Local null-controllability of the 2-D Vlasov-Navier-Stokes system

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The VNS system for (f, u) contains:

- a Vlasov (transport) equation for f,
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- coupling terms relying f and u.

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Let $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, $T > 0$.

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Analysis of the VNS system:

- Mean-field approach: L. Desvillettes, F. Golse and V. Ricci. (J. Stat. Phys., 2008)
- Weak solutions: L. Boudin, L. Desvillettes, C. Grandmont and A. Moussa (DIE, 2008),
- Hydrod. lim.: T. Goudon, P.E. Jabin, A. Vasseur (IUMJ,'08).

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HOW? We want to absorb particles from a subset of $\mathbb{T}^2$, $\omega \subset \mathbb{T}^2$. This amounts to use an internal control in the Vlasov equation, located in the absorption region $\omega \subset \mathbb{T}^2$.

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Goal: an exact controllability result

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$$\begin{array}{ccc} t = 0 & & t = T \\ (f_0, u_0) & \underset{G}{\longrightarrow} & (f_1, u_1) \end{array}$$

We shall answer positively to this question

- for X, U spaces of regular functions (Hölder),
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- for $\omega \subset \mathbb{T}^2$ satisfying a geometric assumption.

Main result: Local null-controllability result

THEOREM (IM, 2016)

Let $\gamma > 2$ and let $\omega \subset \mathbb{T}^2$ satisfy the STRIP ASSUMPTION.
$\exists\ \epsilon > 0$, $M > 0$, $T_0 > 0$ *such that* $\forall T \geq T_0$, *and for every* $f_0 \in \mathscr{C}^1(\mathbb{T}^2 \times \mathbb{R}^2) \cap W^{1,\infty}(\mathbb{T}^2 \times \mathbb{R}^2)$ *and* u_0 *satisfying that*

$$u_0 \in \mathscr{C}^1(\mathbb{T}^2;\mathbb{R}^2) \cap H^2(\mathbb{T}^2;\mathbb{R}^2), \quad \mathrm{div}_x\, u_0 = 0, \quad \|u_0\|_{H^{\frac{1}{2}}(\mathbb{T}^2)} \leq M,$$

$$\|f_0\|_{\mathscr{C}^1(\mathbb{T}^2\times\mathbb{R}^2)} + \|(1+|v|)^{\gamma+2} f_0\|_{\mathscr{C}^0(\mathbb{T}^2\times\mathbb{R}^2)} \leq \epsilon,$$

$$\exists \kappa > 0, \quad \sup_{\mathbb{T}^2\times\mathbb{R}^2} (1+|v|)^{\gamma} \left(|\nabla_x f_0| + |\nabla_v f_0|\right)(x,v) \leq \kappa,$$

there exists a control $G \in \mathscr{C}^0([0,T]\times\mathbb{T}^2\times\mathbb{R}^2)$ *such that a* STRONG SOLUTION *of (VNS) with* $f|_{t=0} = f_0$ *and* $u|_{t=0} = u_0$ *exists, is unique and satisfies*

$$f|_{t=T} = 0, \quad u|_{t=T} = 0.$$

Controllability of kinetic equations

VLASOV EQUATIONS: describe the dynamics of a cloud of particles $f(t,x,v)$ undergoing macroscopic forces F or collisions $\mathscr{C}$,

$$\partial_t f + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = \mathscr{C}(f).$$

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Strategy for the non-collisional cases: local results

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How to eliminate the obstructions? Method of characteristics

Given f_0 and u regular, the transport equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left[(u - v) f \right] = 0, & (0, T) \times \mathbb{T}^2 \times \mathbb{R}^2, \\ f|_{t=0} = f_0(x, v), & \mathbb{T}^2 \times \mathbb{R}^2, \end{cases}$$

has the **explicit solution**

$$f(t, x, v) = e^{2t} f_0((X, V)(0, t, x, v)),$$

where (X, V) are the characteristics associated to $-v + u$.

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- We put the velocity field back to zero thanks to the Coron-Fursikov's theorem.

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Associated distribution function: let $\mathcal{Z}_1, \mathcal{Z}_2 \in \mathscr{S}(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} v_i \mathcal{Z}_j(v)\, \mathrm{d}v = \delta_{ij}, \quad \int_{\mathbb{R}^2} \mathcal{Z}_i(v)\, \mathrm{d}v = 0, \quad i,j = 1,2.$$

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$$\begin{aligned} \overline{w} = j_{\overline{f}} - \rho_{\overline{f}} \overline{u}, &\quad (0,T) \times \mathbb{T}^2, \\ \partial_t \overline{f} + v \cdot \nabla_x \overline{f} + \mathrm{div}_v \left[(\overline{u} - v) \overline{f} \right] = 0, &\quad (0,T) \times (\mathbb{T}^2 \setminus \omega) \times \mathbb{R}^2, \\ \overline{f}|_{t=0} = 0, \quad \overline{f}|_{t=T} = 0. & \end{aligned}$$

PROPOSITION

Let $\omega \subset \mathbb{T}^2$ satisfy the strip assumption. There exists $T_0 > 0$ such that for any $T \geq T_0$, there exists a reference solution $(\overline{f}, \overline{u})$ of the VNS system such that

$$\overline{f} \in \mathscr{C}^\infty([0,T] \times \mathbb{T}^2; \mathscr{S}(\mathbb{R}^2)),$$
$$\overline{u} \in \mathscr{C}^\infty([0,T] \times \mathbb{T}^2; \mathbb{R}^2),$$
$$(\overline{f}, \overline{u})|_{t=0} = (0,0), \quad (\overline{f}, \overline{u})|_{t=T} = (0,0),$$
$$supp(\overline{f}) \subset (0,T) \times \omega \times \mathbb{R}^2,$$

and such that the characteristics associated to $\overline{u}$ satisfy

$$\forall (x,v) \in \mathbb{T}^2 \times \mathbb{R}^2, \exists t \in \left[\frac{T}{12}, \frac{11T}{12}\right] \text{ such that}$$
$$\overline{X}(t,0,x,v) \in \omega, \text{ with } |\overline{V}(t,0,x,v) \cdot n_H| \geq 5.$$

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3. $\tilde{\mathcal{V}}_\epsilon[g] \underset{\Pi-\textit{extension}}{\longmapsto} \mathcal{V}_\epsilon[g] = \bar{f} + \Pi(\tilde{\mathcal{V}}_\epsilon[g])$.

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Thus, choosing ϵ and M **small enough,** (X^{g^*}, V^{g^*}) **meet** ω.

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Thank you very much for your attention!

Joyeux anniversaire, Jean-Michel !