On ‘hard stars’ in general relativity

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Outline

1. *Gravitational Collapse*

Gravitational Collapse

Oppenheimer-Snyder model (1939)

Classical model of spherically symmetric collapse of a dust cloud in general relativity.

$$T = \rho \, u \otimes u$$

$\rho \geq 0$: energy density,
u: fluid speed, unit future-directed vectorfield.
The flow lines of the dust particles are time-like geodesics:

$$\nabla_u u = 0$$

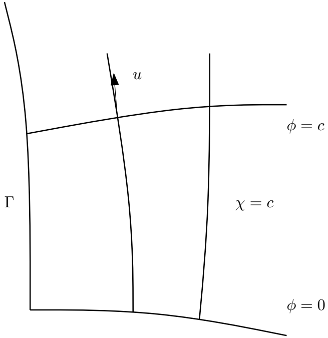
Gravitational Collapse

Initial value problem

Initial data is prescribed at $\phi = 0$:

$$(\rho, r, \frac{\partial r}{\partial \phi}|_\chi)$$

Here ϕ is a time function such that $u(\phi) = 1$ and χ is chosen initially and then constant along the flow lines.
(ϕ, χ) are called *comoving coordinates*.



Gravitational Collapse

Initial value problem

In comoving coordinates, the metric takes the form

$$g = -e^{2\psi} d\phi^2 + e^{2\omega} d\chi^2 + r^2 \overset{\circ}{\gamma}$$

The Hawking mass $m \geq 0$ is defined by $1 - 2m/r = g(\nabla r, \nabla r)$. For the dust model, the Einstein equations then become a system of completely integrable o.d.e.'s for (r, m, ρ, ω) $(\psi = 0)$

$$\frac{\partial^2 r}{\partial \phi^2} = -\frac{m}{r^2}$$

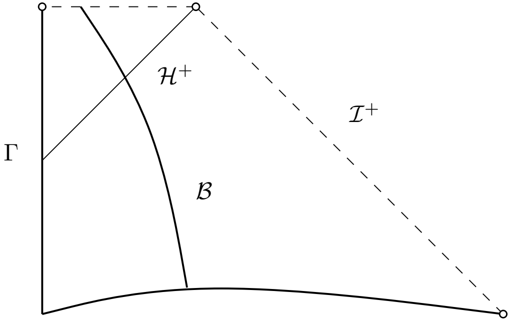
$$\frac{\partial m}{\partial \phi} = 0$$

$$\frac{\partial}{\partial \phi}(r^2 \rho e^{\omega}) = 0$$

Gravitational Collapse

Oppenheimer-Snyder model (1939)

Homogeneous (compactly supported) initial data:
$\rho(0,\chi) = \rho_0$ for $\chi \leq \chi_B$, $\rho(0,\chi) = 0$ for $\chi > \chi_B$.

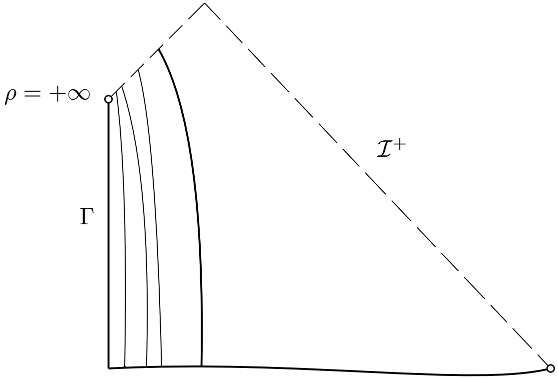


$$\rho(\phi,\chi) = \rho(0,\chi)\frac{r^2(0,\chi)}{r^2(\phi,\chi)}\frac{\partial_\chi r(0,\chi)}{\partial_\chi r(\phi,\chi)}$$

Gravitational Collapse

Violation of cosmic censorship (Christodoulou, 1984)[1]

Inhomogeneous initial data: $\rho(0,\chi) = \rho_0(\chi)$, $\rho_0''(0) < 0$.



Remark: Cosmic censorship does hold if $\rho_0''(0) = 0$.

[1]Numeric results by Eardley-Smarr, 1979

Gravitational Collapse

Christodoulou's two-phase model (1995)[3]

For large densities, the pressure can not be neglected anymore.

Soft phase: For densities below 'nuclear saturation density' ρ_*, the dust model is a good approximation.

Hard phase: For densities above ρ_*, the pressure can no longer be neglected, and the speed of sound approaches the speed of light.[2]

Idealized equation of state for a barotropic fluid:

$$p(\rho) = \begin{cases} 0 & \rho < \rho_* \\ \rho - \rho_* & \rho \geq \rho_* \end{cases}$$

By a choice of units: $\rho_* = 1$. In the hard phase, the speed of sound, $\eta = \sqrt{\frac{\mathrm{d}p}{\mathrm{d}\rho}}$, is equal to the speed of light.

[2] Zel'dovich (1962), Friedman & Pandharipande (1981)

[3] Christodoulou 1995-96 and Christodoulou-Lisibach 2015

Gravitational Collapse

Hard phase

Irrotational barotropic fluid with $p = \rho - 1$:

$$T = (p+\rho)u \otimes u + pg, \qquad u = \frac{V}{\|V\|}, \qquad V^\mu = -g^{\mu\nu}\partial_\nu \phi,$$

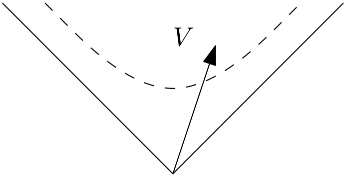
$$T = V \otimes V + \frac{1}{2}\left(\|V\|^2 - 1\right)g, \qquad \|V\|^2 = p + \rho = 2\rho - 1$$

The conservation law is the *linear* wave equation:

$$\Box_g \phi = 0$$

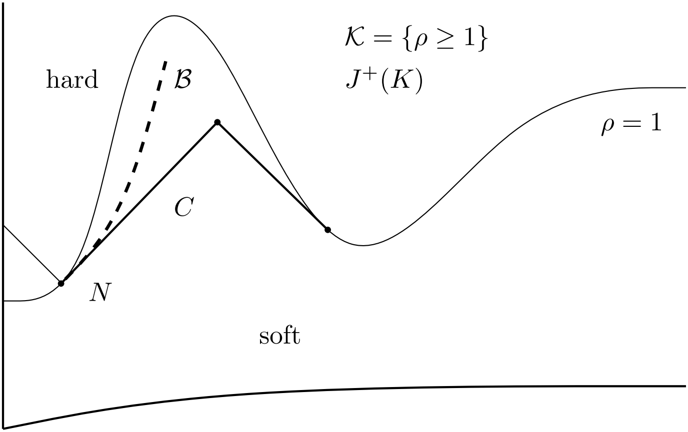
And the condition $\rho \geq 1$ is equivalent to

$$\|V\| = \|d\phi\| := \sqrt{-g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi} \geq 1$$



Gravitational Collapse

Phase transition from soft to hard



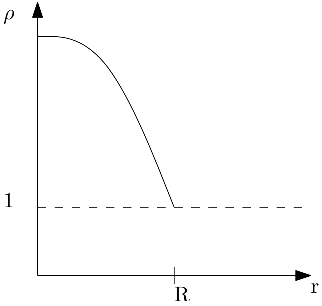
Christodoulou (1996): Analysis of the formation, continuation and termination of a free phase boundary

2. *Hard Stars*

Hard stars

Existence

There exists a 1-parameter family of *static* solutions to the Einstein-Euler equations in the "hard phase," surrounded by vacuum. In particular $\rho \geq 1$ for $0 \leq r \leq R$, the physical boundary condition $p = \rho - 1 = 0$ is satisfied at $r = R$, and the spacetime is isometric to Schwarzschild in the domain $r > R$.



Upper bound on the size of the star: $R \leq R_0 := \sqrt{3/8\pi}$.

Hard Stars

Existence

For *static* solutions the Einstein-Euler equations reduce to the TOLMAN-OPPENHEIMER-VOLKOFF equations (TOVE) of hydrostatic equilibrium,[4] a system of o.d.e.'s for (m, ρ)

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\frac{2\rho - 1}{r - 2m}\left(\frac{m}{r} + 4\pi r^2(\rho - 1)\right), \qquad \rho\big|_{\mathcal{B}} = 1$$

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \qquad m\big|_{\Gamma} = 0$$

with boundary conditions for m at the center, and ρ at the boundary.

Note: The density ρ is manifestly decreasing.[5]

Existence of solutions by standard o.d.e. arguments.

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[4] See ANDERSSON-BURTSCHER 2018 for stars of infinite extent.

[5] No trapped surfaces, $r > 2m$.

Hard Stars

Variations of mass subject to a constraint

- Conservation of mass M and total number of particles N:

$$p\big|_{\mathcal{B}} = 0 : \partial_\phi m\big|_{\mathcal{B}} = 0 : M[\phi] = M[0]$$
$$\mathrm{div}(nu) = 0 : N[\phi] = \int n = N[0]$$

  where n is the particle density given in the hard phase via $n = \sqrt{2\rho - 1}$.

- Variations of M that fix N:

$$\dot{r} := \frac{\mathrm{d}}{\mathrm{d}s} r(s,\chi)\Big|_{s=0},\ r(0,\chi) = r, \ldots \quad N(s,\chi) = N(0,\chi).$$

  For fixed N, M is only a function of r: $M = M[r]$.

Variational characterization of static hard stars[6]:

Let $r, \rho[r], \frac{\partial r}{\partial \phi} = 0$ be an initial configuration at $\{\phi = 0\}$ for a hard star with N particles and non-negative pressure which vanishes only at the boundary $p\big|_{\mathcal{B}} = 0$. Then r is a critical point of the mass functional

$$M[r] = \int_{\{\phi=0\}} 4\pi r^2 \rho \, \mathrm{d}r$$

under variations which preserve the total number of particles N, if and only if the associated density ρ solves the TOVE.

[6]C.f. Harrison, Thorne, Wakano and Wheeler, Gravitation theory and gravitational collapse (1965)

Proposition (Variational properties of small stars)

The above hard stars are local minima of the mass functional, for sufficiently small $R = r\big|_{\mathcal{B}} > 0$, *under variations that preserve* N. *Furthermore, the second variation of* $M[r]$ *for small hard stars is equivalent to the following energy:*

$$\ddot{M}[r_0] \sim \int_{\{\phi=0\}} \dot{r}^2 + r_0^2\left(\frac{\partial \dot{r}}{\partial r_0}\right)^2 + r_0^2\left(\frac{\partial \dot{r}}{\partial \phi}\right)^2 \mathrm{d}r_0$$

and it controls the first variation of the mass aspect function:

$$\left|\left(\frac{m}{r}\right)^{\cdot}\right| \leq C r_0^{\frac{1}{2}} \ddot{M}[r_0].$$

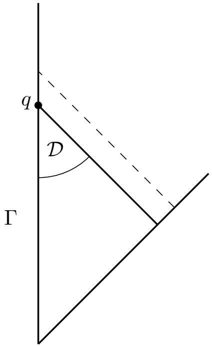
Hard Stars

Mass aspect function and continuation criteria

In spherical symmetry, the *mass aspect function* is defined by

$$\mu = \frac{m}{r}\,.$$

Recall for the static solutions $\frac{m}{r^3} = \mathcal{O}(1)$, because $\rho \geq 1$, however the functional does not provide pointwise control of ρ (lack of compactness).



For the *scalar field* model, Christodoulou showed that if

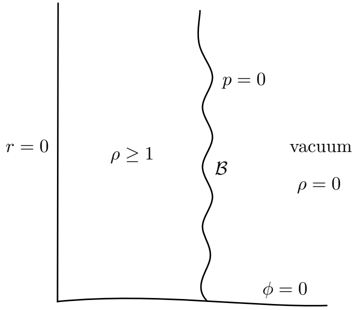
$$\sup_{\mathcal{D}} \mu < \epsilon$$

then the solution can be continued beyond $q \in \Gamma$.

3. *The orbital stability problem*

The orbital stability problem
Conjecture

We *conjecture* that *small* static hard stars are *orbitally stable*, within the hard phase of the two phase model.



The orbital stability problem

Free boundary problems

- Local well-posedness in spherical symmetry for the free phase boundary problem of the two phase model for Einstein-Euler by Christodoulou (1995-1996) and Christodoulou-Lisibach (2015).
- Local well-posedness for the free boundary problem of an incompressible fluid for the non-relativistic Euler equations, without symmetries, by Lindblad (2005).

The orbital stability problem

No asymptotic stability mechanism

- Typical stability mechanism: dispersion, emission of gravitational waves, e.g. stability of Minkowski (no body problem), black hole stability.
- For this problem, the density oscillations are confined, no emission of gravitational waves in spherical symmetry, the exterior is always Schwarzschild of mass $M = M_{\mathrm{star}}$.
- Periodic modes at the linear level; non-linear 'breathers'?[7]
- Newtonian case: 1-body problem for incompressible Euler-Poisson.[8]
- Semi-linear results of orbital stability without asymptotic stability via coercive conserved quantities. In the non-coercive case for Vlassov-Poisson: Lemou-Méhats-Raphaël 2011

[7] See e.g. Chodosh–Shlapentokh-Rothman 2015

[8] Bieri-Miao-Shahshahani-Wu 2015 showed ε^{-2} time interval of existence for ε-perturbations of round balls.

The orbital stability problem
AdS instability type scenario

- Similarities to Ads-scalar field: reflecting boundary condition $p\big|_{\mathcal{B}} = 0$, equation of motion $\Box_g \phi = 0$.
- Numerics for the instability of AdS as a solution to the Einstein-scalar field system in spherical symmetry: Bizon-Rostworoski 2011
- Recent instability proof of AdS for the spherically symmetric Einstein-null dust model:

Theorem (Moschidis '17)

There exists a family of perturbations of AdS which, under evolution by the Einstein–null dust system with reflecting boundary conditions on $\mathcal{I}$ and an inner mirror placed around $r = 0$, lead to trapped surface formation at late enough times.

- Conserved quantity: Renormalised Hawking mass on $\mathcal{I}$, $\tilde{m} = m - \frac{1}{6}\Lambda r^3$. However, it does not control $\frac{m}{r}$.

The orbital stability problem

Continuation criterion

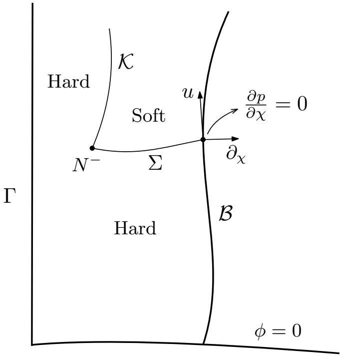
- Christodoulou (1993): The mass aspect function $\frac{m}{r}$ serves as a continuation criterion for the Einstein (massless) scalar field model.
- The hard phase in spherical symmetry reduces to a scalar field model with a positive cosmological constant.
- The second variation of the mass functional about the static hard stars controls the first variation of $\frac{m}{r}$:

$$\left|\left(\frac{m}{r}\right)^{\cdot}\right| \leq C r_0^{\frac{1}{2}} \ddot{M}[r_0].$$

The orbital stability problem

Other breakdown scenarios

- Discontinuities of the fluid, e.g., shocks.
- Degeneracy at the boundary: bubbling phenomenon, transition from hard to soft, see CHRISTODOULOU-LISIBACH (2015).



Future challenges

- Proof of the global-in-time orbital stability of small static hard stars. Numerical simulations?
- Construction of periodic solutions.
- Formation of black holes within the two phase model from the collapse of large dust clouds.
- Asymptotic stability outside spherical symmetry? This is relevant to the set up of the two body problem.[9]

[9] See CHRISTODOULOU's lecture at the IHP in Paris for the celebration of the 100th anniversary of GR, 19 November 2015.
See also MIAO-SHAHSHAHANI 2017 for a new study of the 2-body problem in the Newtonian case.