On the Control of some Galpern-Sobolev Equations

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Outline

- Introduction
  - Barenblatt-Zheltov-Kochina equation
  - Benjamin-Bona-Mahony equation
- Controllability
  - Fixed control domain: lack of null controllability
  - A remedy
- Final Comments

Sobolev-Galpern equations

A partial differential equation is of Sobolev-Galpern type if the highest order terms contain derivatives in both space and time coordinates.

A typical example is

$$\mathcal{M}\partial_t u + \mathcal{L}u = f$$

with $\mathcal{L}$ and $\mathcal{M}$ are linear partial differential operators in the spatial variable of order $2m$ and $l \leq 2m$, respectively (independent of t).

- seepage of fluids through fissured rocks;
- unsteady flows of second-order fluids;
- consolidation;
- surface waves of long wavelength in liquids
- …

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We will focus on the following equations:

- *Barenblatt-Zheltov-Kochina equation*

$$u_t - \Delta u_t - \Delta u = f$$

- *Linearized Benjamin-Bona-Mahony equation*

$$u_t - u_{xxt} + u_x = f$$

Barenblatt-Zheltov-Kochina equation

We are interested in the null controllability of the BZK equation:

$$\left|\begin{array}{lll} u_t - \Delta u_t - \Delta u = f 1_\omega & \text{in} & Q := (0,T) \times \Omega, \\ u = 0 & \text{on} & \Sigma := (0,T) \times \partial\Omega, \\ u(0) = u_0 & \text{in} & \Omega, \end{array}\right.$$

i.e, given u_0, find f such that the solution of BZK satisfies:

$$u(T) = 0.$$

Here, $\Omega \subset \mathbb{R}^{\mathbb{N}}$ and $\omega \subset \Omega$ is a nonempty subset.

Some properties of BZK

$$\left|\begin{array}{lll} u_t - \Delta u_t - \Delta u = 0 & \text{in} & Q := (0,T) \times \Omega, \\ u = 0 & \text{on} & \Sigma := (0,T) \times \partial\Omega, \\ u(0) = u_0 & \text{in} & \Omega, \end{array}\right.$$

- In space, the solution is just as smooth as the initial data allow it to be;
- The solution is very regular in time;
- Null controllability implies exact controllability.

A nice decomposition

The BZK equation can be decomposed as

$$\left|\begin{array}{lll} u - \Delta u = v & \text{in} & Q, \\ v_t + v - u = f 1_\omega & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ v(0) = u_0 - \Delta u_0 & \text{in} & \Omega. \end{array}\right.$$

We have:

$$v(T) = 0 \text{ iff } u(T) = 0.$$

Observability inequality

The null controllability for BZK system is equivalent to the observability inequality for the adjoint solutions, i.e

$$\|\psi(\cdot,0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |\psi|^2,$$

where

$$\left| \begin{array}{lll} \varphi - \Delta\varphi = \psi & \text{in} & Q, \\ -\psi_t + \psi = \varphi & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \psi(T) = \psi_T & \text{in} & \Omega. \end{array} \right.$$

Lack of controllability with fixed control

Remark

For a given (fixed) $\omega \subset \Omega$, null controllability fails for system BZK.

This is due to:

- the spectrum $(I-\Delta)^{-1}\Delta$ is given by $\{\frac{\mu_j}{1+\mu_j} : \mu_j \in \sigma(-\Delta)\}$;
- highly localized solutions (Gaussian beam);

What can we do?

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What can we do?

Remedy: Moving Control

Make the control in the second equation of BZK move on time, i.e. $\omega = \omega(t)$.

The set $\omega(t)$ covers the whole domain Ω in its motion.

This strategy has been previously used for other models:

L. Rosier, B.-Y. Zhang, Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain, J. Differential Equations 254 (2013), 141–178.

P. Martin, L. Rosier, P. Rouchon, Null Controllability of the Structurally Damped Wave Equation with Moving Control, SIAM J. Control Optim., 51 (1) (2013), 660–684.

F. W. Chaves-Silva, L. Rosier, and E. Zuazua, Null Controllability of a System of Viscoelasticity with Moving Control, J. Math. Pures Appl., 101 (9) (2014), 198–222.

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BZK with Moving Control

With moving controls, BZK equation and BZK system read

$$\left|\begin{array}{lll} u_t - \Delta u - \Delta u_t = f 1_{\omega(t)} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ u(\cdot, 0) = u_0 & \text{in} & \Omega \end{array}\right.$$

and

$$\left|\begin{array}{lll} u - \Delta u = v & \text{in} & Q, \\ v_t + v - u = f 1_{\omega(t)} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ v(\cdot, 0) = v_0 & \text{in} & \Omega, \end{array}\right.$$

respectively.

And, still, we want to find a control f such that $u(T) = v(T) = 0$.

1D case

For the $1D$ case, using moment method, Q. Tao et al.[1], showed the null controllability of BZK with periodic boundary condition.

[1] Q. Tao, H. Gao, Z. Yao, Null controllability of a pseudo-parabolic equation with moving control, J. Math. Analysis and Appl., 418 (2)(2014)

N-dimensional case

The adjoint system of BZK reads

$$\left|\begin{array}{lll} \varphi - \Delta\varphi = \psi & \text{in} & Q, \\ -\psi_t + \psi = \varphi & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \psi(T) = \psi_T & \text{in} & \Omega. \end{array}\right.$$

Null controllability of BZK system is equivalent to

$$\|\psi(\cdot,0)\|^2_{L^2(\Omega)} \leq C\int_0^T\int_{\omega(t)} |\psi|^2 dxdt.$$

Carleman inequality

Theorem (Chaves-Silva & S.)
Given $\psi_T \in L^2(\Omega)$, the solution (φ, ψ) of the adj. system of BZK satisfies:

$$\iint_Q \rho_1(x,t)(|\nabla\varphi|^2 + |\varphi|^2)dxdt + \iint_Q \rho_2(x,t)|\psi|^2 dxdt$$
$$+ \iint_Q \rho_3(t)(|\nabla\varphi_t|^2 + |\varphi_t|^2)dxdt$$
$$\leq C \int_0^T \int_{\omega(t)} \rho_4(x,t)|\psi|^2 dxdt,$$

where ρ_i, $i = 1, \ldots, 4$ are appropriate weights.

Idea of the proof

Three main difficulties appear:

1. Carleman inequalities for the Laplace operator and ODE equations with a moving control region[2];
2. We must have the same weight functions in the Carleman for both equations.
3. Eliminate a local integral of φ.

Fortunately, we can handle all these difficulties.

[2] F. W. Chaves-Silva, L. Rosier, and E. Zuazua, Null controllability of a system of viscoelasticity with a moving control, J. Math. Pures Appl., 101 (9) (2014), 198–222.

First controllability result

Theorem (Chaves-Silva & S.)
Given $v_0 \in L^2(\Omega)$*, there exists* $f \in L^2(Q)$ *such that the associated solution* (u, v) *of BZK system satisfies:*

$$v(T) = u(T) = 0.$$

Null controllability for BZK

Corollary

Given $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$*, there exists* $f \in L^2(Q)$ *such that the associated solution* u *of BZK equation satisfies*

$$u(T) = 0.$$

Linearized Benjamin-Bona-Mahony equation

Similar ideas (but not the same!) can be used to study the controllability of

$$
\left|\begin{array}{lll}
u_t - \Delta u_t - div(A(x,t)u) = f 1_\omega & \text{in} & Q, \\
u = 0 & \text{on} & \Sigma, \\
u(0) = u_0 & \text{in} & \Omega,
\end{array}\right.
$$

where A is a regular enough vector function.

A nice decomposition

The BBM equation can be decomposed as

$$\left| \begin{array}{lll} u - \Delta u = v & \text{in} & Q, \\ v_t + \nabla \cdot (A(x,t)u) = f 1_\omega & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ v(\cdot, 0) = v_0 & \text{in} & \Omega. \end{array} \right.$$

We have:

$$v(T) = 0 \text{ iff } u(T) = 0.$$

Observability inequality

The null controllability for BBM system is equivalent to the observability inequality for the adjoint solutions, i.e

$$\|\psi(\cdot,0)\|^2_{L^2(\Omega)} \leq C \iint_{\omega\times(0,T)} |\psi|^2,$$

where

$$\left|\begin{array}{lll} \varphi - \Delta\varphi = A\cdot\nabla\psi & \text{in} & Q, \\ -\psi_t = \varphi & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \psi(T) = \psi_T & \text{in} & \Omega. \end{array}\right.$$

- Lack of controllability [3] [4]: easy to see when $\operatorname{supp}(A)\cap\omega = \emptyset$. ;

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[3] S. Micu, On the controllability of the linearized Benjamin-Bona-Mahony equation, *SIAM J. Control Optim.,* **39** (2001), 1677–1696.

[4] X. Zhang, E. Zuazua, Unique continuation for the linearized Benjamin-Bona-Mahony equation with space-dependent potential, Mathematishe Annalen 325 (2003), 543–582.

Old same remedy: moving controls

With moving controls, BBM system reads

$$\left| \begin{array}{lll} u - \Delta u = v & \text{in} & Q, \\ v_t + \nabla \cdot (A(x,t)u) = f 1_{\omega(t)} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ v(\cdot, 0) = v_0 & \text{in} & \Omega. \end{array} \right.$$

And, still, we want to find a control f such that $u(T) = v(T) = 0$.

The adjoint system of BBM reads

$$\left| \begin{array}{lll} \varphi - \Delta\varphi = A \cdot \nabla\psi & \text{in} & Q, \\ -\psi_t = \varphi & \text{in} & Q, \\ \varphi = 0 & \text{on} & \Sigma, \\ \psi(T) = \psi_T & \text{in} & \Omega. \end{array} \right.$$

Null controllability of BZK system is equivalent to

$$\|\psi(\cdot,0)\|_{L^2(\Omega)}^2 \leq C \int_0^T \int_{\omega(t)} |\psi|^2 dxdt.$$

Theorem (Chaves-Silva & S.)
Given $\psi_T \in L^2(\Omega)$, the solution (φ, ψ) of the adj. system of BZK satisfies:

$$\iint_Q \rho_1^*(x,t)(|\nabla\varphi|^2 + |\varphi|^2)dxdt + \iint_Q \rho_2^*(x,t)|\psi|^2 dxdt$$
$$+ \iint_Q \rho_3^*(t)(|\nabla\varphi_t|^2 + |\varphi_t|^2)dxdt$$
$$\leq C \int_0^T \int_{\omega(t)} \rho_4^*(x,t)|\psi|^2 dxdt,$$

where ρ_i^*, $i = 1, \ldots, 4$ are appropriate weights.

Key ingredients:

- $A \cdot \nabla\psi = \nabla \cdot (A\psi) - \psi\nabla \cdot A$;
- H^{-1} elliptic Carleman inequality;
- energy estimates;

Final comments

- Moving control is a good alternative;
- Neumann boundary conditions:
    - 1D is OK!
    - N-dimensional?
- Controllability results for general Sobolev-Galpern type equations

$$\mathcal{M}\partial_t u + \mathcal{L}u = f1_\omega?$$

Happy birthday Jean-Michel!!!