Local bilinear controllability of the Schrödinger equation.

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Outline

Statement of the problem

Controllability of Schrödinger equation in a neighborhood of an eigenfunction.

Control : **real** potential $\Rightarrow$ bilinear controllability problem.

Ω bounded regular open set of $\mathbb{R}^N$. $\Gamma = \partial\Omega,\ T > 0$.

$$\begin{cases} i\frac{\partial y}{\partial t} + \Delta y + V y = 0 \text{ in } \Omega \times (0, T), \\ y = 0 \text{ on } \Gamma \times (0, T), \\ y(0) = y_0 \text{ in } \Omega. \end{cases}$$

(Real) Eigenfunctions of Laplace operator

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k \text{ in } \Omega, \\ \varphi_k = 0 \text{ on } \Gamma, \\ \int_\Omega \varphi_k\varphi_j dx = \delta_{k,j}\ \forall\, k, j = 1, \cdots, +\infty. \end{cases}$$

The problem

If $V = 0$ (free Schrödinger equation) and $y_0 = \varphi_k$ the solution is

$$\tilde{\varphi}_k(t) = e^{-i\lambda_k t}\varphi_k.$$

Question : Given y_0 (close to φ_k), can we find V **real** such that

$$y(T) = \tilde{\varphi}_k(T) = e^{-i\lambda_k T}\varphi_k\,?$$

V real implies Schrödinger equation preserves the L^2 norm.
Therefore necessary condition

$$\int_\Omega |y_0|^2 dx = 1.$$

Initial data y_0 will be taken on the sphere S of radius 1 in $L^2(\Omega)$.

Previous results Case of dimension 1

Case of dimension 1, and

$$V(x,t) = u(t)\mu(x)$$

μ : prescribed profile. Actual control : amplitude $u(.)$.

Negative result

Bad news :

Negative result by J.Ball-J.Marsden-M.Slemrod SICON 1982 : If X is the space of values of $y(t)$ and if the product by μ is bounded from X to X, no hope to obtain a control $u \in L^r_{loc}(0, T)$. They prove that the set of reachable states is contained in a countable union of compact sets, and therefore has an empty interior.

Positive result

Nevertheless first result by K.Beauchard JMPA 2005 using Nash Moser Theorem.
Real breakthrough.

Then K.Beauchard-C.Laurent JMPA 2010 gave another proof.

Previous results Case of dimension 1

They used the space

$$H_\Delta(\Omega) = \{z \in H_0^1(\Omega), \Delta z \in H_0^1(\Omega)\} \subsetneq H^3(\Omega) \cap H_0^1(\Omega)$$

and μ such that

$$\forall z \in H_\Delta(\Omega), \mu z \in H^3(\Omega) \cap H_0^1(\Omega)$$

but in general

$$\mu z \notin H_\Delta(\Omega).$$

The difference comes from the boundary conditions.

Previous results Case of dimension 1

They proved the following regularity result in 1-dimension :
In the free Schrödinger equation, if the right hand side is in $L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega))$ and $y_0 \in H_\Delta(\Omega)$, then the solution y belongs to $C([0, T]; H_\Delta(\Omega))$.
They could find a control $u \in L^2(0, T)$ using the controllability of the linearized problem and an inverse mapping theorem.

Essential action due to the boundary values.

Case of dimension $N \geq 2$

Regularity result has been extended to dimension N for a regular domain in J-P.P Revista Mat. Complutense 2013.
But if $V(x,t) = u(t)\mu(x)$ the linearized problem is no longer controllable. Same argument cannot be applied.

In dimension $N = 2$, K.Beauchard-C.Laurent (recent result to appear, hal-01333627) obtain a controllability result considering potential V satisfying Poisson equation

$$\begin{cases} -\Delta V(t) + V(t) = 0 \text{ in } \Omega, \\ V(t) = g(t) \text{ on } \Gamma, \end{cases}$$

and some conditions on φ_k, the actual control being here the boundary value $g(.)$.
Here we will consider potentials depending on x and t concentrated near the boundary.

Previous results for linear boundary controllability

Boundary linear exact controllability : given a subset Γ_0 of Γ, for any $y_0 \in H^{-1}(\Omega)$ can we find $g \in L^2(0, T; L^2(\Gamma_0))$ such that the solution y of

$$\begin{cases} i\frac{\partial y}{\partial t} + \Delta y = 0 \text{ in } \Omega \times (0, T), \\ y = g \text{ on } \Gamma_0 \times (0, T), \\ y = 0 \text{ on } (\Gamma \setminus \Gamma_0) \times (0, T), \\ y(0) = y_0 \text{ in } \Omega, \end{cases}$$

satisfies

$$y(T) = 0.$$

Adjoint problem

$$\begin{cases} i\frac{\partial \varphi}{\partial t} + \Delta \varphi = 0 \text{ in } \Omega \times (0, T), \\ \varphi = 0 \text{ on } \Gamma \times (0, T), \\ \varphi(0) = \varphi_0 \text{ in } \Omega, \end{cases}$$

Previous results for linear boundary controllability

Exact controllability is equivalent to boundary observability inequality

$$||\varphi_0||^2_{H^1_0(\Omega)} \leq C \int_{\Gamma_0 \times (0,T)} |\frac{\partial \varphi}{\partial \nu}|^2 d\sigma dt,$$

Inequality proved for any $T > 0$ by E.Machtyngier SICON 1994 with multiplier method when

There exists $x_0 \in \mathbb{R}^N$ such that $\Gamma_0 = \{x \in \Gamma, \, (x - x_0).\nu > 0\}$.

Extended by G.Lebeau JMPA 1992 when Γ_0 satisfies the "geometric control condition" using micro local analysis arguments.

Previous results for linear distributed controllability

Distributed (internal) linear exact controllability : given a non empty open subset ω of Ω, for any $y_0 \in L^2(\Omega)$, can we find a control $h \in L^2(0, T; L^2(\omega))$ such that the solution of

$$\begin{cases} i\frac{\partial y}{\partial t} + \Delta y = h.\mathbb{1}_\omega \text{ in } \Omega \times (0, T), \\ y = 0 \text{ on } \Gamma \times (0, T), \\ y(0) = y_0 \text{ in } \Omega, \end{cases}$$

satisfies

$$y(T) = 0.$$

This is equivalent to internal observability inequality for the adjoint state

$$|\varphi_0|^2_{L^2(\Omega)} \leq C \int_{\omega\times(0,T)} |\varphi|^2 dxdt.$$

Previous results for linear distributed controllability

E.Machtyngier proved that when Γ_0 is such that the boundary observability is true (e.g the GCC) then the internal observability inequality is true when ω is a neighborhood of Γ_0 for example for $\eta > 0$

$$\omega_\eta = \bigcup_{x \in \Gamma_0} (B(x;\eta) \cap \Omega).$$

If $N = 2$ and Ω is a rectangle it is proved in S.Jaffard Port. Math. 1990 that internal observability inequality is valid for any non empty open subset ω of Ω.

Main result. $N \leq 3$

We take potentials concentrated near Γ_0 satisfying the boundary observability inequality. We will consider the space of potentials

$$E = \{V \in H^2(\Omega),\ V\varphi_k \in H^3(\Omega) \cap H_0^1(\Omega)\}.$$

Theorem

$N \leq 3$, Ω of class $C^{3,\alpha}$ with $\alpha > 0$. Γ_0 such that the boundary observability inequality is valid, and (λ_k, φ_k) an eigenpair for the Laplace operator. We assume

(H1) λ_k is a simple eigenvalue.

(H2) $|\dfrac{\partial \varphi_k}{\partial \nu}| > 0$ on $\overline{\Gamma}_0$.

Then there exists $\delta > 0$ such that for every $y_0 \in H_\Delta(\Omega) \cap S$ with $||y_0 - \varphi_k||_{H_\Delta(\Omega)} \leq \delta$, there exists a **real** *potential $V \in C(0, T; E)$ such that the corresponding solution y satisfies $y(T) = e^{-i\lambda_k T}\varphi_k$.*

Case of a rectangle or an interval

In the case of a rectangle for $N = 2$ or an interval for $N = 1$ we have a similar result with simpler assumptions.

Theorem

$N = 2$ and Ω is a rectangle or $N = 1$ and Ω is an interval. Let (λ_k, φ_k) be an eigenpair for the Laplace operator with Dirichlet boundary conditions. We assume hypothesis (H1) in the case $N = 2$ and no hypothesis if $N = 1$ (hypothesis (H1) is automatically satisfied). Then there exists $\delta > 0$ such that for every $y_0 \in H_\Delta(\Omega) \cap S$ with $||y_0 - \varphi_k||_{H_\Delta(\Omega)} \leq \delta$, there exists a **real** *potential $V \in C([0, T]; E)$ such that the corresponding solution y satisfies*

$$y(T) = e^{-i\lambda_k T}\varphi_k.$$

Proof. Technical lemmas

Hypothesis (H1) $\Rightarrow \exists \epsilon_0$ such that $\varphi_k \neq 0$ on ω_{ϵ_0}.
For ϵ and ϵ_1 such that $0 < \epsilon_1 < \epsilon < \epsilon_0$ we define $\chi_{\omega_\epsilon} \in C_0^\infty(\omega_\epsilon \cup (\partial\omega\epsilon \cap \Gamma))$ such that

$$\begin{cases} 0 \leq \chi_{\omega_\epsilon} \leq 1, \\ \chi_{\omega_\epsilon} \geq \mathbb{1}_{\omega_{\epsilon_1}}. \end{cases}$$

We write $\omega = \omega_\epsilon$ and $\chi_\omega = \chi_{\omega_\epsilon}$.

Lemma

If $y \in H^3(\Omega) \cap H_0^1(\Omega)$ then $\chi_\omega . \dfrac{y}{\varphi_k} \in H^2(\Omega)$.

Technical lemmas

Idea of proof.
The result is local. It is enough to prove it in a (small) neighborhood still denoted by ω of any point of $\partial\omega \cap \Gamma$. We can take $\chi_\omega = 1$. Because $\varphi_k(x) > 0$ (and of course vanishes on the boundary), we can make a change of coordinates (x', ξ) where x' is the tangential coodinate and $\xi = \varphi_k(x', x_N)$. The (local) images of Γ and ω can be taken as $\{\xi = 0\}$ and $\{0 < \xi < \frac{\epsilon}{2}\}$. We can write

$$y(x', x_N) = \tilde{y}(x', \xi) = \int_0^1 \frac{d}{dt}\tilde{y}(x', t\xi)dt = \int_0^1 \frac{\partial \tilde{y}}{\partial \xi}(x', t\xi)\xi dt.$$

Therefore

$$\frac{\tilde{y}(x', \xi)}{\xi} = \int_0^1 \frac{\partial \tilde{y}}{\partial \xi}(x', t\xi)dt \in H^2(\omega).$$

Technical lemmas

Lemma

When $N \leq 3$, for any $y \in H_\Delta(\Omega)$ and $V \in E$, then $\chi_\omega.Vy \in H^3(\Omega) \cap H_0^1(\Omega)$ and the mapping

$$(y, V) \in H_\Delta(\Omega) \times E \rightarrow \chi_\omega.Vy \in H^3(\Omega) \cap H_0^1(\Omega)$$

is bilinear continuous.

Proof.
We have $H^2(\Omega) \subset L^\infty(\Omega)$ and $H^1(\Omega) \subset L^6(\Omega)$.
Easy to show that

$$\chi_\omega.Vy \in L^2(\Omega),$$
$$\frac{\partial}{\partial x_i}(\chi_\omega Vy) \in L^2(\Omega),$$
$$\frac{\partial^2}{\partial x_i \partial x_j}(\chi_\omega Vy) \in L^2(\Omega).$$

Technical lemmas

Let us show that $\dfrac{\partial^3}{\partial x_i \partial x_j \partial x_l}(\chi_\omega V y) \in L^2(\Omega)$. The only real difficulty concerns the derivatives of the term

$$\chi_\omega \frac{\partial^2 V}{\partial x_i \partial x_j} y = \frac{\partial^2 V}{\partial x_i \partial x_j} \varphi_k \chi_\omega \frac{y}{\varphi_k}.$$

We know that

$$\chi_\omega \frac{y}{\varphi_k} \in H^2(\Omega).$$

To obtain the result, let us show that

$$\frac{\partial^2 V}{\partial x_i \partial x_j} \varphi_k \in H^1(\Omega).$$

We can write

$$\frac{\partial^2 V}{\partial x_i \partial x_j} \varphi_k = \frac{\partial^2}{\partial x_i \partial x_j}(V \varphi_k) - \frac{\partial V}{\partial x_i} \frac{\partial \varphi_k}{\partial x_j} - \frac{\partial V}{\partial x_j} \frac{\partial \varphi_k}{\partial x_i} - V \frac{\partial^2 \varphi_k}{\partial x_i \partial x_j}$$

and in the right hand side, each term belongs to $H^1(\Omega)$.

Existence result

Lemma

For any $y_0 \in H_\Delta(\Omega) \cap S$ and $V \in C([0,T];E)$, with V real, there exists a unique solution $y \in C([0,T]; H_\Delta(\Omega) \cap S)$ to the Schrödinger equation

$$\left\{ \begin{array}{l} i\frac{\partial y}{\partial t} + \Delta y + \chi_\omega V y = 0 \, in \, \Omega \times (0,T), \\ y = 0 \, on \, \Gamma \times (0,T), \\ y(0) = y_0 \, in \, \Omega. \end{array} \right.$$

Proof.

Take $\tilde{y} \in C([0,T]; H_\Delta(\Omega))$. We know that $\chi_\omega V \tilde{y} \in C([0,T]; H^3(\Omega) \cap H^1_0(\Omega))$. Define z as the solution of

$$\left\{ \begin{array}{l} i\frac{\partial z}{\partial t} + \Delta z + \chi_\omega V \tilde{y} = 0 \text{ in } \Omega \times (0,T), \\ z = 0 \text{ on } \Gamma \times (0,T), \\ z(0) = y_0 \text{ in } \Omega. \end{array} \right.$$

Existence result

From the regularity result (J-P.P. Rev. Compl.), we have in fact $z \in C([0,T]; H_\Delta(\Omega))$ and

$$||z||_{C([0,T];H_\Delta(\Omega))} \le C(||y_0||_{H_\Delta(\Omega)} + ||V\tilde{y}||_{L^2(0,T;H^3(\Omega)\cap H_0^1(\Omega))})$$
$$\le C(||y_0||_{H_\Delta(\Omega)} + \sqrt{T}||V||_{C([0,T];E)}||\tilde{y}||_{C([0,T];H_\Delta(\Omega))}).$$

Classical fixed point method for T small but independent of the initial value, then by simple iterations for any $T > 0$ we obtain a solution

$$y \in C([0,T]; H_\Delta(\Omega)).$$

If $y_0 \in S$, then as V is real, the equation preserves the L^2 norm and $y(t) \in S$ for $t \in [0,T]$.
We can now define the mapping Λ by

$$\Lambda(y_0, V) = (y(T), y_0)$$

and Λ maps continuously $H_\Delta(\Omega) \cap S \times C([0,T]; E)$ into $(H_\Delta(\Omega) \cap S)^2$.
We have

$$\Lambda(\varphi_k, 0) = (\tilde{\varphi}_k(T), \varphi_k) = (e^{-i\lambda_k T}\varphi_k, \varphi_k).$$

Mapping Λ

Define

$$TS_k = \{z \in L^2(\Omega), \mathrm{Re}\,(z, \varphi_k)_{L^2(\Omega)} = 0\}$$

Lemma

The mapping Λ is differentiable at $(\varphi_k, 0)$ and for any $z_0 \in H_\Delta(\Omega) \cap TS_k$ and $W \in C([0,T]; E)$ we have

$$\Lambda'(\varphi_k, 0)[z_0, W] = (\tilde{z}(T), z_0)$$

where

$$\begin{cases} i\frac{\partial \tilde{z}}{\partial t} + \Delta \tilde{z} + \chi_\omega W \tilde{\varphi}_k = 0 \, in \, \Omega \times (0,T), \\ \tilde{z} = 0 \, on \, \Gamma \times (0,T), \\ \tilde{z}(0) = z_0 \, in \, \Omega. \end{cases}$$

Moreover, $\tilde{z}(T) \in H_\Delta(\Omega)$ and $\mathrm{Re}\,(\tilde{z}(T), \tilde{\varphi}_k(T))_{L^2(\Omega)} = 0$.

Strategy

In order to apply an inverse mapping theorem we want to show that $\Lambda'(\varphi_k, 0)$ has a continuous right inverse. Writing

$$z(t) = e^{i\lambda_k t}\tilde{z}(t).$$

it means that we have to solve the null controllability problem for

$$\begin{cases} i\frac{\partial z}{\partial t} + \Delta z + \lambda_k z + \chi_\omega W \varphi_k = 0 \text{ in } \Omega \times (0, T), \\ z = 0 \text{ on } \Gamma \times (0, T), \\ z(0) = z_0 \text{ in } \Omega. \end{cases}$$

More precisely, we want to show that for any $z_0 \in H_\Delta(\Omega) \cap TS_k$, we can find a **real** potential (control) $W \in C([0, T]; E)$ such that the solution z satisfies

$$z(T) = 0.$$

Two difficulties : to find a **real** control W and to show that W can be taken with sufficient regularity.

Existence of a real control

The following result is essentially due to K.Beauchard-C.Laurent.

Proposition

Let $T > 0$. For every $z_0 \in TS_k$, there exists a **real** *control $g \in C([0,T]; L^2(\Omega))$ such that if z is the solution of*

$$\begin{cases} i\frac{\partial z}{\partial t} + \Delta z + \lambda_k z + \chi_\omega g = 0 \, in \, \Omega \times (0,T), \\ z = 0 \, on \, \Gamma \times (0,T), \\ z(0) = z_0 \, in \, \Omega. \end{cases}$$

then we have

$$z(T) = 0.$$

The proof requires several steps.

Real control. Strategy

Let us consider the adjoint equation

$$\begin{cases} i\frac{\partial\psi}{\partial t} + \Delta\psi + \lambda_k\psi = 0 \text{ in } \Omega \times (0, T), \\ \psi = 0 \text{ on } \Gamma \times (0, T), \\ \psi(0) = \psi_0 \text{ in } \Omega, \end{cases}$$

ψ and $\hat{\psi}$ will correspond to initial values ψ_0 and $\hat{\psi}_0$.t
We take T_0 such that $0 < T_0 < T$ and $\delta > 0$ such that $4\delta \leq (T - T_0)$ and we define a function $\eta \in C_0^\infty(\mathbb{R})$ such that

$$\begin{cases} 0 \leq \eta(t) \leq 1, \ \forall t \in \mathbb{R}, \\ \eta(t) = 1, \ \forall t \in [2\delta, T - 2\delta], \\ \text{Supp}(\eta) = [\delta, T - \delta], \ \eta(t) \neq 0 \text{ for } t \in (\delta, T - \delta). \end{cases}$$

Real control. Strategy

We define w as the solution of the (backward) equation

$$\left\{\begin{array}{l} i\frac{\partial w}{\partial t} + \Delta w + \lambda_k w + \eta\chi_\omega \mathrm{Im}\,\psi = 0 \text{ in } \Omega \times (0, T), \\ w = 0 \text{ on } \Gamma \times (0, T), \\ w(T) = 0 \text{ in } \Omega. \end{array}\right.$$

We have $w \in C([0, T]; L^2(\Omega))$ and $w(0) \in TS_k$. Multipying by $\hat{\psi}$ and taking the imaginary part we obtain

$$-\mathrm{Re}\,(w(0), \hat{\psi}_0)_{L^2(\Omega)} = \int_{\omega\times(0,T)} \eta\chi_\omega \mathrm{Im}\,\psi \mathrm{Im}\,\hat{\psi} dxdt = a(\psi_0, \hat{\psi}_0)$$

To solve our problem we want to find $\psi_0 \in TS_k$ such that $w(0) = z_0$ or equivalently a solution $\psi_0 \in TS_k$ of the variational problem

$$a(\psi_0, \hat{\psi}_0) = -\mathrm{Re}\,\int_\Omega z_0 \bar{\hat{\psi}}_0 dx, \;\forall \hat{\psi}_0 \in TS_k$$

Real control. Strategy

From Lax-Milgram Theorem this will be the case If we can prove a coercivity inequality of the form

$$\exists C > 0,\ \forall \psi_0 \in TS_k,\ |\psi_0|^2_{L^2(\Omega)} \leq Ca(\psi_0, \psi_0) = C \int_{\omega\times(0,T)} \eta\chi_\omega |\mathrm{Im}\,\psi|^2 dxdt,$$

This will be done in two steps.

Lemma

There exists $C > 0$ such that for every $\psi_0 \in L^2(\Omega)$

$$|\psi_0|^2_{L^2(\Omega)} \leq C \int_{\omega\times(0,T)} \eta\chi_\omega |\mathrm{Im}\,\psi|^2 dxdt + C||\psi_0||^2_{H^{-2}(\Omega)}.$$

Real control. Proof

We write

$$2\mathrm{Im}\,\psi = \frac{\psi - \bar{\psi}}{i}$$

$$|\psi|^2 = 2|\mathrm{Im}\,\Psi|^2 + \frac{1}{2}((\psi)^2) + (\bar{\psi})^2).$$

From the internal observability inequality we have

$$|\psi_0|^2_{L^2(\Omega)} \leq C \int_{\omega\times(2\delta,T_0+2\delta)} \chi_\omega |\psi|^2 dxdt \leq C \int_{\omega\times(0,T)} \eta\chi_\omega |\psi|^2 dxdt$$

$$\leq 2C \int_{\omega\times(0,T)} \eta\chi_\omega |\mathrm{Im}\,\psi|^2 dxdt$$

$$+\frac{C}{2}\Big|\int_{\omega\times(0,T)} \eta\chi_\omega (\psi)^2 dxdt\Big| + \frac{C}{2}\Big|\int_{\omega\times(0,T)} \eta\chi_\omega (\bar{\psi})^2 dxdt\Big|.$$

Let us show that (analogous for $\bar{\psi}$)

$$\Big|\int_{\omega\times(0,T)} \eta\chi_\omega (\psi)^2 dxdt\Big| \leq C||\psi_0||^2_{H^{-2}(\Omega)}$$

Real control. Proof

If

$$\psi_0 = \sum_{j=1}^{+\infty} a_j \varphi_j$$

we have

$$\psi(t) = \sum_{j=1}^{+\infty} a_j e^{-i(\lambda_j - \lambda_k)t} \varphi_j.$$

Therefore

$$\int_{\omega\times(0,T)} \eta \chi_\omega (\psi)^2 dxdt = \sum_{j,l=1}^{+\infty} \int_{\omega\times(0,T)} (\eta e^{2i\lambda_k t}) e^{-i(\lambda_j+\lambda_l)t} a_j a_l \varphi_j \varphi_l dxdt$$

Now we can integrate by parts $2m$ times the term $\int_0^T (\eta e^{2i\lambda_k t}) e^{-i(\lambda_j+\lambda_l)t} dt$ to obtain

$$\left| \int_0^T (\eta e^{2i\lambda_k t}) e^{-i(\lambda_j+\lambda_l)t} dt \right| \leq \frac{C}{(\lambda_j + \lambda_l)^{2m}}$$

Real control. Proof

We then obtain

$$|\int_{\omega\times(0,T)} \eta\chi_\omega(\psi)^2 dxdt| \leq C \sum_{j,l=1}^{+\infty} \int_\Omega \frac{|a_j||a_l|}{(\lambda_j+\lambda_l)^{2m}} |\varphi_j||\varphi_l| dx$$

$$\leq C \sum_{j,l=1}^{+\infty} \frac{|a_j|}{\lambda_j^m} \frac{|a_l|}{\lambda_l^m} \leq C (\sum_{j=1}^{+\infty} \frac{|a_j|}{\lambda_j^m})^2$$

$$\leq C (\sum_{j=1}^{+\infty} \frac{|a_j|^2}{\lambda_j^2})(\sum_{j=1}^{+\infty} \frac{1}{\lambda_j^{(2m-2)}}) \leq C ||\psi_0||^2_{H^{-2}(\Omega)}$$

from Weyl's Theorem if we choose m large enough.

Real control. Proof

The second step requires $\psi_0 \in TS_k$.

Lemma

There exists $C > 0$ such that for every $\psi_0 \in TS_k$,

$$|\psi_0|^2_{L^2(\Omega)} \leq C \int_{\omega \times (0,T)} \eta \chi_\omega |\mathrm{Im}\,\psi|^2 dxdt = Ca(\psi_0, \psi_0).$$

This is done using a compactness uniqueness argument. We have to show that the set

$$K = \{\psi_0 \in TS_k,\ a(\psi_0, \psi_0) = 0\}$$

is reduced to $\{0\}$.
K is finite dimensional and the operator $-i(\Delta + \lambda_k I)$ maps K into K and is antisymmetric. It can be diagonalized and then we can show that all eigenfunctions have to be equal to 0.

Regularity of control

Now we want to show that we have regularity on our control. This is done following the lines of S.Ervedoza-E.Zuazua DCDS 2010.

Lemma

If $z_0 \in H_\Delta(\Omega) \cap TS_k$, then the solution ψ_0 satisfies

$$\psi_0 \in H_\Delta(\Omega) \cap TS_k.$$

This implies that $\eta\chi_\omega \mathrm{Im}\,\psi \in C([0,T]; H^3(\Omega) \cap H^1_0(\Omega))$.

if $A = -(\Delta + \lambda_k I)$, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$ and $H_\Delta(\Omega) = D(A^{\frac{3}{2}})$.
If $z_0 \in D(A)$, taking for $\tau > 0$ small enough

$$\hat{\psi}_0 = \frac{\psi(\tau) - 2\psi_0 + \psi(-\tau)}{\tau^2}$$

we have

$$\hat{\psi}(t) = \frac{\psi(t+\tau) - 2\psi(t) + \psi(t-\tau)}{\tau^2}.$$

Regularity. Proof

We consider H defined by

$$H = \int_{\omega\times(0,T)} \eta\chi_\omega \mathrm{Im}\,\psi \mathrm{Im}\,\hat{\psi} dxdt = a(\psi_0, \hat{\psi}_0).$$

On the one hand we have

$$H = \mathrm{Re}\,(z_0, \frac{\psi(\tau) - 2\psi_0 + \psi(-\tau)}{\tau^2})_{L^2(\Omega)}$$

$$= \mathrm{Re}\,(z_0, \frac{\psi(\tau) - \psi_0}{\tau^2})_{L^2(\Omega)} - \mathrm{Re}\,(z_0, \frac{\psi_0 - \psi(-\tau)}{\tau^2})_{L^2(\Omega)}.$$

This implies

$$H \leq C|Az_0|_{L^2(\Omega)}|\frac{\psi(\tau) - \psi_0}{\tau}|_{L^2(\Omega)}.$$

On the other hand, using the observability inequality, it can be shown that

$$|\frac{\psi(\tau) - \psi_0}{\tau}|^2_{L^2(\Omega)} \leq C|z_0|_{L^2(\Omega)}|\frac{\psi(\tau) - \psi_0}{\tau}|_{L^2(\Omega)} - CH.$$

Regularity. Proof

Therefore

$$\forall \tau > 0 \,\text{small enough,}\; |\frac{\psi(\tau) - \psi_0}{\tau}|^2_{L^2(\Omega)} \leq C||z_0||^2_{D(A)}$$

which implies

$$\psi_0 \in D(A) \text{ and } ||\psi_0||_{D(A)} \leq C||z_0||_{D(A)}.$$

We can iterate this process exactly in the same way to show that when $z_0 \in D(A^2) \cap TS_k$, then $\psi_0 \in D(A^2)$ and

$$||\psi_0||_{D(A^2)} \leq C||z_0||_{D(A^2)}.$$

Regularity. Proof

By interpolation we see that when $z_0 \in D(A^{\frac{3}{2}}) \cap TS_k$ then $\psi_0 \in D(A^{\frac{3}{2}})$ and

$$||\psi_0||_{D(A^{\frac{3}{2}})} \leq C||z_0||_{D(A^{\frac{3}{2}})}.$$

As $H_\Delta(\Omega) = D(A^{\frac{3}{2}})$ we see that $z_0 \in H_\Delta(\Omega) \cap TS_k$ implies $\psi_0 \in H_\Delta(\Omega)$ which itself implies $\psi \in C([0, T]; H_\Delta(\Omega))$ and therefore $\eta\chi_\omega \, \mathrm{Im}\, \psi \in C([0, T]; H^3(\Omega) \cap H_0^1(\Omega))$.
We can then write the control in the form

$$\eta\chi_\omega \, \mathrm{Im}\, \psi = \eta\chi_\omega W \varphi_k, \text{ with } W \in C([0, T]; E).$$

End of proof

this proves that the derivative of the mapping Λ defined in (22) at the point $(\varphi_k, 0)$ has a continuous right inverse.
Using an inverse mapping theorem, we can find a neighborhood $\mathcal{U}_0$ of 0 in $C([0,T];E)$ and a neighborhood $\mathcal{U}_1$ of $(e^{-i\lambda T}\varphi_k, \varphi_k)$ in $H_\Delta(\Omega)^2$ such that for any $(y_0, y_1) \in \mathcal{U}_1$, there exists $V \in \mathcal{U}_0$ such that the solution y of

$$\begin{cases} i\frac{\partial y}{\partial t} + \Delta y + \eta\chi_\omega V y = 0 \text{ in } \Omega \times (0,T), \\ y = 0 \text{ on } \Gamma \times (0,T), \\ y(0) = y_0 \text{ in } \Omega \end{cases} \tag{1}$$

satisfies

$$y(T) = y_1.$$

THANK YOU FOR YOUR ATTENTION.

HAPPY BIRTHDAY JEAN-MICHEL !

WELCOME TO THE VERSION 6.0 OF LIFE !