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An Invitation to Control Theory of Stochastic Distributed Parameter Systems

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Outline:

1. Introduction
2. Controllability for stochastic DPS
3. Optimal control for stochastic DPS

1. Introduction

◇ What is stochastic DPS and its control?

Systems governed by stochastic differential equations in infinite dimensions.

- Stochastic differential equations with delays;
- Stochastic PDEs;
- Random PDEs, i.e., PDEs with random parameters;
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Control: One hopes to change the system's dynamics, by means of a suitable way.

Two fundamental issues: **Controllability** and **Optimal Control**.

◇ Why should study control theory for stochastic DP-S?

Claim: For control theory, it is now stochastic DPS' turn! Why?

- Control Theory for ODE systems: Relatively mature, many classics.

L.S. Pontryagin: Maximum Principle;

R. Bellman: Dynamic Programming and HJB Equations;

R.E. Kalman: LQ Problem and Filter Theory.

- Control Theory for DPS: Many results (many many papers, many books), still quite active.

Pioneers: Yu V. Egorov, H. O. Fattorini, J.-L. Lions, D. L. Russell.....

Early books:

[1] J.-L. Lions. Optimal Control of Systems Governed by Partial Differential Equations. Springer-Verlag, 1971.

[2] R.F. Curtain and A.J. Pritchard. Infinite Dimensional Linear Systems Theory. Springer-Verlag, 1978.

“Recent” books:

[1] X. Li and J. Yong. Optimal Control Theory for Infinite-Dimensional Systems. Birkhäuser, 1995.

[2] J.-M. Coron. Control and Nonlinearity. American Mathematical Society, 2007.

- Control theory for stochastic systems in finite dimensions (i.e., **Stochastic ODEs**): Many works, closely related to mathematical finance.

Important works: A. Bensoussan, J.-M. Bismut, W. H. Fleming, H.J. Kushner, S. Peng.....

Classical books:

[1] W. H. Fleming and H. M. Soner. Controlled Markov Processes and Viscosity Solutions. Springer-Verlag, 1992.

[2] J. Yong and X. Zhou. Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer-Verlag, 1999.

Controllability theory is NOT well-developed.

- Control theory for stochastic DPS: Almost at its very beginning stage. **Still an ugly duckling!**

Very few works.

Only three books (The first two addressed mainly to some slightly different topics):

[1] A. Bashirov. **Partially Observable Linear Systems Under Dependent Noises**. Birkhäuser Verlag, 2003.

[2] P. S. Knopov and O. N. Deriyeva. **Estimation and Control Problems for Stochastic Partial Differential Equations**. Springer, 2013.

[3] Q. Lü and X. Zhang. **General Pontryagin-Type Stochastic Maximum Principle and Backward Stochastic Evolution Equations in Infinite Dimensions**. Springer, 2014.

The most general control system in the framework of classical physics.

The study of this field may provide some useful hints for that of quantum control systems.

It will eventually become a white swan!

◇ Why the study of control theory for stochastic DPS is difficult?

- Very few are known for stochastic PDEs.
- Both the formulation of stochastic control problems and the tools to solve them may differ considerably from their deterministic counterpart.

- One will meet substantial difficulties in the study of control problems for stochastic DPS.

Unlike the deterministic setting, the solution to an SDE/SPDE is usually non-differentiable with respect to the variable with noise.

The usual compactness embedding result fails to be true for the solution spaces related to SDEs/SPDEs.

The “time” in the stochastic setting is not reversible, even for stochastic hyperbolic equations.

Generally, stochastic control problems cannot be reduced to deterministic ones.

2. Controllability for stochastic DPS

◇ Controllability for stochastic ODEs

- The deterministic setting

Consider the following controlled (ODE) system:

$$\begin{cases} \frac{d}{dt}y = Ay + Bu, & t \in [0, T], \\ y(0) = y_0, \end{cases} \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $T > 0$. System (1) is said to be controllable on $(0, T)$ if for any $y_0, y_1 \in \mathbb{R}^n$, there exists a $u \in L^1(0, T; \mathbb{R}^m)$ such that $y(T) = y_1$.

Put

$$G_T = \int_0^T e^{At} B B^* e^{A^*t} dt.$$

Theorem: If the system (1) is controllable on $(0, T)$, then $\det G_T \neq 0$. Moreover, for any $y_0, y_1 \in \mathbb{R}^n$, the control

$$u^*(t) = -B^* e^{A^*(T-t)} G_T^{-1} (e^{AT} y_0 - y_1)$$

transfers y_0 to y_1 at time T .

Clearly, if (1) is controllable on $(0, T)$ (by means of L^1 -in time) controls), then the same controllability can be achieved by using analytic-(in time) controls. We shall see a completely different phenomenon in the simplest stochastic situation.

- The stochastic setting

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$: a complete filtered probability space on which a one dimensional standard Brownian motion $\{B(t)\}_{t \geq 0}$ is defined.

H : a Banach space, and write $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$.

$L_{\mathbb{F}}^2(0, T; H)$: the Banach space consisting of all H -valued \mathbb{F} -adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|_{L^2(0, T; H)}^2) < \infty$, with the canonical norm;

Similarly, $L_{\mathbb{F}}^\infty(0, T; H)$, $L_{\mathbb{F}}^2(\Omega; C([0, T]; H))$, etc.

The filtration \mathbb{F} plays a crucial role, and it represents the “information” that one has at each time t . For SDE (in the Itô sense), **one needs to use adapted processes $X(\cdot)$** , i.e., $\forall t$, the r.v. $X(t)$ is \mathcal{F}_t -measurable.

Consider a one-dimensional controlled stochastic differential equation:

$$dx(t) = [bx(t) + u(t)]dt + \sigma dB(t), \quad (2)$$

with b and σ being given constants. We say that the system (2) is **exactly controllable** if for any $x_0 \in \mathbb{R}$ and $x_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R})$, there exists a control $u(\cdot) \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}))$ such that the corresponding solution $x(\cdot)$ satisfies $x(0) = x_0$ and $x(T) = x_T$.

Q. Lü, J. Yong and X. Zhang (JEMS, 2012) showed that the system (2) is exactly controllable at any time $T > 0$ (by means of $L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}))$ -controls).

On the other hand, surprisingly, in virtue of a result by S. Peng (Progr. Natur. Sci., 1994), **the system (2) is NOT exactly controllable** if one restricts to use admissible controls $u(\cdot)$ in $L^2_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}))$!

Q. Lü, J. Yong and X. Zhang (JEMS, 2012) showed that the system (2) is **NOT exactly controllable, either** provided that one uses admissible controls $u(\cdot)$ in $L^p_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}))$ for any $p \in (1, \infty]$. This leads to a corrected formulation for the exact controllability of stochastic differential equations, as presented below.

- Definition of exact controllability

Consider a linear stochastic differential equation:

$$\begin{cases} dy = (Ay + Bu)dt + (Cy + Du)dB(t), & t \in [0, T], \\ y(0) = y_0 \in \mathbb{R}^n, \end{cases} \quad (3)$$

where $A, C \in \mathbb{R}^{n \times n}$ and $B, D \in \mathbb{R}^{n \times m}$. No universally accepted notion for stochastic controllability!

Definition: System (3) is said to be exactly controllable if for any $y_0 \in \mathbb{R}^n$ and $y_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$, \exists a control $u(\cdot) \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}^m))$ such that $Du(\cdot, \omega) \in L^2(0, T; \mathbb{R}^n)$, a.e. $\omega \in \Omega$ and the corresponding solution $y(\cdot)$ to (3) satisfies $y(T) = y_T$.

Though the above definition seems to be a reasonable notion for exact controllability of stochastic differential equations, a complete study on this problem is still under consideration and it does not seem to be easy.

When $n > 1$, the controllability for the linear system (5) is in general unclear.

Compared to the deterministic case, the controllability/ observability for stochastic differential equations is at its “enfant” stage.

- How about the null controllability?

We consider the following 2 – d system:

$$\begin{cases} dx = ydt + \varepsilon ydB(t), & t \in [0, T] \\ dy = udt, & t \in [0, T], \\ (x(0), y(0)) = (x_0, y_0) \in \mathbb{R}^2, \end{cases} \quad (4)$$

where $u(\cdot) \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathbb{R}))$ is the control variable, ε is a parameter.

When $\varepsilon = 0$, (4) is null controllable.

When $\varepsilon \neq 0$ (no matter how small it is), (4) is NOT null controllable.

This indicates: NO hope to establish a Kalman-type rank condition for null controllability of stochastic ODEs, even for two dimensions.

◇ Controllability for stochastic parabolic equations

- Null controllability of stochastic parabolic equations with two controls:

$$\left\{ \begin{array}{l} dy - \sum_{i,j=1}^n (a^{ij} y_{x_i})_{x_j} dt = [\langle \alpha, \nabla y \rangle + \beta y + \chi_{G_0} \gamma] dt \\ \quad + (qy + \Gamma) dB(t) \quad \text{in } Q \equiv (0, T) \times G, \\ y = 0 \quad \text{on } \Sigma \equiv (0, T) \times \partial G, \\ y(0) = y_0 \quad \text{in } G (\subset \mathbb{R}^n). \end{array} \right. \quad (5)$$

The null controllability of (5): For any given $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, one can find a control $(\gamma, \Gamma) \in L^2_{\mathbb{F}}(0, T; L^2(G_0)) \times L^2_{\mathbb{F}}(0, T; L^2(G))$ such that the solution y to (5) satisfying $y(T) = 0$ in G , P -a.s.

By means of the classical duality argument, the null controllability of (5) may be reduced to an observability estimate for the following backward stochastic parabolic equation:

$$\left\{ \begin{array}{ll} dz + \sum_{i,j=1}^n (a^{ij} z_{x_i})_{x_j} dt \\ \quad = [\langle a, \nabla z \rangle + bz + cZ] dt + Z dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = z_T & \text{in } G, \end{array} \right. \quad (6)$$

i.e., to find a constant $C > 0$ such that all solutions to (6) satisfy

$$\begin{aligned} |z(0)|_{L^2(\Omega, \mathcal{F}_0, P; L^2(G))} &\leq C \left(|z|_{L^2_{\mathcal{F}}(0, T; L^2(G_0))} + |Z|_{L^2_{\mathcal{F}}(0, T; L^2(G))} \right), \\ \forall z_T &\in L^2(\Omega, \mathcal{F}_T, P; L^2(G)). \end{aligned} \quad (7)$$

S. Tang and X. Zhang (SICON, 2009) proved (7), by means of the following identity for a stochastic parabolic-like operator:

Theorem Let $b^{ij} = b^{ji} \in C^{1,2}$ ($i, j = 1, 2, \dots, m$), $\ell \in C^{1,3}$, u be a $C^2(\mathbb{R}^m)$ -valued semimartingale. Set $\theta = e^\ell$ and $v = \theta u$. Then, for a suitable function \mathcal{M} ,

$$\begin{aligned}
 & 2 \int_0^T \theta \left[- \sum_{i,j=1}^m (b^{ij} v_{x_i})_{x_j} + Av \right] \left[du - \sum_{i,j=1}^m (b^{ij} u_{x_i})_{x_j} dt \right] \\
 & + \int_0^T \sum_{j=1}^m \left[\dots \right]_{x_j} dt + 2 \int_0^T \sum_{i,j=1}^m (b^{ij} v_{x_i} dv)_{x_j} \\
 & = \int_0^T \sum_{i,j=1}^m \left\{ \dots \right\} v_{x_i} v_{x_j} dt + \int_0^T (\dots) v^2 dt + \dots \\
 & + (\dots) \Big|_0^T - \int_0^T \theta^2 \sum_{i,j=1}^m b^{ij} du_{x_i} du_{x_j} - \int_0^T \theta^2 \mathcal{M}(du)^2.
 \end{aligned}$$

- Controllability of stochastic parabolic equations with one control:

$$\begin{cases} dy - \sum_{i,j=1}^n (a^{ij}(x)y_{x_i})_{x_j} dt = q(t, x)y dB(t) + \chi_{G_0} u dt & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0 & \text{in } G, \end{cases} \quad (8)$$

where $q(\cdot, \cdot) \in L_{\mathbb{F}}^{\infty}(0, T; L^{\infty}(G))$, $y_0 \in L^2(\Omega, \mathcal{F}_0, P; L^2(G))$, the control u belongs to $L_{\mathbb{F}}^2(0, T; L^2(G_0))$.

The only known controllability result for (16) is for the special case that $q(t, x) \equiv q(t)$ (Q. Lü, JFA, 2011).

By the duality method, the null controllability of (16) is equivalent to an observability estimate for the following backward stochastic parabolic equation:

$$\left\{ \begin{array}{ll} dz + \sum_{i,j=1}^n (a^{ij} z_{x_i})_{x_j} dt = -q(t, x)Z dt + Z dB(t) & \text{in } Q, \\ z = 0 & \text{on } \Sigma, \\ z(T) = z_T & \text{in } G, \end{array} \right. \quad (9)$$

i.e., to find a constant $C > 0$ such that all solutions to (9) satisfy, for any $z_T \in L^2(\Omega, \mathcal{F}_T, P; L^2(G))$,

$$|z(0)|_{L^2(\Omega, \mathcal{F}_0, P; L^2(G))} \leq C |z|_{L^2_{\mathcal{F}}(0, T; L^2(G_0))}. \quad (10)$$

It is an unsolved problem to prove the observability estimate (10), or the null controllability of (16), even for the one space dimension.

• Controllability of some coupled stochastic parabolic systems by one control:

$$\left\{ \begin{array}{ll} dy - \Delta y dt = [a(t)y + b(t)z + \chi_{G_0}u] dt \\ \quad \quad \quad + [c(t)y + f(t)z] dB(t) & \text{in } Q, \\ dz - \Delta z dt = [d(t)y + e(t)z] dt \\ \quad \quad \quad + [g(t)z + h(t)y] dB(t) & \text{in } Q, \\ y = 0, z = 0 & \text{on } \Sigma, \\ y(0) = y_0, z(0) = z_0 & \text{in } G, \end{array} \right. \quad (11)$$

where u is the control variable, (y, z) is the state variable, $(y_0, z_0) \in (L^2(\Omega, \mathcal{F}_0, \mathcal{P}; L^2(G)))^2$ is any given initial value. Moreover, $a, b, c, d, e, f, g, h \in L_{\mathcal{F}}^\infty(0, T; \mathbb{R})$ are given coefficients.

Suppose that the coefficients d and h satisfy:

(H_1) There exist a positive constant d_0 and an interval $I \subset [0, T]$ such that $d(t) \geq d_0$ or $d(t) \leq -d_0$ on I ;

(H_2) There exists an interval $\tilde{I} \subseteq I$ such that $h(t) = 0$ on \tilde{I} .

Theorem (X. Liu, SICON, 2014): Under the conditions (\mathbf{H}_1) and (\mathbf{H}_2) , the system (11) is null controllable.

It deserves to emphasize that **neither (\mathbf{H}_1) nor (\mathbf{H}_2) in the above Theorem could be dropped**, as shown by X. Liu (SICON, 2014).

Especially, by virtue of some counterexamples by X. Liu (SICON, 2014), it is found that the controllability of the coupled system (11) is **not robust with respect to the coupling coefficient** in the diffusion terms.

Indeed, when this coefficient equals to zero on $[0, T]$, the system is null controllable. However, **if it is a nonzero bounded function, no matter how small this function is, the corresponding system is uncontrollable any more!**

This indicates that **the Carleman-type estimates cannot be used to** study the controllability of the coupled stochastic parabolic system (11). Note however that, one can use the Carleman estimate to prove the controllability of the deterministic version of (11).

In view of S. Tang and X. Zhang (SICON, 2009) (addressed to the null controllability of stochastic parabolic equations with two controls), it seems that (12) is exactly controllable, at least when $G_1 = G_2 = G$.

Under some geometric conditions on G_1 , Q. Lü (JDE, 2013) showed the exact controllability of stochastic Schrödinger equations; Q. Lü (SICON, 2014) showed the exact controllability of stochastic transport equations.

Surprisingly, (12) is NOT exactly controllable even if $G_1 = G_2 = G$, i.e., the controls are active everywhere in both drift and diffusion terms!

Why? The point is the “indirect control” on y . Rewrite (12) as

$$\left\{ \begin{array}{ll} y_t = z & \text{in } Q, \\ dz - \sum_{i,j=1}^n (a^{ij} y_{x_i})_{x_j} dt = \gamma dt + \Gamma dB(t) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad z(0) = y_1 & \text{in } G. \end{array} \right. \quad (13)$$

In (13), γ and Γ control perfectly the evolution of z . Nevertheless, the “control variable” for the first equation in (13) is z , which is always continuous (rather than $L^1_{\mathbb{F}}$) in time, and therefore, it is NOT enough to control y (for the same reason as that in the stochastic ODE setting)!

- Approximate controllability of stochastic hyperbolic equations.

J. U. Kim (AMO, 2004) proved the approximate controllability of (12) with $G_1 = G$ and $G_2 = \emptyset$. Kim's result can be easily obtained because the unique continuation of the dual system of (12) is obvious when $G_1 = G$ (i.e., controlling everywhere).

Nothing is known when $G_1 \neq G$.

The classical “compactness-uniqueness argument” (for the wave equation) fails for (12)!

- Null controllability of stochastic hyperbolic equations.

By means of the classical duality argument, the null controllability of (12) may be reduced to the observability estimates for the following backward stochastic hyperbolic equations:

$$\left\{ \begin{array}{ll} dp = -qdt + PdB(t) & \text{in } Q, \\ dq + \sum_{i,j=1}^n (a^{ij} p_{x_i})_{x_j} dt & \\ = (ap + bP + cQ)dt + QdB(t) & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(T) = p_T, \quad q(T) = q_T & \text{in } G. \end{array} \right. \quad (14)$$

That is, to find a constant $C > 0$ such that all solutions to (6) satisfy

$$\begin{aligned} & |p(0)|_{L^2(\Omega, \mathcal{F}_0, P; L^2(G))} + |q(0)|_{L^2(\Omega, \mathcal{F}_0, P; H^{-1}(G))} \\ & \leq C \left(|p|_{L^2_{\mathcal{F}}(0, T; L^2(G_1))} + |P|_{L^2_{\mathcal{F}}(0, T; L^2(G_2))} \right), \end{aligned} \quad (15)$$

$$\forall (p_T, q_T) \in L^2(\Omega, \mathcal{F}_T, P; L^2(G) \times H^{-1}(G)).$$

One of the difficulty: One cannot reduce (14) to an equation of second order, as one does in the deterministic setting.

Nothing is published for the estimate (15), even when $G_1 = G_2 = G!$

Observability estimate for forward stochastic hyperbolic equations and applications in **Inverse Problems**: X. Zhang (SIMA, 2007), Q. Lü (IP, 2013), Q. Lü and X. Zhang (CPAM, 2015), G. Yuan (IP, 2015).

- An inverse stochastic hyperbolic problem with three unknowns.

Consider a stochastic hyperbolic equation:

$$\left\{ \begin{array}{l} dz_t - \sum_{i,j=1}^n (a^{ij} z_{x_i})_{x_j} dt \\ = (b_1 z_t + b_2 \cdot \nabla z + b_3 z) dt + (b_4 z + g) dB(t) \text{ in } Q, \\ z = 0 \text{ on } \Sigma, \\ z(0) = z_0, z_t(0) = z_1 \text{ in } G. \end{array} \right. \quad (16)$$

Here, $z_t = \frac{\partial z}{\partial t}$; b_k ($k = 1, \dots, 4$) are known; while $(z_0, z_1) \in L^2(\Omega, \mathcal{F}_0, P; H_0^1(G) \times L^2(G))$ and $g \in L^2_{\mathcal{F}}(0, T; L^2(G))$ are unknown. Physically, g stands for the intensity of a random force.

In (16), the random force $\int_0^t g dB$ is assumed to cause the random vibration starting from some initial state (z_0, z_1) . We expect to determine the unknown random force intensity g , the unknown initial displacement z_0 and the initial velocity z_1 from the (partial) boundary observation $\frac{\partial z}{\partial \nu} \Big|_{(0,T) \times \Gamma_0}$ and the measurement on the terminal displacement $z(T)$.

Theorem. (Q. Lü and X. Zhang, CPAM, 2015) *Let $(a_{ij}(\cdot))$, b_k ($k = 1, 2, 3, 4$), T , G and Γ_0 satisfy suitable conditions. Assume that the solution z to (16) satisfies*

$$\frac{\partial z}{\partial \nu} \Big|_{(0,T) \times \Gamma_0} = 0, \quad z(T) = 0 \text{ in } G, \quad P - a.s.$$

Then $g = 0$ in Q and $z_0 = z_1 = 0$ in G , P -a.s.

However the same conclusion as that in the above theorem does not hold true **even for the deterministic wave equation**. Indeed, we choose any $y \in C_0^\infty(Q)$ so that it does not vanish in some subdomain of Q . Putting $f = y_{tt} - \Delta y$, we see that y solves the following wave equation

$$\begin{cases} y_{tt} - \Delta y = f & \text{in } Q, \\ y = 0, & \text{on } \Sigma, \\ y(0) = 0, \quad y_t(0) = 0 & \text{in } G. \end{cases}$$

It is easy to show that $y(T) = 0$ in G and $\frac{\partial y}{\partial \nu} = 0$ on Σ . However, f does not vanish in Q ! This counterexample shows that **the formulation of the stochastic inverse problem may differ considerably from its deterministic counterpart**.

◇ Full of open problems for controllability of stochastic DPS.

- **Stochastic Return Method**: Remains to be done;
- **Stochastic Micro-local Analysis Method**: Also remains to be done. For some very preliminary work, see “X. Liu and X. Zhang, *The theory of stochastic pseudo-differential operators and its applications, I*, Preprint”;
- Unlike the deterministic setting, the duality argument does not work well. Any direct method?
-

3. Optimal control for stochastic DPS

◇ Optimal control problems for stochastic DPS

Consider the following controlled stochastic evolution equation

$$\begin{cases} dx(t) = [Ax(t) + a(t, x(t), u(t))] dt \\ \quad + b(t, x(t), u(t)) dB(t), \quad t \in (0, T], \\ x(0) = x_0, \end{cases} \quad (17)$$

where A is **an unbounded linear operator** (on a Hilbert space H), generating a C_0 -semigroup. Put

$$\mathcal{U}[0, T] \triangleq \left\{ u(\cdot) : [0, T] \rightarrow U \mid u(\cdot) \text{ is } \mathbb{F}\text{-adapted} \right\}.$$

Define a cost functional $\mathcal{J}(\cdot)$ as follows:

$$\mathcal{J}(u(\cdot)) \triangleq \mathbb{E} \left[\int_0^T g(t, x(t), u(t)) dt + h(x(T)) \right].$$

We consider the following optimal control problem:

Problem (P): Find $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ such that

$$\mathcal{J}(\bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[0, T]} \mathcal{J}(u(\cdot)). \quad (18)$$

Any $\bar{u}(\cdot) \in \mathcal{U}[0, T]$ satisfying (18) is called an *optimal control*, the corresponding $\bar{x}(\cdot) \equiv x(\cdot; \bar{u}(\cdot))$ and $(\bar{x}(\cdot), \bar{u}(\cdot))$ are called an *optimal state process/trajectory* and *optimal pair*, respectively.

◇ Our goal is to give a Pontryagin-type maximum principle for the above general stochastic optimal control problem.

- The case when $\dim H < \infty$ is now well-understood, see S. Peng (SICON, 1990).

- The case when the control does NOT appear in the diffusion term or the control set is convex: A. Bensoussan (J. Franklin Inst., 1983), Y. Hu and S. Peng (Stoch. Stoch. Rep., 1990), etc.

- The case when the control appears in the diffusion term and the control set is nonconvex: X.Y. Zhou (SICON, 1993) addressing the linear problem, and S. Tang and X. Li (LNPAM, 1994) for the problem with very special data.

◇ **Main difficulty:** How to define the solution to the following operator-valued backward stochastic evolution equation (BSEE)?

$$\begin{cases} dP = -(A^* + J^*(t))Pdt - P(A + J(t))dt - K^*PKdt \\ \quad - (K^*Q + QK)dt + Fdt + QdB(t) \quad \text{in } [0, T), \\ P(T) = P_T. \end{cases} \quad (19)$$

Here $F \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))$, $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))$, and $J, K \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))$.

• When $H = \mathbb{R}^n$, an $\mathbb{R}^{n \times n}$ (matrix)-valued equation can be regarded as an \mathbb{R}^{n^2} (vector)-valued equation.

- When $\dim H = \infty$, $\mathcal{L}(H)$ (with the uniform operator topology) is still a Banach space. Nevertheless, it is neither reflexive nor separable even if H itself is separable.
- There exist no satisfactory stochastic integration/evolution equation theories in general Banach spaces, say how to define the Itô integral $\int_0^T Q(t)dB(t)$ (for an operator-valued process $Q(\cdot)$)? The existing result on stochastic integration/evolution equation in UMD Banach spaces does not fit the present case because, if a Banach space is UMD, then it is reflexive.
- We employ the **Stochastic Transposition Method** introduced by Q. Lü and X. Zhang (JDE, 2013).

◇ Stochastic Transposition Method.

- The classical transposition method for **deterministic non-homogeneous boundary value problems** (J.-L. Lions and E. Magenes, 1972).
- The **main idea** in the above method: To interpret the solution to a **less understood equation** by means of another **well understood one**.
- By the above idea, we can establish the well-posedness of **vector-valued** (or more precisely, Hilbert space-valued) BSEE with general filtration, without using the Martingale Representation Theorem (Q. Lü and X. Zhang, 2014).

◇ To solve the operator-valued BSEE (19), we need another idea from Distribution Theory, i.e., to transfer the differentiation operation to the test functions. Here, we transfer the Itô integral operation to two test equations.

More precisely, we introduce the following two stochastic differential equation:

$$\begin{cases} dx_1 = (A + J)x_1 ds + u_1 ds + Kx_1 dB + v_1 dB & \text{in } (t, T], \\ x_1(t) = \xi_1, \end{cases} \quad (20)$$

$$\begin{cases} dx_2 = (A + J)x_2 ds + u_2 ds + Kx_2 dB + v_2 dB & \text{in } (t, T], \\ x_2(t) = \xi_2. \end{cases} \quad (21)$$

Here $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1, u_2 \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ and $v_1, v_2 \in L^4_{\mathbb{F}}(t, T; L^4(\Omega; H))$.

Definition. We call $(P(\cdot), Q(\cdot)) \in D_{\mathbb{F},w}([0, T]; L^2(\Omega; \mathcal{L}(H))) \times L^2_{\mathbb{F},w}(0, T; L^2(\Omega; \mathcal{L}(H)))$ a transposition solution to (19) if for any $t \in [0, T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^4_{\mathbb{F}}(t, T; L^4(\Omega; H))$, it holds that

$$\begin{aligned}
& \mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s) x_1(s), x_2(s) \rangle_H ds \\
&= \mathbb{E} \langle P(t) \xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P(s) u_1(s), x_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle P(s) x_1(s), u_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) K(s) x_1(s), v_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle P(s) v_1(s), K x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s) v_1(s), v_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle Q(s) v_1(s), x_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle Q(s) x_1(s), v_2(s) \rangle_H ds.
\end{aligned}$$

Denote by $\mathcal{L}_2(H)$ the set of the Hilbert-Schmidt operators on H .

Theorem. (Q. Lü and X. Zhang, 2014) *Assume that H is a separable Hilbert space and $L^p_{\mathcal{F}_T}(\Omega)$ ($1 \leq p < \infty$) is a separable Banach space. Then, for any $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}_2(H))$, $F \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}_2(H)))$ and $J, K \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))$, the equation (19) admits one and only one transposition solution (P, Q) with the regularity $(P(\cdot), Q(\cdot)) \in D_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{L}_2(H))) \times L^2_{\mathbb{F}}(0, T; \mathcal{L}_2(H))$. Furthermore,*

$$\begin{aligned} & |(P, Q)|_{D_{\mathbb{F}}([0, T]; L^2(\Omega; \mathcal{L}_2(H))) \times L^2_{\mathbb{F}}(0, T; \mathcal{L}_2(H))} \\ & \leq C \left[|F|_{L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}_2(H)))} + |P_T|_{L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}_2(H))} \right]. \end{aligned} \quad (22)$$

- The above Theorem indicates that, in some sense, our definition of transposition solution is a reasonable notion for the solution to (19).
- Unfortunately, we are unable to prove the existence of transposition solution to (19) in the general case.
- In Q. Lü and X. Zhang (2014), we introduced a weaker version of solution, i.e., **relaxed transposition solution** (to (19)), which looks awkward but it suffices to establish the Pontryagin-type stochastic maximum principle for Problem (P) in the general setting.

Definition. We call $(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)}) \in D_{\mathbb{F}, w}([0, T]; L^{\frac{4}{3}}(\Omega; \mathcal{L}(H))) \times \mathcal{Q}[0, T]$ a relaxed transposition solution to (19) if for any $t \in [0, T]$, $\xi_1, \xi_2 \in L^4_{\mathcal{F}_t}(\Omega; H)$, $u_1(\cdot), u_2(\cdot) \in L^2_{\mathbb{F}}(t, T; L^4(\Omega; H))$ and $v_1(\cdot), v_2(\cdot) \in L^4_{\mathbb{F}}(t, T; L^4(\Omega; H))$, it holds that

$$\begin{aligned}
& \mathbb{E} \langle P_T x_1(T), x_2(T) \rangle_H - \mathbb{E} \int_t^T \langle F(s)x_1(s), x_2(s) \rangle_H ds \\
&= \mathbb{E} \langle P(t)\xi_1, \xi_2 \rangle_H + \mathbb{E} \int_t^T \langle P(s)u_1(s), x_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle P(s)x_1(s), u_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)K(s)x_1(s), v_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle P(s)v_1(s), Kx_2(s) \rangle_H ds + \mathbb{E} \int_t^T \langle P(s)v_1(s), v_2(s) \rangle_H ds \\
&+ \mathbb{E} \int_t^T \langle v_1(s), \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s) \rangle_H ds + \mathbb{E} \int_t^T \langle Q^{(t)}(\xi_1, u_1, v_1)(s), v_2(s) \rangle_H ds.
\end{aligned}$$

- It is easy to see that, if $(P(\cdot), Q(\cdot))$ is a transposition solution to (19), then one can find a relaxed transposition solution $(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)})$ to the same equation (from $(P(\cdot), Q(\cdot))$). Indeed, they are related by

$$Q(s)x_1(s) = Q^{(t)}(\xi_1, u_1, v_1)(s),$$

$$Q(s)^*x_2(s) = \widehat{Q}^{(t)}(\xi_2, u_2, v_2)(s).$$

This means that, we know only the action of $Q(s)$ (or $Q(s)^*$) on the solution processes $x_1(s)$ (or $x_2(s)$).

- However, it is unclear how to obtain a transposition solution $(P(\cdot), Q(\cdot))$ to (19) by means of its relaxed transposition solution $(P(\cdot), Q^{(\cdot)}, \widehat{Q}^{(\cdot)})$. It seems that this is possible but we cannot do it at this moment.

- Well-posedness result for the equation (19) in the sense of relaxed transposition solution:

Theorem. (Q. Lü and X. Zhang, 2014) *Assume that H is a separable Hilbert space, and $L^p_{\mathcal{F}_T}(\Omega; \mathbb{C})$ ($1 \leq p < \infty$) is a separable Banach space. Then, for any $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))$, $F \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))$ and $J, K \in L^4_{\mathbb{F}}(0, T; L^\infty(\Omega; \mathcal{L}(H)))$, the equation (19) admits one and only one relaxed transposition solution $(P(\cdot), Q(\cdot), \widehat{Q}(\cdot))$. Furthermore,*

$$\begin{aligned}
& \|P\|_{\mathcal{L}(L^2_{\mathbb{F}}(0, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(0, T; L^{\frac{4}{3}}(\Omega; H)))} \\
& + \sup_{t \in [0, T]} \left\| (Q^{(t)}, \widehat{Q}^{(t)}) \right\|_{\left(\mathcal{L}(L^4_{\mathcal{F}_t}(\Omega; H) \times L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)) \times L^2_{\mathbb{F}}(t, T; L^4(\Omega; H)), L^2_{\mathbb{F}}(t, T; L^{\frac{4}{3}}(\Omega; H)) \right)}^2 \\
& \leq C \left[|F|_{L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))} + |P_T|_{L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))} \right].
\end{aligned} \tag{23}$$

- The relaxed transposition solution works well for Pontryagin-type stochastic maximum principle because, as its finite-dimensional counterpart, the “term $Q(\cdot)$ ” does not appear in the optimality condition.

Further application of relaxed transposition solution: Since the action of $Q(s)$ (or $Q(s)^*$) on the solution processes is known, it can be employed to derive an integral-type second order necessary optimality condition.

- Nevertheless, it is still quite interesting to establish the well-posedness of (19) in the sense of transposition solution.

Indeed, sometimes one does need $Q(\cdot)$, say the feedback control of general LQ problems, the pointwise higher-order necessary optimality conditions, etc.

It is an unsolved problem to prove the existence of transposition solution to (19), even for the following special case:

$$\begin{cases} dP = -A^* P dt - P A dt + F dt + Q dB(t) & \text{in } [0, T), \\ P(T) = P_T. \end{cases} \quad (24)$$

Here $F \in L^1_{\mathbb{F}}(0, T; L^2(\Omega; \mathcal{L}(H)))$, $P_T \in L^2_{\mathcal{F}_T}(\Omega; \mathcal{L}(H))$.

The same can be said for (24) even when $A = -\Delta$, the Laplacian with homogenous Dirichlet boundary condition, again even for the one space dimension.

Happy Birthday,
Jean-Michel!

福如东海(Fu Ru Dong Hai) (May your happiness
be as immense as the East Sea)

寿比南山(Shou Bi Nan Shan) (Live as long as the
Zhongnan Mountains)!