Motivation

- Deterministic control interpretation via “two persons repeated games” for a broad class of fully nonlinear equations: [1]
- Parabolic setting: the whole space case.

- Motivation: to adapt their approach to the Neumann problem in both settings. In the parabolic case we can study \( \{u - (f(x, t, Du, D^2u)) = 0, \text{ for } x \in \Omega \text{ and } t < T, \}
\]
\( \{Du(u(x, t)) = h(x), \text{ for } x \in \Gamma_0 \text{ and } t < T, \}
\]
\( u(0, T) = g(x) \text{ for } x \in \Gamma \}
\[ \text{for } x \in \Omega \text{ and } t \geq 0. \]

- The oblique and mixed type Dirichlet-Neumann boundary conditions can also be treated by this analysis.

The game

- Two players Mark and Helen in a repeated game.
- Let \( \alpha, \beta, \gamma \in [0, 1] \) satisfy some algebraic relations.

Rules of the game

Helen’s goal is to maximize her score at time \( T \), and Mark’s is to work against her. If at time \( t_j = t_0 = \alpha x^2 \), the position is \( x_j \) and Helen’s score is \( y_j \), then

1. Helen chooses a vector \( p_j \in \mathbb{R}^n \) and an \( N \times N \) symmetric matrix \( \Gamma_j \) restricted by
\[ \|p_j\| \leq \varepsilon \gamma, \|\Gamma_j\| \leq \varepsilon \gamma. \]

2. Taking Helen’s choice into account, Mark chooses the stock price \( x_{j+1} \) so as to degrade Helen’s outcome. Mark chooses an intermediate point \( \bar{x}_{j+1} = x_j + \Delta x_j \in \mathbb{R}^n \) such that
\[ \|\Delta x_j\| \leq \varepsilon^m. \]

This position \( \bar{x}_{j+1} \) determines the new position \( x_{j+1} = x_0 + \Delta x_j \in \mathbb{R}^n \) at time \( t_{j+1} \) by the rule
\[ x_{j+1} = \text{proj}_{\bar{x}_{j+1}}(p_j). \]

3. Helen’s score changes to
\[ y_{j+1} = y_{j} - p_j \cdot \Delta x_j - \frac{1}{2} \sum \Gamma_{j}(\bar{x}_{j+1}, \Delta x_j) - \frac{1}{2} f(t_j, x_j, p_j, \Gamma_j) + \|\Delta x_j\| \cdot h(x_j + \Delta x_j). \]

4. The clock steps forward to \( t_{j+1} = t_j + \varepsilon \gamma \); the process repeats, stopping when \( t_K = T \). At \( t_K = T \), Helen receives \( g(x_K) \).

Dynamic programming principle

Helen’s value function \( u^* \) is determined by:

- the final-time condition \( u^*(x, T) = g(x) \),
- and the dynamic programming principle:
\[ u^*(x, t) = \max_{p_j, \Gamma_j} \left\{ u^* (x + \Delta x_j, t_0) - p_j \cdot \Delta x_j - \frac{1}{2} \sum \Gamma_{j}(\bar{x}_{j+1}, \Delta x_j) - \frac{1}{2} f(t_j, x_j, p_j, \Gamma_j) + \|\Delta x_j\| \cdot h(x_j + \Delta x_j) \right\} \tag{2} \]
\[ \text{for } x \in \Omega, \text{ and } t \leq T. \]

Main result

**Theorem 1.** Consider the final-value problem (1). If the PDE has a comparison principle (for uniformly bounded solutions) then it follows that \( u^* \) converge locally uniformly to the unique viscosity solution of (1).

Viscosity solutions for the parabolic PDE

**Definition 2.** A real-valued lower-semicontinuous function \( u(x, t) \) defined on \( \Omega \times (0, T) \) is a viscosity supersolution (resp. sub-solution) of the final value problem (1) if
- for any \( (x_0, t_0) \) with \( x_0 \in \Omega \) and \( t_0 \leq t < T \) and any smooth \( \phi(x, t) \) such that \( u - \phi \) has a local minimum at \( x_0 \), we have
\[ \langle \partial_t \phi(x_0, t_0) - f(x_0, t_0, Du(x_0, t_0), D^2u(x_0, t_0)), \rangle \leq 0, \]
- for any \( (x_0, t_0) \) with \( x_0 \in \Omega \) and \( t_0 \leq t < T \) and any smooth \( \phi(x, t) \) such that \( u - \phi \) has a local maximum on \( \mathbb{R}^n \) at \( x_0 \), we have
\[ \langle (\partial_t \phi(x_0, t_0) - f(x_0, t_0, Du(x_0, t_0), D^2u(x_0, t_0)), \rangle \geq 0. \]

**Comparison principle**

When we construct some good substitution \( \psi \) and supersolution \( v \) of the PDE, we can compare \( u \) and \( v \).
- **Reversing inequality** to comparison principle:
- if \( v \) is a sub-solution and \( u \) is a supersolution then \( u \leq v \).
- If the PDE has a comparison principle:
- \( u \) is a viscosity solution of the PDE.

To determine the optimal choice for Helen on \( p \), consider the order- \( m \) optimization problem by neglecting the second order- \( n \) terms
\[ M = \max_{p} \min_{f \in F} \left\{ Dv - p \cdot \Delta x - \|\Delta x\| \cdot d \right\}. \]

If \( m > 0 \), Helen chooses \( p = Dv \) whereas if \( m = 0 \), Helen chooses
\[ p = -Dv \left( 1 - \frac{\|h\|}{\|\Delta x\|} \right). \]

A further analysis provides
\[ 0 \geq \|v^* - d \| \geq \|v^* - d \|_{C^{m, \alpha}} \geq \|v^* - d \|_{C^{m, \alpha}} \]
\[ \geq c^m \|v^* - d \|_{C^{m, \alpha}} \]
\[ \|v^* - d \|_{C^{m, \alpha}} \geq \|v^* - d \|_{C^{m, \alpha}}. \]

We distinguish two cases:
- if \( d \in \partial \Omega \), we get \( v^* - d \) bounded when \( d \in \partial \Omega \).
- if \( d \in \Omega \), we get \( v^* - d \) bounded when \( d \in \Omega \).

**Sketch of the rigorous proof**

**Landmark theorem of Barles and Souganidis [2] states:**

- “If a numerical scheme is monotone, stable, and consistent, then the associated ‘lower semi-relaxed limit’ is a viscosity supersolution and the associated ‘upper semi-relaxed limit’ is a viscosity sub-solution.”

**Characteristics**

- \( S_t \) is monotone, i.e. if \( \phi_1 \geq \phi_2 \), then
\[ S_t [x, \phi_2] \leq S_t [x, \phi_1]. \]

**Stability**

- Show that if the final-time data are uniformly bounded, then \( u^*(x, t) \) remains bounded when \( t \geq 0 \) (no blow-up).

**References**

