of Navier-Stokes equations is then:

\[
\begin{cases}
\text{Given } f; \text{ find } (u, p) \text{ such that } \\
\rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + \mu \Delta u + f . \\
\text{div} u = 0
\end{cases}
\]  

(20)
The Navier-Stokes equations (3)

\[ \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla)u \right) = -\nabla p + \mu \Delta u + f \]

1. The inertial or non-linear term \((u \cdot \nabla)u\) is related to convective acceleration: it contributes to the transfer of kinetic energy.

2. The term \(-\nabla p\) is the pressure gradient:
   A fluid reacts to attempt to change its volume according to a pressure. This term arises from the isotropic part of the Cauchy-stress tensor \((6) \ p = 1/3 \ \text{tr}(\sigma)\). The pressure is only involved through its gradient in Navier-Stokes equations: accelerating the fluid from high to low pressure.
   Pressure can be interpreted as a Lagrange multiplier corresponding to \(\text{div} \ u = 0\).
The Navier-Stokes equations (4)

\[ \rho \left( \frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + \mu \Delta u + f \]

3. The term \( \mu \Delta u \) is the dissipative viscous term;

\( \mu \) is the dynamic viscosity of the fluid.

Viscosity operates in a diffusion of a momentum, much alike the diffusion of heat in the heat transfer equation (a Laplacian in space). Diffusion of fluid momentum is the result of friction between fluid particles moving at uneven speed that tend gradually to become the same.

4. The vector field \( f \) represents the body forces (forces per unit volume) such as gravity.

**Remark 1.** The NS equations are among the very few equations of mathematical physics where the nonlinearity does not occur from physical assumption but just from mathematical (kinematics) arguments.
1. **Initial condition:**
   - initial velocity field must be divergence-free in $\Omega_t$ and at $\partial \Omega_t$:
   \[
   u(x, 0) = u_0(x), \quad \text{div} \, u_0 = 0.
   \]

2. **Boundary conditions for viscous fluids**
   - Need to consider molecular interactions between fluid and surface of a solid body;
   - **No-slip** condition: normal and tangential components of the velocity vanish:
   \[
   u(x, t) = 0, \quad \forall x \in \partial \Omega_t, \quad \forall t \in I;
   \]
   - General situation: $\partial \Omega_t$ moving at speed $v(x, t)$, then solid and fluid share the same velocity at the boundary, *i.e.*
   \[
   u(x, t) = v(x, t), \quad \forall x \in \partial \Omega_t, \quad \forall t \in I.
   \]
The Reynolds number

- Write the equations in a non-dimensional form, independent of the system of units.

- Define the dimensionless variables as follows:
  \[ u' = \frac{u}{U}, \quad x' = \frac{x}{L}, \quad t' = \frac{tU}{L}, \quad p' = \frac{p}{\rho U^2}. \]

- Substitute the new variables in the Navier-Stokes equations:
  \[
  \begin{cases}
  \frac{\partial u'}{\partial t'} + (u' \cdot \nabla)u' = -\nabla p' + \frac{1}{Re} \Delta u' + f', \\
  \text{div} u' = 0
  \end{cases}
  \]

- New dimensionless parameter: the Reynolds number of the flow:
  \[ Re = \frac{UL}{\nu} = \frac{\rho U^2}{\mu} = \frac{\text{acceleration forces}}{\text{viscosity forces}}. \]

  provides a measure of the viscosity of the flow.
Energy Inequality

- consider the flow of a fluid in a region $\Omega \subset \mathbb{R}^d$ bounded by walls and driven by a body force $f(x, t)$. Let pose $\nu = \mu/\rho$, we have the set of equations:

  - the fluid velocity and pressure are functions which satisfy:

    $$\begin{cases}
    \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = f \\
    \text{div}(u) = 0 \quad \text{and} \quad \int_{\Omega} p \, dx = 0 \quad \text{for} \ t \in (0, T) \\
    u(x, 0) = u_0(x) \quad \text{and} \quad u = 0 \quad \text{on} \ \partial\Omega
    \end{cases}$$

    (22)

  - the key idea of the analysis is due to Leray and is based on the notion of weak solution. The calculation behind Leray’s theory is the energy inequality.

  - If $u, p$ is a smooth solution to (22), then multiplying the momentum equation by $u$, integrating over $\Omega$, integrating by parts, and integrating in time gives:

    $$\frac{1}{2} \|u(t)\|^2 + \int_0^t \nu \|\nabla u(\tau)\| \, d\tau = \frac{1}{2} \|u_0\|^2 + \int_0^t (f'(\tau), u(\tau)) \, d\tau ;$$

  - the energy equality has the physical interpretation:

    kinetic energy(t) + total energy dissipated over $[0, t]$
    
    = initial kinetic energy + total power input.
Historical landmarks

- **Archimedes of Syracuse** (287-212 BC) discovered the principle of buoyancy: an immersed body is acted upon by a force equal to the weight of water it displaces (Eureka).

- **Leonardo da Vinci** (1452-1519) stated the idea of continuity of the fluid continuum, who also performed detailed studies of waves, jets, and interacting eddies. The concept of momentum in physics was introduced by **Galileo** (1564–1642).

- **Isaac Newton** (1642-1727) introduced conservation of linear momentum (force = mass \(\times\) acceleration). He also studied and introduced the first concept of viscosity for laminar flow as a special case of a linear stress–strain relation.

- **Daniel Bernoulli** (1700-1782) introduced the first equations of fluid motion coupling the velocity and pressure and the kinetic theory of gasses, jet propulsion, and manometers.

- **Leonhard Euler** (1707-1783) the father of fluid mechanics as a mathematical discipline. He derived the correct mathematical equations of inviscid flow (the Euler equations). Lagrange wrote, “Euler did not contribute to fluid mechanics but created it.”
Historical Landmarks

- **C. L. M. H. Navier** (1785–1836) and **G. Stokes** (1819–1903)
  Navier-Stokes equations first written in their modern form. Navier is credited for developing the equations of viscous flows. Stokes gave the first clear and correct derivation of the viscous terms in the Navier-Stokes equations based on the **A. Cauchy** (1789–1857) stress principle.

- **J. C. Maxwell** (1831–1879)
  Derived the fundamental equations of electricity and magnetism and the continuum fluid flow equations by a limiting process beginning with Daniel Bernoulli’s kinetic theory of gases, and he studied correct conditions at a boundary.

- **O. Reynolds** (1842–1912)
  Studied turbulence in real fluids experimentally and showed the correct path to be prediction of turbulent flow averages rather than pointwise values.

- **H. Poincaré** (1854–1912)
  Introduced the method of sweeping - an early numerical method for solving partial differential equations - and forecasted the future arithmetization of analysis.
**Historical Landmarks**

- **J. Leray** (1906-1998)
  founder of the mathematical theory of the Navier–Stokes equations; introduced the idea of weak solution of PDEs (turbulent solutions), fixed point and degree theory, connected topology to PDEs, and developed one of the major conjectures concerning turbulence.

- **A. N. Kolmogorov** (1903-1987)
  introduced the theory of homogeneous, isotropic turbulence, which explains many universal features observed in turbulent flows.

- **J. von Neumann** (1903–1957)
  he and R. D. Richtmeyer developed numerical methods for shock problems.

- **O. A. Ladyzhenskaya** (1922–2004)
  developed complete, beautiful, and fully rigorous mathematical theory for the Navier-Stokes system, with a correction term to the equations to account for possible nonlinear effects in a stress–strain relation.