Stabilization of normal type equation connected with 3D Helmholtz equation by start control

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3D Navier-Stokes system

3D Navier-Stokes system with periodic boundary conditions:

$$\partial_t v(t,x) - \Delta v(t,x) + (v,\nabla)x + \nabla p(t,x) = 0,\ \mathrm{div}\, v = 0, \qquad (1)$$

$$v(t,...,x_i,...) = v(t,...,x_i+2\pi,...),\ i = 1,2,3 \qquad (2)$$

$$v(t,x)|_{t=0} = v_0(x) \qquad (3)$$

$t \in \mathbb{R}_+,\ x = (x_1,x_2,x_3) \in \mathbb{R}^3,\ v(t,x) = (v_1,v_2,v_3),$
∇p – the gradient of pressure, Δ – the Laplace operator,
$(v,\nabla)v = \sum_{j=1}^3 v_j \partial_{x_j} v.$

Phase space:

$$V^m = V^m(\mathbb{T}^3) = \{v(x) \in (H^m(\mathbb{T})^3)^3 : \mathrm{div} v = 0, \int_{\mathbb{T}^3} v(x)dx = 0\}$$

where $H^m(\mathbb{T}^3)$ is the Sobolev space, $m \geq 0,\ m \in \mathbb{Z}$.

Energy estimate

$$\int_{\mathbb{T}^3} (v(t,x),\nabla)v(t,x)\cdot v(t,x)dx = 0 \Rightarrow$$

energy estimate in $V^0(\mathbb{T}^3)$:

$$\int_{\mathbb{T}^3} |v(t,x)|^2 dx + 2\int_0^t \int_{\mathbb{T}^3} |\nabla_x v(\tau,x)|^2 dx d\tau \le \int_{\mathbb{T}^3} |v_0(x)|^2 dx$$

$\Rightarrow \exists$ weak solution for (1)-(3).

$((v,\nabla)v\cdot v)_{V^1(\mathbb{T}^3)} \neq 0 \Rightarrow$ no energy estimate in $V^1(\mathbb{T}^3)$

Operator curl^{-1}

$$\omega(x) = \mathrm{curl}\, v(x)$$

$$v(x) = \sum_{k\in\mathbb{Z}^3} \hat{v}(k)e^{i(k,x)}, \;\; \hat{v}(k) = (2\pi)^{-3}\int_{\mathbb{T}^3} v(x)e^{-i(k,x)}dx,$$

$$\mathrm{curl}\,\mathrm{curl}\, v = -\Delta v, \;\; \text{if } \mathrm{div}\, v = 0$$

$$\Rightarrow \text{operator } \mathrm{curl}^{-1} \text{ is well-defined and}$$

$$\mathrm{curl}^{-1}\omega(x) = i\sum_{k\in\mathbb{Z}^3} \frac{k\times\hat{\omega}(k)}{|k|^2}e^{i(k,x)} \Rightarrow$$

$$\mathrm{curl} : V^1 \mapsto V^0 \text{ realizes isomorphism of the spaces}$$

.

Helmholtz equation

$$v(t,x) \to \omega(t,x) = \operatorname{curl} v(t,x) \Rightarrow$$

Helmholtz system:

$$\partial_t \omega(t,x) - \Delta\omega + \underbrace{(v,\nabla)\omega - (\omega,\nabla)v}_{B(\omega)} = 0$$

$$\omega(t,x)|_{t=0} = \omega_0(x) \quad (= \operatorname{curl} v_0)$$

$$(B(\omega),\omega)_{V_0} = -\int_{\mathbb{T}^3} \sum_{j,k=1}^{3} \omega_j \partial_j \omega_k \omega_k dx \neq 0$$

$\Rightarrow$ no energy estimate for 3-D Helmholtz system

Derivation of Normal Parabolic Equations (NPE)

$$\Sigma(\|\omega\|_{V^0}) = \{u \in V^0 : \|u\|_{V^0} = \|\omega\|_{V^0}\}$$

Non-linear term in Helmholtz equation

$$B(\omega)$$

$$\swarrow \searrow$$

othogonal to $\Sigma(\|\omega\|_{V^0})$ at ω $\longrightarrow B_n(\omega) + B_\tau(\omega) \longleftarrow$ tangent to $\Sigma(\|\omega\|_{V^0})$ at ω

$B(\omega) \to B_n(\omega) \Rightarrow$ Normal Parabolic Equations (NPE):

$$\partial_t \omega(t,x) - \Delta\omega - \Phi(\omega)\omega = 0, \ \operatorname{div}\omega = 0,$$

$$\Phi(\omega)\omega = \begin{cases} \frac{\int_{\mathbb{T}^3} (\omega(x),\nabla)\, \text{curl}^{-1}\, \omega(x)\cdot\omega(x)dx}{\int_{\mathbb{T}^3} |\omega(x)|^2} dx, & \omega \neq 0, \\ 0, & \omega \equiv 0. \end{cases}$$

Explicit formula for solution of NPE

Lemma

Let $\mathbf{S}(t,x;\omega_0)$ *be the solution of the following Stokes system with periodic boundary conditions:*

$$\partial_t\omega - \Delta\omega = 0;$$
$$\omega(t,\ldots,x_i+2\pi,\ldots) = \omega(t,x), \quad i = 1,2,3;$$
$$\omega(0,x) = \omega_0,$$

i.e. $\mathbf{S}(t,x;\omega_0) = \omega(t,x)$. *Then the solution of problem* (6) *with periodic boundary conditions and initial condition* $\omega(0,x) = \omega_0(x)$ *has the form*

$$\omega(t,x;\omega_0) = \frac{\mathbf{S}(t,x;\omega_0)}{1 - \int_0^t \Phi(\mathbf{S}(\tau,x;\omega_0))d\tau}$$

The structure of NPE dynamics

Definition

$M_- = \{\omega_0 \in V^0 : \|\omega(t,\cdot;\omega_0)\|_0 \le \alpha\|\omega_0\|_0 e^{-t}\ \forall t > 0\}$ – *the set of stability.*

$M_+ = \{\omega_0 \in V^0 : \exists\omega(t,\cdot;\omega_0), t < t_0, \lim_{t\to t_0} \|\omega(t,\cdot;\omega_0)\|_0 = \infty\}$ – *the set of explosions.*

$M_g = \{\omega_0 \in V^0 : \exists\omega(t,x;\omega_0)\forall t \in \mathbb{R}_+, \lim_{t\to\infty} \|\omega(t,\cdot;\omega_0)\|_0 = \infty\}$ – *the set of growth.*

Theorem

The sets of stability, explosions and growth are not emty:

$$M_- \neq \emptyset, \quad M_+ \neq \emptyset, \quad M_g \neq \emptyset$$

Moreover

$$M_- \cup M_+ \cup M_g = V^0.$$

Formulation of the main result on stabilization

We consider semilinear parabolic equations:

$$\begin{aligned} &\partial_t y(t,x) - \Delta y(t,x) - \Phi(y)y = 0, \\ &y(t, ...x_i + 2\pi, ...) = y(t,x),\ i = 1,2,3, \\ &y(t,x)|_{t=0} = y_0(x) + v(x). \end{aligned} \qquad (4)$$

where

$$\Phi(y) = \begin{cases} \frac{\int_{\mathbb{T}^3} (y(x),\nabla)\,\mathrm{rot}^{-1}\, y(x)\cdot y(x)dx}{\int_{\mathbb{T}^3} |y(x)|^2 dx}, & y \neq 0 \\ 0, & y \equiv 0 \end{cases}$$

Here $y_0(x) \in V^0$ is an arbitrary given initial datum and $v(x) \in V^0$ is a control.

We assume that

$$\mathrm{supp}\, v \subset [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \subset \mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3 \qquad (5)$$

Main stabilization result

Theorem

Let $y_0 \in V^0$ be given. Then there exists a control $v \in V^0$ satisfying (5) *such that there exists unique solution $y(t, x; y_0 + v)$ of* (4), *and this solution satisfies estimate*

$$\|y(t, \cdot; y_0 + v)\|_0 \leq \alpha \|y_0 + v\|_0 e^{-t} \quad \forall t > 0$$

with a certain $\alpha > 1$.

Control function

$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \to \Omega = [-\rho_1, \rho_1] \times [-\rho_2, \rho_2] \times [-\rho_3, \rho_3]$
We choose $p \in \mathbb{N} : \frac{\pi}{p} < \rho_i, \ i = 1, 2, 3$

$$\chi_{\frac{\pi}{p}} = \begin{cases} 1, & |x| \le \frac{\pi}{p}, \\ 0, & \frac{\pi}{p} < |x| \le \pi. \end{cases}$$

Control function:

$$v(x) = \lambda u(x),$$

$$u(x) = \operatorname{curl}\operatorname{curl}(\chi_{\frac{\pi}{p}}(x_1)\chi_{\frac{\pi}{p}}(x_2)\chi_{\frac{\pi}{p}}(x_3)w(px_1, px_2, px_3), 0, 0),$$

where $w(x_1, x_2, x_3) =$
$\sum_{\substack{i,j,k=1 \\ i<j, k\neq i,j}}^{3} a_k(1 + \cos x_k)(\sin x_i + \frac{1}{2}\sin 2x_i)(\sin x_j + \frac{1}{2}\sin 2x_j)$
$a_1, a_2, a_3 \in \mathbb{R}$

Principal estimate

$$\partial_t \mathbf{S}(t,x) - \Delta \mathbf{S}(t,x) = 0, \qquad \mathbf{S}(t,x)|_{t=0} = u(x)$$

The following theorem is true:

Theorem

For each $\rho \in (0,\pi)$ function $u(x)$ defined in (11), satisfies conditions:

$$u(x) \in H^0(\mathbb{T}^3), \qquad supp\, u \subset ([-\rho,\rho])^3$$

and

$$\int_{T^3} ((\mathbf{S}(t,x;u),\nabla)\operatorname{rot}^{-1}\mathbf{S}(t,x;u), \mathbf{S}(t,x;u))dx > \beta e^{-18t}$$

$$\forall\ t \geq 0$$

with a positive constant β.

Reduction to 1D estimates

$$\int_{T^3}((\mathbf{S}(t,x;u),\nabla)\operatorname{curl}^{-1}\mathbf{S}(t,x;u),\mathbf{S}(t,x;u))dx =$$
$$\frac{5}{4}(a_3^2a_1 - a_2^2a_1)J_1(t)J_3(t)\left(J_2(t)+J_4(t)\right),$$

$$J_1(t) = \int_{-\pi}^{\pi} S^2(t,x;\chi_{\frac{\pi}{p}}(1+\cos p\xi))S(t,x;\chi_{\frac{\pi}{p}}(\cos p\xi+\cos 2p\xi))dx$$

$$J_2(t) = \int_{-\pi}^{\pi} S(t,x;\chi_{\frac{\pi}{p}}(1+\cos p\xi))S^2(t,x;\chi_{\frac{\pi}{p}}(\cos p\xi+\cos 2p\xi))dx$$

$$J_3(t) = \int_{-\pi}^{\pi} S^3(t,x;\chi_{\frac{\pi}{p}}(\cos p\xi+\cos 2p\xi))dx$$

$$J_4(t) = \int_{-\pi}^{\pi} S(t,x;\chi_{\frac{\pi}{p}}(\cos p\xi+\cos 2p\xi))\cdot$$
$$S(t,x;\chi_{\frac{\pi}{p}}(\sin p\xi))S(t,x;\chi_{\frac{\pi}{p}}(\sin p\xi+\tfrac{1}{2}\sin 2p\xi))dx$$

1D estimates

The following estimates are true:

$$J_1(t) > C_1 \cdot e^{-6t}, \tag{6}$$

$$J_2(t) + J_4(t) > C_2 \cdot e^{-6t}, \tag{7}$$

$$J_3 > C_3 \cdot e^{-6t} \tag{8}$$

for all $t > 0$, where C_1, C_2 and C_3 are some positive constants. Estimate (8) was proved in

- A.V.Fursikov, *Stabilization of the simplest normal parabolic equation*, 2014
- A. V. Fursikov, L. S. Shatina, *On an estimate connected with the stabilization of normal parabolic equation by start*, 2014

Estimate (6): Fourier decomposition

$S(t, x; \chi_{\frac{\pi}{p}}(1 + \cos p\xi)) = \frac{1}{p} + \sum_{k=1}^{\infty} c(k) \cos kx e^{-k^2 t}$,
$S(t, x; \chi_{\frac{\pi}{p}}(\cos p\xi + \cos 2p\xi)) = \sum_{k=1}^{\infty} d(k) \cos kx e^{-k^2 t}$,
where

$$c(k) = \begin{cases} \dfrac{2p^2 \sin \frac{\pi k}{p}}{\pi k(p^2 - k^2)}, k \neq p, \\ \dfrac{1}{p}, k = p, \end{cases}$$

$$d(k) = \begin{cases} \dfrac{6p^2 k \sin \frac{\pi k}{p}}{\pi(p^2 - k^2)(4p^2 - k^2)}, k \neq p, 2p, \\ \dfrac{1}{p}, k = p, 2p \end{cases}$$

Estimate (6): Fourier decomposition

$$J_1(t) = \frac{2\pi}{p} J_{10}(t) + \frac{\pi}{2}(2J_{11}(t) + J_{12}(t)),$$

where

$$J_{10}(t) = \sum_{m=1}^{\infty} c(m)d(m)e^{-2m^2 t},$$

$$J_{11}(t) = \sum_{m,l=1}^{\infty} c(m)d(l)c(m+l)e^{-2(m^2+l^2+ml)t} =: \sum_{m,l=1}^{\infty} F_{11}(m,l;t),$$

$$J_{12}(t) = \sum_{m,l=1}^{\infty} c(m)c(l)d(m+l)e^{-2(m^2+l^2+ml)t} =: \sum_{m,l=1}^{\infty} F_{12}(m,l;t).$$

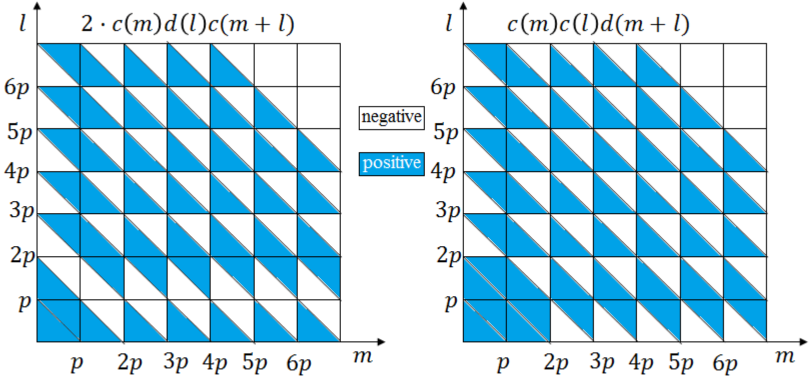
Positiveness of J_{10}

$$J_{10}(t) = \underbrace{\sum_{m=1}^{p-1} c(m)d(m)e^{-2m^2 t} + \frac{1}{p^2}e^{-2p^2 t}}_{+} + \underbrace{\sum_{m=p+1}^{2p-1} c(m)d(m)e^{-2m^2 t}}_{-}$$

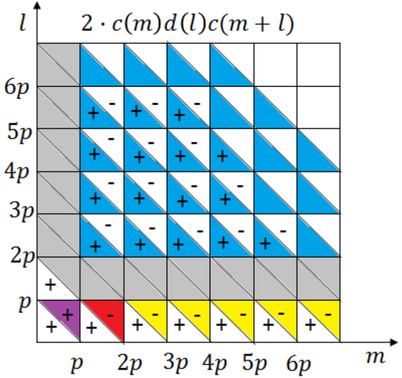
$$|\text{Negative part}| \le \sum_{m=2p+1}^{+\infty} \frac{12p^4 \sin^2 \frac{\pi m}{p}}{(m^2-p^2)^2(m^2-4p^2)} e^{-2(2p+1)^2 t} \le$$

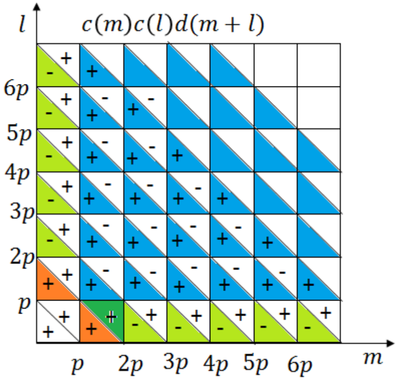
$$\frac{12 \cdot 0.035}{p^2} e^{-2(2p+1)^2 t} < \frac{1}{p^2} e^{-2p^2 t} \quad \forall t > 0$$

Sign distribution in J_{11} and J_{12}



Positiveness of J_{12} + certain part of J_{11}

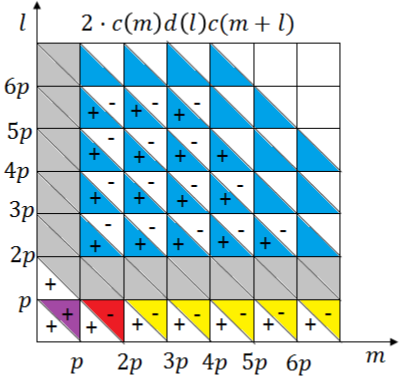


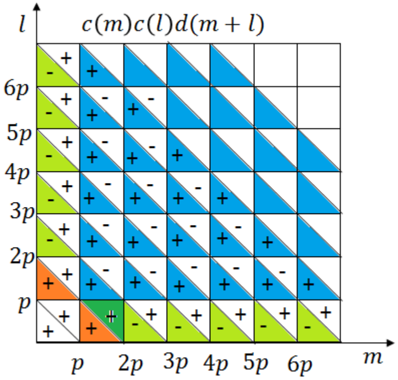


$J_{12}({\color{cyan}\blacksquare}) + J_{12}(\square) > 0$ $\quad J_{12}({\color{orange}\blacksquare}) + J_{12}({\color{lime}\blacksquare}) > 0$ $\quad J_{11}({\color{purple}\blacksquare}) + J_{11}({\color{yellow}\blacksquare}) > 0$

$J_{11}({\color{cyan}\blacksquare}) + J_{11}(\square) > 0$ $\quad J_{12}({\color{green}\blacksquare}) + J_{11}({\color{red}\blacksquare}) > 0$ $\quad |J_{11}({\color{gray}\blacksquare})| < \frac{0.81}{p^3} \cdot e^{-3p^2 t}$

Estimate for $J_1(t)$: last step





$$|J_{11}(\square)| < \frac{0.81}{p^3} \cdot e^{-3p^2 t}$$

$$\begin{cases} F_{11}(\frac{p-1}{2}, \frac{p+1}{2}; t) + F_{11}(\frac{p+1}{2}, \frac{p-1}{2}; t) > \frac{0.81}{p^3} \cdot e^{-3p^2 t}, & p \geq 3 \text{ is odd} \\ F_{11}(\frac{p}{2}, \frac{p}{2}; t) + F_{11}(\frac{p}{2} - 1, \frac{p}{2} + 1; t) > \frac{0.81}{p^3} \cdot e^{-3p^2 t}, & p \geq 4 \text{ is even} \end{cases} \quad \square$$

Estimate for $J_2(t) + J_4(t)$: preliminary lemmas

Lemma

Let $S(t, x; \chi_{\frac{\pi}{p}}\varphi(p\xi))$ be the solution of the heat equation $\partial_t S - \partial_{xx} S = 0$, $S(t, x)|_{t=0} = \chi_{\frac{\pi}{p}}\varphi(px)$ with periodic boundary condition, where $\chi_{\frac{\pi}{p}}$ is the characteristic function of interval $[-\pi/p, \pi/p]$, $p \in \mathbb{N}$. Then

1. *For all $t \geq 0$,*
   $\int_{-\pi}^{\pi} S(t, x; \chi_{\frac{\pi}{p}}\varphi(p\xi))dx = \int_{-\pi}^{\pi} S(0, x; \chi_{\frac{\pi}{p}}\varphi(p\xi))dx = const;$
2. *If function $\varphi(x) \in C^1[-\pi, \pi]$, and $\varphi(\pm\pi) = 0$, then*
   $\partial_x S(t, x; \chi_{\frac{\pi}{p}}\varphi(p\xi)) = pS(t, x; \chi_{\frac{\pi}{p}}\varphi'(p\xi))$,
3. *If function $\varphi(x) \in C^2[-\pi, \pi]$, $\varphi(\pm\pi) = 0$ and $\varphi'(\pm\pi) = 0$ then*
   $\partial_t S(t, x; \chi_{\frac{\pi}{p}}\varphi(p\xi)) = p^2 S(t, x; \chi_{\frac{\pi}{p}}\varphi''(p\xi))$.

Equation for $J_2(t) + J_4(t)$

$$\frac{d}{dt}(J_2(t) + J_4(t)) + 24p^2(J_2 + J_4) = 8p^2 J_2(t) + \underbrace{9p^2 \int_{-\pi}^{\pi} S^2(t,x;\chi_{\frac{\pi}{p}}(\sin p\xi))S(t,x;\chi_{\frac{\pi}{p}}(\cos p\xi + \cos 2p\xi))dx}_{R(t)} +$$

$$\underbrace{12p^2 \int_{-\pi}^{\pi} S(t,x;\chi_{\frac{\pi}{p}}(1+\cos p\xi))S(t,x;\chi_{\frac{\pi}{p}}(\cos p\xi + \cos 2p\xi)) \cdot S(t,x;\chi_{\frac{\pi}{p}}(\cos p\xi))dx}_{Q(t)} + \text{Positive Function}(t)$$

and

$$\frac{d}{dt}(12R(t) + 9Q(t)) + 8p^2(12R(t) + Q(t)) = PositiveFunction(t) + \underbrace{48p^2 \int_{-\pi}^{\pi} S^2(t,x;\chi_{\frac{\pi}{p}}(\sin p\xi))S(t,x;\chi_{\frac{\pi}{p}}(-\cos 2p\xi))dx}_{\tilde{J}(t)}.$$

Estimate for $\tilde{J}(t)$

Theorem

Let $\tilde{J}(t)$ be the function

$$\tilde{J}(t) := \int_{-\pi}^{\pi} S^2(t, x; \chi_{\frac{\pi}{p}}(\sin p\xi)) S(t, x; \chi_{\frac{\pi}{p}}(-\cos 2p\xi)) dx$$

Then

$$\tilde{J}(t) > \alpha \cdot e^{-6t} \ \forall t > 0,$$

where

$$\alpha = \frac{144 \sin^2 \frac{\pi}{p} \sin \frac{2\pi}{p}}{p^4}.$$

Fourier decomposition for $\tilde{J}(t)$

$$\tilde{J}(t) = \tfrac{4p^2}{\pi^2} \underbrace{(I(t) + II(t))}_{J(t)},$$

where

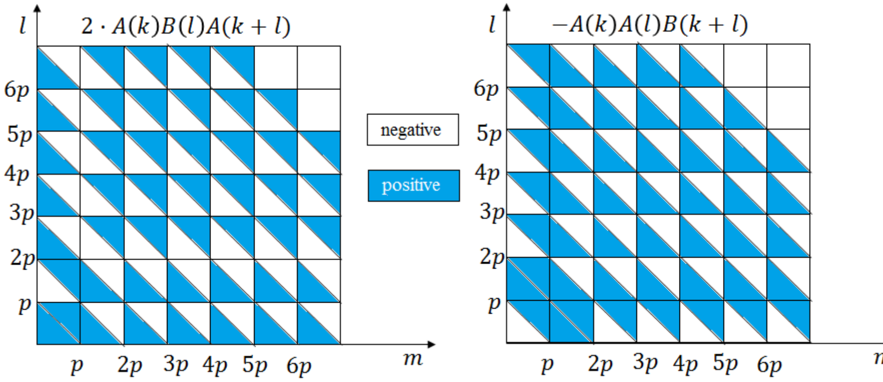
$I(t) = \sum_{k,l=1}^{\infty} 2A(k)A(k+l)B(l)e^{-2(k^2+l^2+kl)t}$

$II(t) = \sum_{k,l=1}^{\infty} (-A(k)A(l)B(k+l))e^{-2(k^2+l^2+kl)t}$

and

$$A(k) = \begin{cases} \dfrac{\sin\frac{\pi k}{p}}{p^2 - k^2}, k \neq p, \\ \dfrac{\pi}{2p^2}, k = p \end{cases} \quad B(k) = \begin{cases} \dfrac{k\sin\frac{\pi k}{p}}{4p^2 - k^2}, k \neq 2p, \\ -\dfrac{\pi}{2p}, k = 2p \end{cases}$$

Sign distribution in $J(t)$



Positiveness of $J(t)$

The sum of summands with symmetrical coordinates:

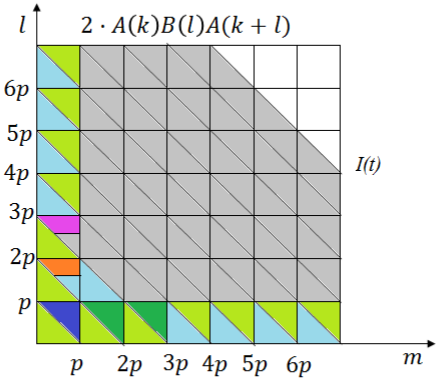
$$(2A(k)A(k+l)B(l) - A(k)A(l)B(k+l) + 2A(l)A(l+k)B(k) - A(l)A(k)B(l+k))e^{-2(k^2+l^2+kl)t} = H(k,l)\cdot e^{-2(k^2+l^2+kl)t}$$

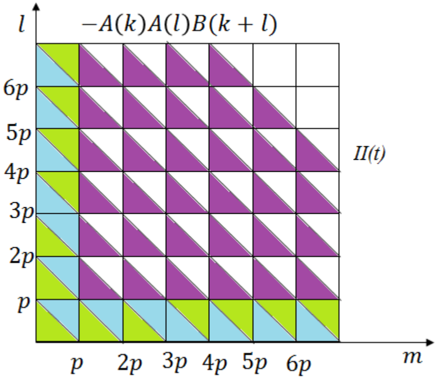
where

$$H(k,l) = \frac{6p^2kl(k+l)(12p^2+kl-(k+l)^2)\sin\frac{\pi k}{p}\sin\frac{\pi l}{p}\sin\frac{\pi(k+l)}{p}}{(p^2-k^2)(p^2-l^2)(p^2-(k+l)^2)(4p^2-k^2)(4p^2-l^2)(4p^2-(k+l)^2)}$$

Signs of $\left(\sin\frac{\pi k}{p}\sin\frac{\pi l}{p}\sin\frac{\pi(k+l)}{p}\right)$: 

Positiveness of $J(t)$





$J(\square) > 0$

$J(\square) > 0$

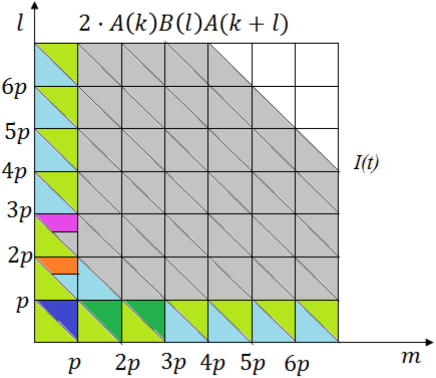
$II(\square)+II(\square) > 0$

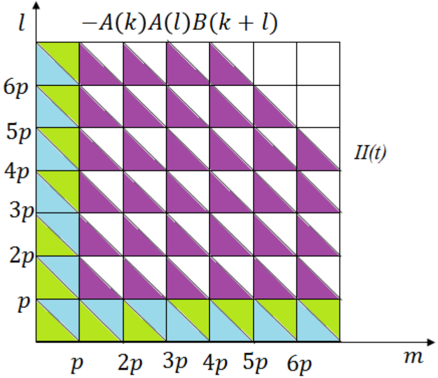
$I(\square)+I(\square) > 0$

$I(\square)+I(\square) > 0$

$|I(\square)| < \frac{1.55}{p^5} \cdot e^{-\frac{27p^2}{4}t}$

Positiveness of $J(t)$





$$|I(\square)| < \frac{1.55}{p^5} \cdot e^{-\frac{27p^2}{4}t}$$

$$-A(p)A(p)B(2p) = \frac{\pi^3}{8p^2} \cdot e^{-6p^2 t} > \frac{1.55}{p^5} \cdot e^{\frac{-27p^2}{4}t} \ \forall t > 0 \quad \square$$

The End

Thank you for your attention