Mean Field Limits for Ginzburg-Landau Vortices

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Jean-Michel Coron's 60th birthday, June 20, 2016

The Ginzburg-Landau equations

$u : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$

$$\boxed{-\Delta u = \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{Ginzburg-Landau equation (GL)}$$

$$\boxed{\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{parabolic GL equation (PGL)}$$

$$\boxed{i\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{Gross-Pitaevskii equation (GP)}$$

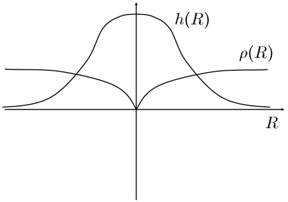
Associated energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}$$

Models: superconductivity, superfluidity, Bose-Einstein condensates, nonlinear optics

Vortices

- in general $|u| \leq 1$, $|u| \simeq 1$ = superconducting/superfluid phase, $|u| \simeq 0$ = normal phase
- u has zeroes with nonzero degrees = **vortices**
- $u = \rho e^{i\varphi}$, characteristic length scale of $\{\rho < 1\}$ is ε = vortex core size



- degree of the vortex at x_0:

$$\frac{1}{2\pi} \int_{\partial B(x_0, r)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

- In the limit $\varepsilon \to 0$ vortices become *points*, (or curves in dimension 3).

Solutions of (GL), bounded number N of vortices

- minimal energy

$$\min E_\varepsilon = \pi N |\log \varepsilon| + \min W + o(1) \quad \text{as} \quad \varepsilon \to 0$$

- u_ε minimizing E_ε has vortices all of degree $+1$ (or all -1) which converge to a minimizer of

$$W((x_1, d_1), \dots, (x_N, d_N)) = -\pi \sum_{i \neq j} d_i d_j \log |x_i - x_j| + \text{boundary terms...}$$

"renormalized energy", Kirchhoff-Onsager energy (in the whole plane) [Bethuel-Brezis-Hélein '94]

- Some boundary condition needed to obtain nontrivial minimizers
- nonminimizing solutions: u_ε has vortices which converge to a critical point of W:

$$\nabla_i W(\{x_i\}) = 0 \quad \forall i = 1, \cdots N$$

[Bethuel-Brezis-Hélein '94]

- stable solutions converge to stable critical points of W [S. '05]

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Dynamics, bounded number N of vortices

- For well-prepared initial data, $d_i = \pm 1$, solutions to (PGL) have vortices which converge (after some time-rescaling) to solutions to

$$\frac{dx_i}{dt} = -\nabla_i W(x_1, \dots, x_N)$$

[Lin '96, Jerrard-Soner '98, Lin-Xin '99, Spirn '02, Sandier-S '04]

- For well-prepared initial data, $d_i = \pm 1$, solutions to (GP)

$$\frac{dx_i}{dt} = -\nabla_i^\perp W(x_1, \dots, x_N) \qquad \nabla^\perp = (-\partial_2, \partial_1)$$

[Colliander-Jerrard '98, Spirn '03, Bethuel-Jerrard-Smets '08]

- All these hold up to collision time
- For (PGL), extensions beyond collision time and for ill-prepared data [Bethuel-Orlandi-Smets '05-07, S. '07]

A word about dimension 3 (or higher)

- Leading order of the energy becomes $\pi|d||L||\log\varepsilon|$ where L= length (or area) of vortex line (integer multiplicity rectifiable current)
- Minimizers/solutions to (GL) converge to length minimizing / stationary currents (= straight lines)
  [Rivière '95, Lin-Rivière '01, Sandier '01, Bethuel-Brezis-Orlandi '01, Jerrard-Soner '02, Alberti-Baldo-Orlandi '03, Bourgain-Brezis-Mironescu '04]
- (PGL) $\rightarrow$ mean curvature motion (Brakke)
  [Bethuel-Orlandi-Smets '06]
- (GP) $\rightarrow$ binormal flow (partial results)
  [Jerrard '02]

Vorticity

- In the case $N_\varepsilon \to \infty$, describe the vortices via the **vorticity** :

  supercurrent

  $$j_\varepsilon := \langle iu_\varepsilon, \nabla u_\varepsilon \rangle \qquad \langle a, b \rangle := \frac{1}{2}(a\bar{b} + \bar{a}b)$$

  vorticity

  $$\mu_\varepsilon := \mathrm{curl}\, j_\varepsilon$$

- $\simeq$ vorticity in fluids, but quantized: $\mu_\varepsilon \simeq 2\pi \sum_i d_i \delta_{a_i^\varepsilon}$
- $\frac{\mu_\varepsilon}{2\pi N_\varepsilon} \to \mu$ signed measure, or probability measure,

Mean-field limit for stationary solutions

If u_ε is a solution to (GL) and $N_\varepsilon \gg 1$ then $\mu_\varepsilon / N_\varepsilon \to \mu$ solution to

$$\mu \nabla h = 0 \qquad h = -\Delta^{-1}\mu$$

in a suitable weak sense ($\simeq$ Delort):

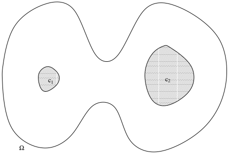
$$T_\mu := -\nabla h \otimes \nabla h + \frac{1}{2}|\nabla h|^2 \delta_i^j$$

Weak relation is

$$\mathrm{div}\ T_\mu = 0 \quad \text{in “finite parts"}$$

[Sandier-S '04]

$\rightsquigarrow$ h is constant on the support of μ



Dynamics in the case $N_\varepsilon \gg 1$

Back to

$$\frac{N_\varepsilon}{|\log \varepsilon|}\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \quad \text{in } \mathbb{R}^2 \qquad \text{(PGL)}$$

$$iN_\varepsilon \partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \quad \text{in } \mathbb{R}^2 \qquad \text{(GP)}$$

- For (GP), by Madelung transform, the limit dynamics is expected to be the 2D incompressible Euler equation. Vorticity form

$$\partial_t \mu - \mathrm{div}\,(\mu \nabla^\perp h) = 0 \qquad h = -\Delta^{-1}\mu \quad \text{(EV)}$$

- For (PGL), formal model proposed by [Chapman-Rubinstein-Schatzman '96], [E '95]: if $\mu \geq 0$

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Study of the Chapman-Rubinstein-Schatzman-E equation

- [Lin-Zhang '00, Du-Zhang '03, Masmoudi-Zhang '05] existence of weak solutions (à la Delort) by vortex approximation method, existence and uniqueness of L^∞ solutions, which decay in $1/t$ (uses pseudo-differential operators)
- [Ambrosio-S '08] variational approach in the setting of a bounded domain. The equation is formally the gradient flow of $F(\mu) = \frac{1}{2}\int_\Omega |\nabla\Delta^{-1}\mu|^2$ for the 2-Wasserstein metric (à la [Otto, Ambrosio-Gigli-Savaré]).
- [S-Vazquez '13] PDE approach in all dimension. Existence via limits in fractional diffusion $\partial_t\mu + \mathrm{div}\,(\mu\nabla\Delta^{-s}\mu)$ when $s\to 1$, uniqueness in the class L^∞, propagation of regularity, asymptotic self-similar profile

$$\mu(t) = \frac{1}{\pi t}\mathbf{1}_{B_{\sqrt{t}}}$$

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Previous rigorous convergence results

- (PGL) case : [Kurzke-Spirn '14] convergence of $\mu_\varepsilon/(2\pi N_\varepsilon)$ to μ solving (CRSE) under assumption $N_\varepsilon \leq (\log\log|\log\varepsilon|)^{1/4}$ + well-preparedness
- (GP) case: [Jerrard-Spirn '15] convergence to μ solving (EV) under assumption $N_\varepsilon \leq (\log|\log\varepsilon|)^{1/2}$ + well-preparedness
- both proofs "push" the fixed N proof (taking limits in the evolution of the energy density) by making it more quantitative
- difficult to go beyond these dilute regimes without controlling distance between vortices, possible collisions, etc

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Alternative method: the "modulated energy"

- Exploits the regularity and stability of the solution to the limit equation
- Works for dissipative as well as conservative equations
- Works for gauged model as well
- Works for model with "pinning" weight [Duerinckx-S]

Let $\mathrm{v}(t)$ be the expected limiting velocity field (such that $\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle \rightharpoonup \mathrm{v}$ and $\mathrm{curl}\,\mathrm{v} = 2\pi\mu$). Define the modulated energy

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2)}{2\varepsilon^2},$$

modelled on the Ginzburg-Landau energy.
Analogy with "modulated entropy" methods in kinetic to fluid limits [Brenier '00].

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Main result: Gross-Pitaevskii case

Theorem (S. '15)

Assume u_ε *solves (GP) and let* N_ε *be such that* $|\log \varepsilon| \ll N_\varepsilon \ll \frac{1}{\varepsilon}$*. Let* v *be a* $L^\infty(\mathbb{R}_+, C^{0,1})$ *solution to the incompressible Euler equation*

$$\begin{cases} \partial_t \mathrm{v} = 2\mathrm{v}^\perp \mathrm{curl}\, \mathrm{v} + \nabla \mathrm{p} & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases} \qquad \text{(IE)}$$

with $\mathrm{curl}\, \mathrm{v} \in L^\infty(L^1)$.
Let $\{u_\varepsilon\}_{\varepsilon>0}$ *be solutions associated to initial conditions* u_ε^0*, with* $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \le o(N_\varepsilon^2)$*. Then, for every* $t \ge 0$*, we have*

$$\frac{1}{N_\varepsilon} \langle \nabla u_\varepsilon, i u_\varepsilon \rangle \to \mathrm{v} \quad \text{in } L^1_{loc}(\mathbb{R}^2).$$

Implies of course the convergence of the vorticity $\mu_\varepsilon / N_\varepsilon \to \mathrm{curl}\, \mathrm{v}$
Works in 3D as well

Main result: parabolic case

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Assume u_ε *solves (PGL) and let* N_ε *be such that* $1 \ll N_\varepsilon \leq O(|\log \varepsilon|)$. *Let* v *be a* $L^\infty([0,T], C^{1,\gamma})$ *solution to*

- *if* $N_\varepsilon \ll |\log \varepsilon|$

$$\begin{cases} \partial_t \mathrm{v} = -2\mathrm{v}\,\mathrm{curl}\,\mathrm{v} + \nabla \mathrm{p} & \textit{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \textit{in } \mathbb{R}^2, \end{cases} \tag{L1}$$

- *if* $N_\varepsilon \sim \lambda |\log \varepsilon|$

$$\partial_t \mathrm{v} = \frac{1}{\lambda} \nabla \mathrm{div}\ \mathrm{v} - 2\mathrm{v}\,\mathrm{curl}\,\mathrm{v} \quad \textit{in } \mathbb{R}^2. \tag{L2}$$

Assume $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \leq \pi N_\varepsilon |\log \varepsilon| + o(N_\varepsilon^2)$ *and* $\mathrm{curl}\,\mathrm{v}(0) \geq 0$. *Then* $\forall t \leq T$ *we have*

$$\frac{1}{N_\varepsilon} \langle \nabla u_\varepsilon, i u_\varepsilon \rangle \to \mathrm{v} \quad \textit{in } L^1_{loc}(\mathbb{R}^2).$$

Taking the curl of the equation yields back the (CRSE) equation if $N_\varepsilon \ll |\log \varepsilon|$, but *not* if $N_\varepsilon \propto |\log \varepsilon|$!

Long-time existence for the limiting equations [Duerinckx '16]

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Proof method

- Go around the question of minimal vortex distances by using instead the modulated energy and showing a Gronwall inequality on $\mathcal{E}$.
- the proof relies on algebraic simplifications in computing $\frac{d}{dt}\mathcal{E}_\varepsilon(u_\varepsilon(t))$ which reveal only quadratic terms
- Uses the regularity of v to bound corresponding terms
- An insight is to think of v as a spatial gauge vector and $\mathrm{div}\ \mathrm{v}$ (resp. p) as a temporal gauge

Quantities and identities

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2}{2\varepsilon^2} \quad \text{(modulated energy)}$$

$$j_\varepsilon = \langle iu_\varepsilon, \nabla u_\varepsilon\rangle \qquad \mathrm{curl}\, j_\varepsilon = \mu_\varepsilon \quad \text{(supercurrent and vorticity)}$$

$$V_\varepsilon = 2\langle i\partial_t u_\varepsilon, \nabla u_\varepsilon\rangle \quad \text{(vortex velocity)}$$

$$\partial_t j_\varepsilon = \nabla\langle iu_\varepsilon, \partial_t u_\varepsilon\rangle + V_\varepsilon$$

$$\partial_t \mathrm{curl}\, j_\varepsilon = \partial_t \mu_\varepsilon = \mathrm{curl}\, V_\varepsilon \quad (V_\varepsilon^\perp \text{ transports the vorticity}).$$

$$S_\varepsilon := \langle \partial_k u_\varepsilon, \partial_l u_\varepsilon\rangle - \frac{1}{2}\left(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\delta_{kl} \quad \text{(stress-energy tensor)}$$

$$\tilde{S}_\varepsilon = \langle \partial_k u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_k, \partial_l u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_l\rangle - \frac{1}{2}\left(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\delta_{kl} \quad \text{“modulated stress tensor”}$$

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The Gross-Pitaevskii case

Time-derivative of the energy (if u_ε solves (GP) and v solves (IE))

$$\frac{d\mathcal{E}_\varepsilon(u_\varepsilon(t),t))}{dt} = \int_{\mathbb{R}^2} N_\varepsilon \underbrace{(N_\varepsilon \mathrm{v} - j_\varepsilon)}_{\text{linear term}} \cdot \underbrace{\partial_t \mathrm{v}}_{2\mathrm{v}^\perp \mathrm{curl}\,\mathrm{v} + \nabla \mathrm{p}} - N_\varepsilon V_\varepsilon \cdot \mathrm{v}$$

linear term a priori controlled by $\sqrt{\mathcal{E}} \rightsquigarrow$ unsufficient

But

$$\mathrm{div}\ \tilde{S}_\varepsilon = -N_\varepsilon (N_\varepsilon \mathrm{v} - j_\varepsilon)^\perp \mathrm{curl}\,\mathrm{v} - N_\varepsilon \mathrm{v}^\perp \mu_\varepsilon + \frac{1}{2} N_\varepsilon V_\varepsilon$$

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Multiply by $2\mathrm{v}$

$$\int_{\mathbb{R}^2} 2\mathrm{v} \cdot \mathrm{div}\ \tilde{S}_\varepsilon = \int_{\mathbb{R}^2} -N_\varepsilon (N_\varepsilon \mathrm{v} - j_\varepsilon) \cdot 2\mathrm{v}^\perp \mathrm{curl}\, \mathrm{v} + N_\varepsilon V_\varepsilon \cdot \mathrm{v}$$

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2 \underbrace{\tilde{S}_\varepsilon}_{\text{controlled by } \mathcal{E}_\varepsilon} : \underbrace{\nabla \mathrm{v}}_{\text{bounded}}$$

$\rightsquigarrow$ Gronwall OK: if $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \leq o(N_\varepsilon^2)$ it remains true (vortex energy is $\pi N_\varepsilon |\log \varepsilon| \ll N_\varepsilon^2$ in the regime $N_\varepsilon \gg |\log \varepsilon|$)

The parabolic case

If u_ε solves (PGL) and v solves (L1) or (L2)

$$\frac{d\mathcal{E}_\varepsilon(u_\varepsilon(t),t))}{dt} = -\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 + \int_{\mathbb{R}^2} (N_\varepsilon(N_\varepsilon \mathrm{v} - j_\varepsilon)\cdot \partial_t \mathrm{v} - N_\varepsilon V_\varepsilon \cdot \mathrm{v})$$

$$\mathrm{div}\ \tilde{S}_\varepsilon = \frac{N_\varepsilon}{|\log\varepsilon|}\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle + N_\varepsilon(N_\varepsilon\mathrm{v} - j_\varepsilon)^\perp \mathrm{curl}\,\mathrm{v} - N_\varepsilon \mathrm{v}^\perp \mu_\varepsilon.$$

$$\phi = \mathrm{p} \quad \text{if } N_\varepsilon \ll |\log\varepsilon| \qquad \phi = \lambda\,\mathrm{div}\ \mathrm{v} \quad \text{if not}$$

Multiply by $\mathrm{v}^\perp$ and insert:

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2\tilde{S}_\varepsilon : \nabla \mathrm{v}^\perp - N_\varepsilon V_\varepsilon \cdot \mathrm{v} - 2N_\varepsilon|\mathrm{v}|^2\mu_\varepsilon$$

$$-\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\mathrm{v}^\perp \cdot \frac{N_\varepsilon}{|\log\varepsilon|}\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle.$$

The parabolic case

If u_ε solves (PGL) and v solves (L1) or (L2)

$$\frac{d\mathcal{E}_\varepsilon(u_\varepsilon(t),t))}{dt} = -\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 + \int_{\mathbb{R}^2} \left(N_\varepsilon(N_\varepsilon \mathrm{v} - j_\varepsilon)\cdot \partial_t \mathrm{v} - N_\varepsilon V_\varepsilon \cdot \mathrm{v}\right)$$

$$\mathrm{div}\ \tilde{S}_\varepsilon = \frac{N_\varepsilon}{|\log\varepsilon|}\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle + N_\varepsilon (N_\varepsilon \mathrm{v} - j_\varepsilon)^\perp \mathrm{curl}\,\mathrm{v} - N_\varepsilon \mathrm{v}^\perp \mu_\varepsilon.$$

$\phi = \mathrm{p}$ if $N_\varepsilon \ll |\log\varepsilon|$ $\qquad$ $\phi = \lambda\,\mathrm{div}\ \mathrm{v}$ if not

Multiply by $\mathrm{v}^\perp$ and insert:

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2\tilde{S}_\varepsilon : \nabla \mathrm{v}^\perp - N_\varepsilon V_\varepsilon \cdot \mathrm{v} - 2N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon$$
$$-\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\mathrm{v}^\perp \cdot \frac{N_\varepsilon}{|\log\varepsilon|}\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle.$$

The vortex energy $\pi N_\varepsilon |\log \varepsilon|$ is no longer negligible with respect to N_ε^2. We now need to prove

$$\frac{d\mathcal{E}_\varepsilon}{dt} \le C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + o(N_\varepsilon^2).$$

Need all the tools on vortex analysis:

- vortex ball construction [Sandier '98, Jerrard '99, Sandier-S '00, S-Tice '08]: allows to bound the energy of the vortices from below in disjoint vortex balls B_i by $\pi |d_i| |\log \varepsilon|$ and deduce that the energy outside of $\cup_i B_i$ is controlled by the excess energy $\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|$
- "product estimate" of [Sandier-S '04] allows to control the velocity:

$$\left| \int V_\varepsilon \cdot \mathrm{v} \right| \le \frac{2}{|\log \varepsilon|} \left(\int |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 \int |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}|^2 \right)^{\frac{1}{2}}$$

$$\le \frac{1}{|\log \varepsilon|} \left(\frac{1}{2} \int |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 + 2 \int |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}|^2 \right)$$

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2 \underbrace{\tilde{S}_\varepsilon}_{\leq C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|)} : \underbrace{\nabla \mathrm{v}^\perp}_{\text{bounded}} - \underbrace{N_\varepsilon V_\varepsilon \cdot \mathrm{v}}_{\text{controlled by prod. estimate}} - 2N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon$$

$$- \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log \varepsilon|} |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 + \underbrace{2\mathrm{v}^\perp \cdot \frac{N_\varepsilon}{|\log \varepsilon|} \langle \partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v} \rangle}_{\text{bounded by Cauchy-Schwarz}} .$$

$$\frac{d\mathcal{E}_\varepsilon}{dt} \leq C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log \varepsilon|} (\frac{1}{2} + \frac{1}{2} - 1) |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2$$

$$+ \frac{2N_\varepsilon}{|\log \varepsilon|} \int_{\mathbb{R}^2} |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}^\perp|^2 + |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}|^2 - 2N_\varepsilon \int_{\mathbb{R}^2} |\mathrm{v}|^2 \mu_\varepsilon$$

$$= C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + \underbrace{\frac{2N_\varepsilon}{|\log \varepsilon|} \int_{\mathbb{R}^2} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 |\mathrm{v}|^2 - 2N_\varepsilon \int_{\mathbb{R}^2} |\mathrm{v}|^2 \mu_\varepsilon}_{\text{bounded by } C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) \text{ by ball construction estimates}}$$

$\rightsquigarrow$ Gronwall OK

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2 \underbrace{\tilde{S}_\varepsilon}_{\leq C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|)} : \underbrace{\nabla \mathrm{v}^\perp}_{\text{bounded}} - \underbrace{N_\varepsilon V_\varepsilon \cdot \mathrm{v}}_{\text{controlled by prod. estimate}} - 2N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon$$

$$- \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log \varepsilon|} |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 + \underbrace{2\mathrm{v}^\perp \cdot \frac{N_\varepsilon}{|\log \varepsilon|} \langle \partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v} \rangle}_{\text{bounded by Cauchy-Schwarz}} .$$

$$\frac{d\mathcal{E}_\varepsilon}{dt} \leq C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log \varepsilon|} \left(\frac{1}{2} + \frac{1}{2} - 1\right) |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2$$

$$+ \frac{2N_\varepsilon}{|\log \varepsilon|} \int_{\mathbb{R}^2} |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}^\perp|^2 + |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}|^2 - 2N_\varepsilon \int_{\mathbb{R}^2} |\mathrm{v}|^2 \mu_\varepsilon$$

$$= C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + \underbrace{\frac{2N_\varepsilon}{|\log \varepsilon|} \int_{\mathbb{R}^2} |\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}|^2 |\mathrm{v}|^2 - 2N_\varepsilon \int_{\mathbb{R}^2} |\mathrm{v}|^2 \mu_\varepsilon}_{\text{bounded by } C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) \text{ by ball construction estimates}}$$

$\rightsquigarrow$ Gronwall OK

Joyeux anniversaire, Jean-Michel!