Analyticity of solutions to parabolic equations and observability

Coron60: Conference in honor of Jean-Michel Coron.

Santiago Montaner

University of the Basque Country
(UPV/EHU)

June, 21st 2016

A joint work with
Luis Escauriaza (UPV/EHU) and Can Zhang (UPMC Paris 6)

Interior observability inequality over open sets

The interior null-controllability property for the Heat equation is equivalent to the *interior observability*, i.e., there exists a constant $N = N(\omega, \Omega, T)$ s.t. the solution to

$$\begin{cases} \partial_t v - \Delta v = 0, & \text{in } \Omega \times (0, T], \\ v = 0, & \text{on } \partial\Omega \times (0, T], \\ v(0) = v_0. & \text{in } \Omega, \end{cases}$$

satisfies the *observability inequality*

$$\|v(T)\|_{L^2(\Omega)} \leq N \|v\|_{L^2(\omega \times (0,T))}.$$

The null-controllability property for the Heat equation and other second-order parabolic equations was obtained by Fattorini-Russell (1971), Imanuvilov, Lebeau-Robbiano (1995). Also some results for 4th-order parabolic equations by Le Rousseau-Robbiano (2015).

An interior observability inequality over measurable sets

Theorem (J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, 2014)

Let $0 < T < 1$, $\mathcal{D} \subset \Omega \times (0, T)$ ($\partial\Omega$ Lipschitz) be a measurable set, $|\mathcal{D}| > 0$. Then $\exists\ N = N(\mathcal{D}, \Omega, T)$ s.t.

$$\|u(T)\|_{L^2(\Omega)} \leq N \int_{\mathcal{D}} |u(x,t)|\, dxdt$$

holds for all solutions to

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0, T], \\ u = 0 & \text{on } \partial\Omega \times (0, T], \\ u(0) = u_0, & u_0 \in L^2(\Omega). \end{cases}$$

Null-controllability of a parabolic equations from measurable sets

Corollary (J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, 2014)

Let $0 < T < 1$ and $\mathcal{D} \subseteq \Omega \times (0, T)$ ($\partial\Omega$ Lipschitz) be a measurable set, $|\mathcal{D}| > 0$. Then for each $u_0 \in L^2(\Omega)$ exists $f \in L^\infty(\Omega \times (0, T))$ s.t.

$$\|f\|_{L^\infty(\mathcal{D})} \leq N(\mathcal{D}, \Omega, T)\|u_0\|_{L^2(\Omega)}$$

and the solution to

$$\begin{cases} \partial_t u - \Delta u = \chi_{\mathcal{D}} f, & \text{in } \Omega \times (0, T], \\ u = 0, & \text{on } \partial\Omega \times (0, T], \\ u(0) = u_0. & \text{in } \Omega, \end{cases}$$

satisfies $u(T) \equiv 0$.

In *Observation from measurable sets for parabolic analytic evolutions and applications* (Escauriaza, Montaner, Zhang (2015)), these results are extended to some equations and systems with real-analytic coefficients not depending on time such as:

- higher-order parabolic evolutions,
- strongly coupled second-order systems with a possibly non-symmetric structure,
- one-component control of a weakly coupled system of two equations,

In this work, the real-analyticity of coefficients is quantified as:

$$|\partial_x^\gamma a_\alpha(x)| \leq {\rho_0}^{-1-|\gamma|}|\gamma|! \ \text{ in } \overline{\Omega} \times [0, T].$$

The proof of these results rely on:

- An inequality of *propagation of smallness from measurable sets* by S. Vessella (1999).

The proof of these results rely on:

- An inequality of *propagation of smallness from measurable sets* by S. Vessella (1999).
- New quantitative estimates of space-time analyticity of the form

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}} \rho^{-|\gamma|-p}\, |\gamma|!\, p!\, t^{-p} \|u_0\|_{L^2(\Omega)},$$

  $0 < t \leq 1$, $\gamma \in \mathbb{N}^n$, $p \geq 0$ and $2m$ is the order of the parabolic problem solved by u. These estimates are obtained quantifying each step of a reasoning developed by Landis and Oleinik (1974) which reduces the strong UCP within characteristic hyperplanes of parabolic equations to its elliptic counterpart and is based on a spectral representation of solutions.

The proof of these results rely on:

- An inequality of *propagation of smallness from measurable sets* by S. Vessella (1999).
- New quantitative estimates of space-time analyticity of the form

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq e^{1/\rho t^{1/(2m-1)}} \rho^{-|\gamma|-p}\,|\gamma|!\,p!\,t^{-p}\|u_0\|_{L^2(\Omega)},$$

  $0 < t \leq 1$, $\gamma \in \mathbb{N}^n$, $p \geq 0$ and $2m$ is the order of the parabolic problem solved by u. These estimates are obtained quantifying each step of a reasoning developed by Landis and Oleinik (1974) which reduces the strong UCP within characteristic hyperplanes of parabolic equations to its elliptic counterpart and is based on a spectral representation of solutions.
- The so-called *telescoping series method* (L. Miller; K. D. Phung, G. Wang).

S. Vessella. *A continuous dependence result in the analytic continuation problem.* Forum Math. **11**, 6 (1999), 695–703.

Lemma. (Propagation of smallness from measurable sets)

Let $\omega \subset B_R$ be a measurable set $|\omega| > 0$. Let f be a real-analytic function in B_{2R} s.t. there exist numbers M and ρ for which

$$|\partial_x^\gamma f(x)| \leq M(\rho R)^{-|\gamma|}|\gamma|!$$

holds when $x \in B_{2R}$ and $\gamma \in \mathbb{N}^n$. Then, there are $N = N(B_R, \rho, |\omega|)$ and $\theta = \theta(B_R, \rho, |\omega|)$, $0 < \theta < 1$, such that

$$\|f\|_{L^\infty(B_R)} \leq NM^{1-\theta}\left(\frac{1}{|\omega|}\int_\omega |f|dx\right)^\theta.$$

Some remarks on the quantitative estimates

The quantitative estimate of space-time real-analyticity

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq e^{t^{-\frac{1}{2m-1}}} \rho^{-1-|\gamma|-p}\, t^{-p}\, |\gamma|!\, p! \|u_0\|_{L^2(\Omega)}$$

Some remarks on the quantitative estimates

The quantitative estimate of space-time real-analyticity

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq e^{t^{-\frac{1}{2m-1}}} \rho^{-1-|\gamma|-p}\, t^{-p}\, |\gamma|!\, p! \|u_0\|_{L^2(\Omega)}$$

- yields a positive lower bound ρ for the radius of convergence of the Taylor series in the spatial variables independent of t,

Some remarks on the quantitative estimates

The quantitative estimate of space-time real-analyticity

$$|\partial_x^\gamma \partial_t^p u(x,t)| \le e^{t^{-\frac{1}{2m-1}}} \rho^{-1-|\gamma|-p}\, t^{-p}\, |\gamma|!\, p! \|u_0\|_{L^2(\Omega)}$$

- yields a positive lower bound ρ for the radius of convergence of the Taylor series in the spatial variables independent of t,
- if $p = 0$, it blows up like $e^{t^{-\frac{1}{2m-1}}}$ when $t \to 0^+$.

Some remarks on the quantitative estimates

The quantitative estimate of space-time real-analyticity

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq e^{t^{-\frac{1}{2m-1}}} \rho^{-1-|\gamma|-p}\, t^{-p}\, |\gamma|!\, p! \|u_0\|_{L^2(\Omega)}$$

- yields a positive lower bound ρ for the radius of convergence of the Taylor series in the spatial variables independent of t,
- if $p = 0$, it blows up like $e^{t^{-\frac{1}{2m-1}}}$ when $t \to 0^+$.

These features of the quantitative estimates of analyticity are essential in the proof of the interior observability estimate over measurable sets.

Parabolic operators with time dependent coefficients

In order to deal with time-dependent coefficients, we cannot adapt the reasoning by Landis and Oleinik!

Parabolic operators with time dependent coefficients

In order to deal with time-dependent coefficients, we cannot adapt the reasoning by Landis and Oleinik!

Consider the $2m$-th order operator

$$L = \sum_{|\alpha|\leq 2m} a_\alpha(x,t)\partial_x^\alpha = \sum_{|\alpha|,|\beta|\leq m} \partial_x^\alpha (A_{\alpha\beta}(x,t)\partial_x^\beta) + \sum_{|\gamma|\leq m} A_\gamma(x,t)\partial_x^\gamma,$$

assume that for some ρ_0, $0 < \rho_0 < 1$

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha,\beta}(x,t)\xi^{\alpha+\beta} \geq \rho_0|\xi|^{2m} \quad \forall \xi \in \mathbb{R}^n, \text{ in } \overline{\Omega}\times[0,T],$$

Parabolic operators with time dependent coefficients

In order to deal with time-dependent coefficients, we cannot adapt the reasoning by Landis and Oleinik!

Consider the $2m$-th order operator

$$L = \sum_{|\alpha|\le 2m} a_\alpha(x,t)\partial_x^\alpha = \sum_{|\alpha|,|\beta|\le m} \partial_x^\alpha(A_{\alpha\beta}(x,t)\partial_x^\beta) + \sum_{|\gamma|\le m} A_\gamma(x,t)\partial_x^\gamma,$$

assume that for some ρ_0, $0<\rho_0<1$

$$\sum_{|\alpha|=|\beta|=m} A_{\alpha,\beta}(x,t)\xi^{\alpha+\beta} \ge \rho_0|\xi|^{2m} \;\; \forall \xi\in\mathbb{R}^n, \text{ in } \overline{\Omega}\times[0,T],$$

$$|\partial_x^\gamma\partial_t^p a_\alpha(x,t)| \le {\rho_0}^{-1-|\gamma|-p}|\gamma|!p! \;\; \text{in } \overline{\Omega}\times[0,T].$$

As far as we know, the best estimate that follows from the works of S. D. Eidelman, A. Friedman, D. Kinderlehrer, L. Nirenberg, G. Komatsu and H. Tanabe is:

Theorem

There is $0 < \rho \leq 1$, $\rho = \rho(\rho_0, n, \partial\Omega)$ *such that* $\forall \alpha \in \mathbb{N}^n, p \in \mathbb{N}$

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\gamma|}{2m}-p} |\gamma|!\ p!\ t^{-\frac{|\gamma|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\Omega)},$$

in $\overline{\Omega} \times (0, T]$ when u solves

$$\begin{cases} \partial_t u + (-1)^m L u = 0, & \text{in } \Omega \times (0, T], \\ u = Du = \ldots = D^{m-1} u = 0, & \text{in } \partial\Omega \times (0, T], \\ u(\cdot, 0) = u_0, & u_0 \in L^2(\Omega). \end{cases}$$

and $\partial\Omega$ is a real-analytic hypersurface.

If u satisfies

$$|\partial_x^\gamma \partial_t^p u(x,t)| \le \rho^{-1-\frac{|\gamma|}{2m}-p} |\gamma|!\ p!\ t^{-\frac{|\gamma|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\Omega)},$$
$$\forall \gamma \in \mathbb{N}^n, p \in \mathbb{N},$$

we observe that:

- the space analyticity estimate blows up as t tends to zero, which is unavoidable if u_0 is an arbitrary $L^2(\Omega)$ function;

If u satisfies

$$|\partial_x^\gamma \partial_t^p u(x,t)| \le \rho^{-1-\frac{|\gamma|}{2m}-p} |\gamma|!\ p!\ t^{-\frac{|\gamma|}{2m}-p-\frac{n}{4m}} \|u_0\|_{L^2(\Omega)},$$
$$\forall \gamma \in \mathbb{N}^n, p \in \mathbb{N},$$

we observe that:

- the space analyticity estimate blows up as t tends to zero, which is unavoidable if u_0 is an arbitrary $L^2(\Omega)$ function;
- for each fixed $t > 0$, the radius of convergence in the space variable is greater than or equal to $\sqrt[2m]{\rho t}$.

If u satisfies

$$|\partial_x^\gamma \partial_t^p u(x,t)| \leq \rho^{-1-\frac{|\gamma|}{2m}-p}|\gamma|!\; p!\; t^{-\frac{|\gamma|}{2m}-p-\frac{n}{4m}}\|u_0\|_{L^2(\Omega)},$$
$$\forall \gamma \in \mathbb{N}^n, p \in \mathbb{N},$$

we observe that:

- the space analyticity estimate blows up as t tends to zero, which is unavoidable if u_0 is an arbitrary $L^2(\Omega)$ function;
- for each fixed $t > 0$, the radius of convergence in the space variable is greater than or equal to $\sqrt[2m]{\rho t}$.

This estimate is useless for applications to observability inequalities from measurable sets.

Main result

Theorem (L. Escauriaza, S. Montaner, C. Zhang, 2015)

Let $T \in (0,1]$ *and* $\partial\Omega$ *be a real-analytic hypersurface. There are constants* ρ *and* N *s.t. for any* $\alpha \in \mathbb{N}^n$ *and* $p \in \mathbb{N}$

$$|\partial_x^\alpha \partial_t^p u(x,t)| \leq N e^{N t^{-\frac{1}{2m-1}}} \rho^{-|\alpha|-p} t^{-p} |\alpha|! p! \|u\|_{L^2(\Omega\times(0,T))} \text{ in } \overline{\Omega} \times (0, T],$$

if u *solves*

$$\begin{cases} \partial_t u + (-1)^m L u = 0, & \text{in } \Omega \times (0, T], \\ u = Du = \ldots = D^{m-1} u = 0 & \text{in } \partial\Omega \times (0, T], \\ u(0) = u_0, & u_0 \in L^2(\Omega). \end{cases}$$

This estimate is adequate to prove the interior observabililty estimate over measurable sets when the coefficients of L are space-time real-analytic.

Idea of the proof of the quantitative estimates of analyticity: case of 2nd order equations

We prove a L^2 estimate by induction on $|\gamma|$ and p, let $B_r \subseteq B_1 \subseteq$ s.t. $B_r \cap \overline{\Omega} \neq \emptyset$:

$$(1-r)^2 \| t^{p+1} e^{-\frac{\theta}{t}} \partial_t^{p+1} \partial_x^\gamma u \|_{L^2(\Omega \cap B_r \times (0,T)} + \sum_{k=0}^{2} (1-r)^k \| t^{p+\frac{k}{2}} e^{-\frac{\theta}{t}} D^k \partial_t^p \partial_x^\gamma u \|_{L^2(\Omega \cap B_r \times (0,T))} \leq \rho^{-1-|\gamma|-p} \theta^{-\frac{|\gamma|}{2}} (1-r)^{-|\gamma|} |\gamma|! p! \|u\|_{L^2(\Omega \times (0,T))}. \quad (1)$$

Idea of the proof of the quantitative estimates of analyticity: case of 2nd order equations

We prove a L^2 estimate by induction on $|\gamma|$ and p, let $B_r \subseteq B_1 \subseteq$ s.t. $B_r \cap \overline{\Omega} \neq \emptyset$:

$$(1-r)^2 \| t^{p+1} e^{-\frac{\theta}{t}} \partial_t^{p+1} \partial_x^\gamma u \|_{L^2(\Omega \cap B_r \times (0,T)} + \sum_{k=0}^{2} (1-r)^k \, \| t^{p+\frac{k}{2}} e^{-\frac{\theta}{t}} D^k \partial_t^p \partial_x^\gamma u \|_{L^2(\Omega \cap B_r \times (0,T))} \leq \rho^{-1-|\gamma|-p} \theta^{-\frac{|\gamma|}{2}} (1-r)^{-|\gamma|} |\gamma|! p! \|u\|_{L^2(\Omega \times (0,T))}. \quad (1)$$

The precise form of the weights $t^{p+1} e^{-\frac{\theta}{t}}$ is crucial to obtain:

Idea of the proof of the quantitative estimates of analyticity: case of 2nd order equations

We prove a L^2 estimate by induction on $|\gamma|$ and p, let $B_r \subseteq B_1 \subseteq$ s.t. $B_r \cap \overline{\Omega} \neq \emptyset$:

$$(1-r)^2 \| t^{p+1} e^{-\frac{\theta}{t}} \partial_t^{p+1} \partial_x^{\gamma} u \|_{L^2(\Omega \cap B_r \times (0,T)} + \sum_{k=0}^{2} (1-r)^k \, \| t^{p+\frac{k}{2}} e^{-\frac{\theta}{t}} D^k \partial_t^p \partial_x^{\gamma} u \|_{L^2(\Omega \cap B_r \times (0,T))} \leq \rho^{-1-|\gamma|-p} \theta^{-\frac{|\gamma|}{2}} (1-r)^{-|\gamma|} |\gamma|! p! \| u \|_{L^2(\Omega \times (0,T))}. \quad (1)$$

The precise form of the weights $t^{p+1} e^{-\frac{\theta}{t}}$ is crucial to obtain:

- the lower bound $\rho \theta^{\frac{1}{2}} (1-r)$, (not depending on t) for the spatial radius of convergence of the Taylor series of u.

Idea of the proof of the quantitative estimates of analyticity: case of 2nd order equations

We prove a L^2 estimate by induction on $|\gamma|$ and p, let $B_r \subseteq B_1 \subseteq$ s.t. $B_r \cap \overline{\Omega} \neq \emptyset$:

$$(1-r)^2 \| t^{p+1} e^{-\frac{\theta}{t}} \partial_t^{p+1} \partial_x^{\gamma} u \|_{L^2(\Omega \cap B_r \times (0,T)}$$
$$+ \sum_{k=0}^{2} (1-r)^k \, \| t^{p+\frac{k}{2}} e^{-\frac{\theta}{t}} D^k \partial_t^p \partial_x^{\gamma} u \|_{L^2(\Omega \cap B_r \times (0,T))}$$
$$\leq \rho^{-1-|\gamma|-p} \theta^{-\frac{|\gamma|}{2}} (1-r)^{-|\gamma|} |\gamma|! p! \|u\|_{L^2(\Omega \times (0,T))}. \quad (1)$$

The precise form of the weights $t^{p+1} e^{-\frac{\theta}{t}}$ is crucial to obtain:

- the lower bound $\rho \theta^{\frac{1}{2}} (1-r)$, (not depending on t) for the spatial radius of convergence of the Taylor series of u.
- the adequate factors $|\gamma|! p!$ in the right hand side of (1).

This allows us to prove

$$\|t^{p+1}e^{-\frac{\theta}{t}}\partial_t^p\partial_x^\gamma u\|_{L^2(B_{1/2}\times(0,T)} \leq \rho^{-1-|\gamma|-p}\theta^{-\frac{|\gamma|}{2}}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))},$$

This allows us to prove

$$\|t^{p+1}e^{-\frac{\theta}{t}}\partial_t^p\partial_x^\gamma u\|_{L^2(B_{1/2}\times(0,T)} \leq \rho^{-1-|\gamma|-p}\theta^{-\frac{|\gamma|}{2}}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))},$$

therefore for some $N > 0$ and ρ, $0 < \rho < 1$

$$\|\partial_t^p\partial_x^\gamma u\|_{L^2(B_{1/2}\times(T/2,T))} \leq e^{\frac{N}{T}}\rho^{-1-|\gamma|-p}T^{-p}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))},$$

This allows us to prove

$$\|t^{p+1}e^{-\frac{\theta}{t}}\partial_t^p\partial_x^\gamma u\|_{L^2(B_{1/2}\times(0,T)} \leq \rho^{-1-|\gamma|-p}\theta^{-\frac{|\gamma|}{2}}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))},$$

therefore for some $N>0$ and ρ, $0<\rho<1$

$$\|\partial_t^p\partial_x^\gamma u\|_{L^2(B_{1/2}\times(T/2,T))} \leq e^{\frac{N}{T}}\rho^{-1-|\gamma|-p}T^{-p}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))},$$

and using Sobolev's embedding:

$$|\partial_x^\gamma\partial_t^p u(x,t)| \leq e^{\frac{N}{t}}\rho^{-1-|\gamma|-p}t^{-p}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))}$$

in $B_{1/4}\times(0,T]$ for some ρ, $0<\rho<1$.

This allows us to prove

$$\|t^{p+1}e^{-\frac{\theta}{t}}\partial_t^p\partial_x^\gamma u\|_{L^2(B_{1/2}\times(0,T)} \leq \rho^{-1-|\gamma|-p}\theta^{-\frac{|\gamma|}{2}}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))},$$

therefore for some $N > 0$ and ρ, $0 < \rho < 1$

$$\|\partial_t^p\partial_x^\gamma u\|_{L^2(B_{1/2}\times(T/2,T))} \leq e^{\frac{N}{T}}\rho^{-1-|\gamma|-p}T^{-p}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))},$$

and using Sobolev's embedding:

$$|\partial_x^\gamma\partial_t^p u(x,t)| \leq e^{\frac{N}{t}}\rho^{-1-|\gamma|-p}t^{-p}|\gamma|!p!\|u\|_{L^2(\Omega\times(0,T))}$$

in $B_{1/4} \times (0, T]$ for some ρ, $0 < \rho < 1$.

This finishes the proof of space-time analyticity in the *interior* of Ω.

Observability estimate: case of 2nd order equations

Let $t \in (0, T)$, we set

$$\mathcal{D}_t = \{x \in \Omega : (x, t) \in \mathcal{D}\} \quad , \quad E = \{t \in (0, T) : |\mathcal{D}_t| \geq |\mathcal{D}|/(2T)\}.$$

Let $t \in (0,T)$, we set

$$\mathcal{D}_t = \{x \in \Omega : (x,t) \in \mathcal{D}\} \quad , \quad E = \{t \in (0,T) : |\mathcal{D}_t| \geq |\mathcal{D}|/(2T)\}.$$

Now *analyticity estimates*,

Observability estimate: case of 2nd order equations

Let $t \in (0, T)$, we set

$$\mathcal{D}_t = \{x \in \Omega : (x,t) \in \mathcal{D}\} \quad , \quad E = \{t \in (0,T) : |\mathcal{D}_t| \geq |\mathcal{D}|/(2T)\}.$$

Now *analyticity estimates*, *propagation of smallness from measurable sets*,

Observability estimate: case of 2nd order equations

Let $t \in (0, T)$, we set

$$\mathcal{D}_t = \{x \in \Omega : (x,t) \in \mathcal{D}\} \quad , \quad E = \{t \in (0,T) : |\mathcal{D}_t| \geq |\mathcal{D}|/(2T)\}.$$

Now *analyticity estimates*, *propagation of smallness from measurable sets*, and *energy inequality*

Let $t \in (0, T)$, we set

$$\mathcal{D}_t = \{x \in \Omega : (x, t) \in \mathcal{D}\} \quad , \quad E = \{t \in (0, T) : |\mathcal{D}_t| \geq |\mathcal{D}|/(2T)\}.$$

Now *analyticity estimates*, *propagation of smallness from measurable sets*, and *energy inequality* imply

$$\exists\ N = N(\Omega, |\mathcal{D}|/T, \rho) \text{ and } \theta = \theta(\Omega, |\mathcal{D}|/T, \rho) \in (0, 1)$$

such that

$$\|u(T_2)\|_{L^2(\Omega)} \leq \left(N e^{\frac{N}{T_2 - T_1}} \int_{E \cap (T_1, T_2)} \|u(t)\|_{L^1(\mathcal{D}_t)}\, dt \right)^{\theta} \|u(T_1)\|_{L^2(\Omega)}^{1-\theta}$$

for any two times T_1 and T_2 such that $0 < T_1 < T_2 \leq T$.

Given a density point $l \in E$ and a number $z > 1$ we can find a monotone decreasing sequence $l < \ldots < l_{k+1} < l_k < \ldots < l_1 \leq T$ such that

$$l_k - l_{k+1} = z(l_{k+1} - l_{k+2}), \qquad |E \cap (l_{k+1}, l_k)| \geq \frac{1}{3}(l_k - l_{k+1}).$$

Setting $T_2 = l_k$ and $T_1 = l_{k+1}$ in

$$\|u(T_2)\|_{L^2(\Omega)} \leq \left(N e^{\frac{N}{T_2 - T_1}} \int_{E \cap (T_1, T_2)} \|u(t)\|_{L^1(\mathcal{D}_t)} \, dt \right)^{\theta} \|u(T_1)\|_{L^2(\Omega)}^{1-\theta},$$

it turns into

$$\|u(l_k)\|_{L^2(\Omega)} \leq \left(N e^{\frac{N}{l_k - l_{k+1}}} \int_{E \cap (l_{k+1}, l_k)} \|u(t)\|_{L^1(\mathcal{D}_t)} \, dt \right)^{\theta} \|u(l_{k+1})\|_{L^2(\Omega)}^{1-\theta}.$$

We write the previous inequality as

$$A_k \leq e^{\frac{N}{l_k - l_{k+1}}} B_k^{\theta} A_{k+1}^{1-\theta}$$

We write the previous inequality as

$$A_k \leq e^{\frac{N}{l_k - l_{k+1}}} B_k^{\theta} A_{k+1}^{1-\theta} \leq e^{\frac{N}{l_k - l_{k+1}}} B_k \varepsilon^{-\theta} + \varepsilon^{1-\theta} A_{k+1}$$

We write the previous inequality as

$$A_k \leq e^{\frac{N}{l_k - l_{k+1}}} B_k^{\theta} A_{k+1}^{1-\theta} \leq e^{\frac{N}{l_k - l_{k+1}}} B_k \varepsilon^{-\theta} + \varepsilon^{1-\theta} A_{k+1},$$

where

$$A_k = \|u(l_k)\|_{L^2(\Omega)}, \qquad B_k = \int_{E\cap(l_{k+1},l_k)} \|u(t)\|_{L^1(\mathcal{D}_t)}\, dt.$$

We write the previous inequality as

$$A_k \le e^{\frac{N}{l_k - l_{k+1}}} B_k^{\theta} A_{k+1}^{1-\theta} \le e^{\frac{N}{l_k - l_{k+1}}} B_k \varepsilon^{-\theta} + \varepsilon^{1-\theta} A_{k+1},$$

where

$$A_k = \|u(l_k)\|_{L^2(\Omega)}, \qquad B_k = \int_{E\cap(l_{k+1},l_k)} \|u(t)\|_{L^1(\mathcal{D}_t)}\, dt.$$

Using $l_{k+1} - l_k = z(l_{k+1} - l_{k+2})$ we arrive to

$$\varepsilon^{\theta} A_k e^{-\frac{N}{l_k - l_{k+1}}} - \varepsilon A_{k+1} e^{-\frac{N}{z(l_{k+1} - l_{k+2})}} \le B_k.$$

We write the previous inequality as

$$A_k \leq e^{\frac{N}{l_k - l_{k+1}}} B_k^{\theta} A_{k+1}^{1-\theta} \leq e^{\frac{N}{l_k - l_{k+1}}} B_k \varepsilon^{-\theta} + \varepsilon^{1-\theta} A_{k+1},$$

where

$$A_k = \|u(l_k)\|_{L^2(\Omega)}, \qquad B_k = \int_{E\cap(l_{k+1},l_k)} \|u(t)\|_{L^1(\mathcal{D}_t)}\, dt.$$

Using $l_{k+1} - l_k = z(l_{k+1} - l_{k+2})$ we arrive to

$$\varepsilon^{\theta} A_k e^{-\frac{N}{l_k - l_{k+1}}} - \varepsilon A_{k+1} e^{-\frac{N}{z(l_{k+1} - l_{k+2})}} \leq B_k.$$

A suitable choice of z and ε yields a *telescoping series*:

$$e^{-\frac{N}{l_k - l_{k+1}}} A_k - e^{-\frac{N}{l_{k+1} - l_{k+2}}} A_{k+1} \leq B_k, \quad \|u(l_1)\|_{L^2(\Omega)} = A_1 \leq \sum_{k=1}^{\infty} B_k.$$

We write the previous inequality as

$$A_k \leq e^{\frac{N}{l_k - l_{k+1}}} B_k^{\theta} A_{k+1}^{1-\theta} \leq e^{\frac{N}{l_k - l_{k+1}}} B_k \varepsilon^{-\theta} + \varepsilon^{1-\theta} A_{k+1},$$

where

$$A_k = \|u(l_k)\|_{L^2(\Omega)}, \qquad B_k = \int_{E\cap(l_{k+1},l_k)} \|u(t)\|_{L^1(\mathcal{D}_t)}\, dt.$$

Using $l_{k+1} - l_k = z(l_{k+1} - l_{k+2})$ we arrive to

$$\varepsilon^{\theta} A_k e^{-\frac{N}{l_k - l_{k+1}}} - \varepsilon A_{k+1} e^{-\frac{N}{z(l_{k+1} - l_{k+2})}} \leq B_k.$$

A suitable choice of z and ε yields a *telescoping series*:

$$e^{-\frac{N}{l_k - l_{k+1}}} A_k - e^{-\frac{N}{l_{k+1} - l_{k+2}}} A_{k+1} \leq B_k, \quad \|u(l_1)\|_{L^2(\Omega)} = A_1 \leq \sum_{k=1}^{\infty} B_k.$$

The resulting *telescoping series* and the energy inequality gives

$$\|u(T)\|_{L^2(\Omega)} \leq N\|u(l_1)\|_{L^2(\Omega)} \leq N\|u\|_{L^1(D)}.$$

Merci pour votre attention et joyeaux anniversaire!