Controllability and stability of difference equations and applications

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Nonlinear Partial Differential Equations and Applications

A conference in the honor of Jean-Michel Coron for his 60th birthday

Paris – June 20$^{\text{th}}$, 2016

CMAP, École Polytechnique
Team GECO, Inria Saclay
France

Outline

Introduction

Linear difference equations

1. Stability analysis of the difference equation

$$\Sigma_{\text{stab}}: \quad x(t) = \sum_{j=1}^{N} A_j(t) x(t - \Lambda_j), \quad t \geq 0.$$

2. Relative controllability of the difference equation

$$\Sigma_{\text{contr}}: \quad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + B u(t), \quad t \geq 0.$$

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- $\Lambda_1, \dots, \Lambda_N$: (*rationally independent*) positive delays ($\Lambda_{\min} = \min_j \Lambda_j$, $\Lambda_{\max} = \max_j \Lambda_j$).
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Motivation:

- Applications to some hyperbolic PDEs.
- Generalization of previous results.

Introduction

Motivation: hyperbolic PDEs

Hyperbolic PDEs → difference equations: [Cooke, Krumme, 1968], [Slemrod, 1971], [Greenberg, Li, 1984], [Coron, Bastin, d'Andréa Novel, 2008], [Fridman, Mondié, Saldivar, 2010], [Gugat, Sigalotti, 2010]...

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$$\begin{cases} \partial_t u_i(t,\xi) + \partial_\xi u_i(t,\xi) + \alpha_i(t,\xi) u_i(t,\xi) = 0, \\ \hfill t \in \mathbb{R}_+,\ \xi \in [0, \Lambda_i],\ i \in [\![1, N]\!], \\ u_i(t, 0) = \displaystyle\sum_{j=1}^N m_{ij}(t) u_j(t, \Lambda_j), \quad t \in \mathbb{R}_+,\ i \in [\![1, N]\!]. \end{cases}$$

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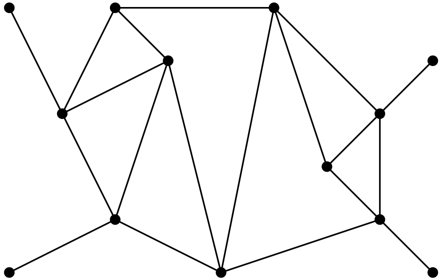
Method of characteristics: for $t \geq \Lambda_{\max}$,

$$u_i(t,0) = \sum_{j=1}^N m_{ij}(t) u_j(t, \Lambda_j) = \sum_{j=1}^N m_{ij}(t) e^{-\int_0^{\Lambda_j} \alpha_j(t-s, \Lambda_j - s) ds} u_j(t - \Lambda_j, 0).$$

Set $x(t) = (u_i(t,0))_{i \in [\![1,N]\!]}$. Then x satisfies a difference equation.

Introduction

Motivation: hyperbolic PDEs



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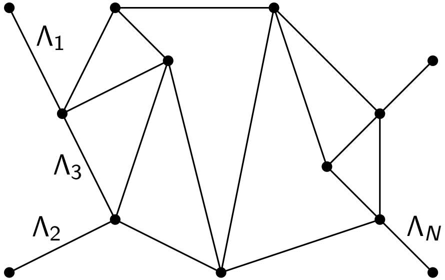


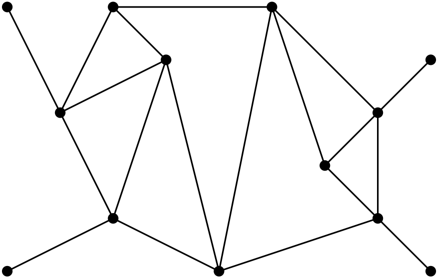
Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$

$$\partial_{tt}^2 u_i(t, \xi) = \partial_{\xi\xi}^2 u_i(t, \xi)$$
$$u_i(t, q) = u_j(t, q), \qquad \forall q \in \mathcal{V},\ \forall i, j \in \mathcal{E}_q$$

$+$ conditions on vertices.

Introduction

Motivation: hyperbolic PDEs



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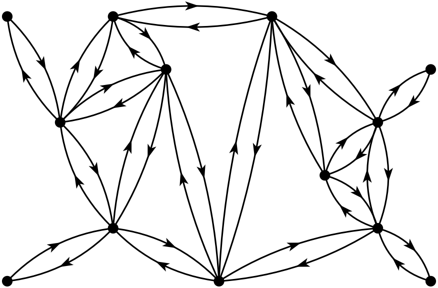
Motivation: hyperbolic PDEs

D'Alembert decomposition on travelling waves:



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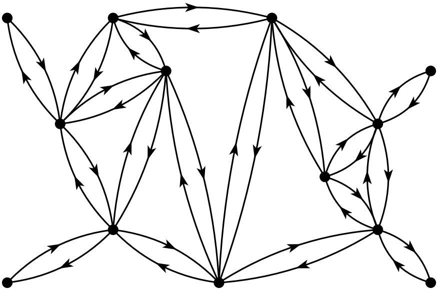

System of $2N$ transport equations.

Can be reduced to a system of difference equations.

Introduction

Motivation: previous stability results (cf. [Cruz, Hale, 1970], [Henry, 1974], [Michiels et al., 2009])

$$\Sigma_{\text{stab}}^{\text{aut}}: \quad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j), \quad t \geq 0.$$

Stability for rationally independent $\Lambda_1, \dots, \Lambda_N$ characterized by $\rho_{\text{HS}}(A) = \max_{(\theta_1,\dots,\theta_N)\in[0,2\pi]^N} \rho\left(\sum_{j=1}^N A_j e^{i\theta_j}\right)$.

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Theorem (Hale, 1975; Silkowski, 1976)

The following are equivalent:

- $\rho_{\text{HS}}(A) < 1$;
- $\Sigma_{\text{stab}}^{\text{aut}}$ *is exponentially stable for some* $\Lambda \in (0, +\infty)^N$ *with rationally independent components;*
- $\Sigma_{\text{stab}}^{\text{aut}}$ *is exponentially stable for every* $\Lambda \in (0, +\infty)^N$.

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Motivation: previous controllability results

$$\Sigma_{\text{contr}}: \quad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), \quad t \geq 0.$$

- Stabilization by linear feedbacks $u(t) = \sum_{j=1}^{N} K_j x(t - \Lambda_j)$: [Hale, Verduyn Lunel, 2002 and 2003].

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- Relative controllability in time $T > 0$: for any initial condition $x_0 : [-\Lambda_{\max}, 0] \to \mathbb{C}^d$ and final target state $x_1 \in \mathbb{C}^d$, find $u : [0, T] \to \mathbb{C}^m$ such that the solution x with initial condition x_0 and control u satisfies $x(T) = x_1$.

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  Case of two *integer* delays: [Diblík, Khusainov, Růžičková, 2008], [Pospíšil, Diblík, Fečkan, 2015].

Stability analysis and applications
Stability analysis

$$\Sigma_{\mathrm{stab}}: \quad x(t) = \sum_{j=1}^{N} A_j(t) x(t - \Lambda_j), \quad t \geq 0.$$

- $\mathsf{X}_p^\delta = L^p([-\Lambda_{\max}, 0], \mathbb{C}^d)$, $p \in [1, +\infty]$.
- Exponential stability of Σ_{stab} uniformly with respect to a given set $\mathcal{A}$ of functions $A : \mathbb{R} \to \mathcal{M}_d(\mathbb{C})^N$.

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- In this talk, to simplify, $\mathcal{A} = L^\infty(\mathbb{R}, \mathfrak{B})$ for some bounded $\mathfrak{B} \subset \mathcal{M}_d(\mathbb{C})^N$ (more general $\mathcal{A}$: see [Chitour, M., Sigalotti, 2015]).
- RI: set of all $\Lambda = (\Lambda_1, \dots, \Lambda_N) \in (0, +\infty)^N$ with rationally independent components.

Stability analysis and applications
Stability analysis

Let $\mu(\mathfrak{B}) = \limsup_{\substack{|\mathbf{n}|_1 \to +\infty \\ \mathbf{n} \in \mathbb{N}^N}} \sup_{\substack{B^r \in \mathfrak{B} \\ \text{for } r \in \mathcal{L}_{\mathbf{n}}(\Lambda)}} \left| \sum_{v \in V_{\mathbf{n}}} \prod_{k=1}^{|\mathbf{n}|_1} B_{v_k}^{\Lambda_{v_1} + \dots + \Lambda_{v_{k-1}}} \right|^{\frac{1}{\Lambda \cdot \mathbf{n}}}$,

where $\mathcal{L}_{\mathbf{n}}(\Lambda) = \{\Lambda \cdot \mathbf{k} \mid \mathbf{k} \in \mathbb{N}^N,\ \Lambda \cdot \mathbf{k} < \Lambda \cdot \mathbf{n}\}$ and $V_{\mathbf{n}}$ is the set of all permutations of $(\underbrace{1, \dots, 1}_{n_1 \text{ times}}, \underbrace{2, \dots, 2}_{n_2 \text{ times}}, \dots, \underbrace{N, \dots, N}_{n_N \text{ times}})$.

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The following statements are equivalent:

- $\mu(\mathfrak{B}) < 1$;
- Σ_{stab} *is uniformly exponentially stable in* X_p^δ *for some* $p \in [1, +\infty]$ *and* $\Lambda \in \mathrm{RI}$;
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Stability analysis and applications

Technique of the proof

To simplify, consider $\Sigma_{\text{stab}}^{\text{aut}}: \quad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j)$.

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Lemma (Explicit solution)

Let $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$. *The solution* $x : [-\Lambda_{\max}, +\infty) \to \mathbb{C}^d$ *of* $\Sigma_{\mathrm{stab}}^{\mathrm{aut}}$ *is, for* $t \geq 0$,

$$x(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ t < \Lambda \cdot \mathbf{n} \leq t + \Lambda_{\max}}} \sum_{\substack{j \in [\![1, N]\!] \\ \Lambda \cdot \mathbf{n} - \Lambda_j \leq t}} \Xi_{\mathbf{n} - e_j} A_j x_0(t - \Lambda \cdot \mathbf{n}),$$

where the matrices $\Xi_{\mathbf{n}}$ *are defined recursively for* $\mathbf{n} \in \mathbb{N}^N$ *by*

$$\Xi_{\mathbf{n}} = \sum_{\substack{k=1 \\ n_k \geq 1}}^{N} A_k \Xi_{\mathbf{n} - e_k}, \qquad \Xi_0 = \mathrm{Id}_d \,.$$

Stability analysis and applications

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To simplify, consider $\Sigma^{\mathrm{aut}}_{\mathrm{stab}}$: $x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j)$.

Lemma (Explicit solution)

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- Can be easily adapted to time-dependent matrices.
- Exponential stability can be analyzed through Ξ.
- Rational independence: all $\Lambda \cdot \mathbf{n}$ are different.

Stability analysis and applications

Applications

Using the previous transformations of hyperbolic PDEs into difference equations: under arbitrary switching, exponential stability for some $\Lambda \in \mathrm{RI}$ and $p \in [1, +\infty]$ $\iff$ exponential stability for all $\Lambda \in (0, +\infty)^N$ and $p \in [1, +\infty]$.

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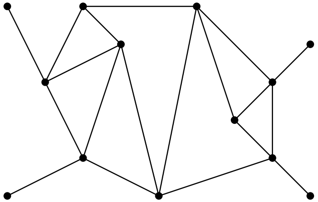
Example: wave propagation on networks.



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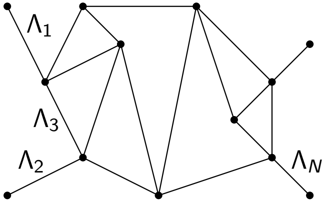

Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$

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$$u_i(t, q) = u_j(t, q), \quad \forall q \in \mathcal{V},\ \forall i, j \in \mathcal{E}_q,$$

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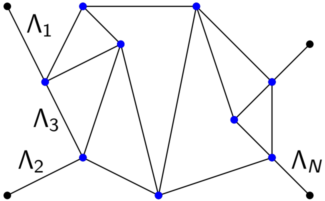

Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$

Interior vertices: $\mathcal{V}_{\mathrm{int}}$

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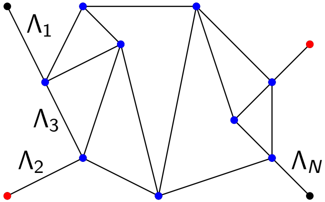

Edges: $\mathcal{E}$
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Damped vertices: $\mathcal{V}_{\mathrm{d}}$

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$$\begin{aligned} &\textstyle\sum_{i \in \mathcal{E}_q} \partial_n u_i(t, q) = 0, && \forall q \in \mathcal{V}_{\mathrm{int}}, \\ &\partial_t u_i(t, q) = -\eta_q(t) \partial_n u_i(t, q), && \forall q \in \mathcal{V}_{\mathrm{d}}, \end{aligned}$$

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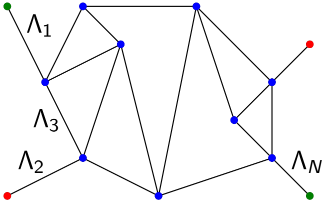

Edges: $\mathcal{E}$
Vertices: $\mathcal{V}$
$\mathcal{V} = \mathcal{V}_{\mathrm{int}} \cup \mathcal{V}_{\mathrm{d}} \cup \mathcal{V}_{\mathrm{u}}$
Interior vertices: $\mathcal{V}_{\mathrm{int}}$
Damped vertices: $\mathcal{V}_{\mathrm{d}}$
Undamped vertices: $\mathcal{V}_{\mathrm{u}}$

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Stability analysis and applications

Applications

We assume $(\eta_q)_{q \in \mathcal{V}_{\mathrm{d}}} \in L^\infty(\mathbb{R}, \mathfrak{D})$ for some bounded $\mathfrak{D} \subset \mathbb{R}_+^{\mathcal{V}_{\mathrm{d}}}$.

Theorem

The previous system is uniformly exponentially stable in $W_0^{1,p} \times L^p$ for some p if and only if the network is a tree, $\mathcal{V}_{\mathrm{u}}$ contains only one point, and $\overline{\mathfrak{D}} \subset (0, +\infty)^d$.

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$\Longleftarrow$: classical methods based on an energy estimate and an observability inequality (see, e.g., [Dáger, Zuazua, 2006]).

Stability analysis and applications
Applications

$\Longrightarrow$: (only for the case $\Lambda \in \mathrm{RI}$)

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Stability analysis and applications

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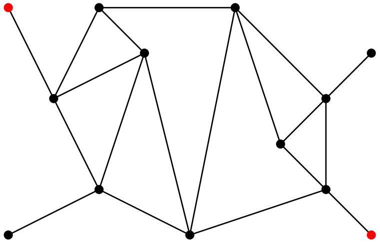

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Stability analysis and applications

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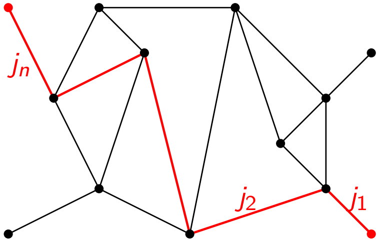

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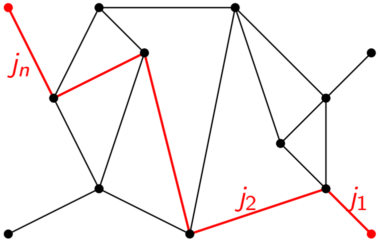

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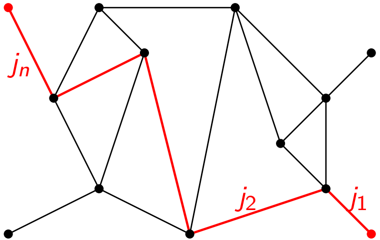

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Not exponentially stable for L, then not exponentially stable for Λ either.

Relative controllability

Definition

$$\Sigma_{\text{contr}}: \quad x(t) = \sum_{j=1}^{N} A_j x(t - \Lambda_j) + Bu(t), \quad t \geq 0.$$

For every initial condition $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ and control $u : [0, T] \to \mathbb{C}^m$, Σ_{contr} admits a unique solution $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ (no regularity assumptions!).

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We say that Σ_{contr} is relatively controllable in time $T > 0$ if, for every $x_0 : [-\Lambda_{\max}, 0) \to \mathbb{C}^d$ and $x_1 \in \mathbb{C}^d$, there exists $u : [0, T] \to \mathbb{C}^m$ such that the unique solution x of Σ_{contr} with initial condition x_0 and control u satisfies $x(T) = x_1$.

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Explicit formula

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Let $u : [0, T] \to \mathbb{C}^m$*. The solution* $x : [-\Lambda_{\max}, T] \to \mathbb{C}^d$ *of* Σ_{contr} *with zero initial condition and control* u *is, for* $t \in [0, T]$*,*

$$x(t) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^N \\ \Lambda \cdot \mathbf{n} \leq t}} \Xi_{\mathbf{n}} B u(t - \Lambda \cdot \mathbf{n}),$$

where the matrices $\Xi_{\mathbf{n}}$ *are defined as before.*

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where the matrices $\Xi_{\mathbf{n}}$ are defined as before.

- By linearity, solution with initial condition x_0 and control u is the sum of this formula with the previous one.
- Rational independence: all $\Lambda \cdot \mathbf{n}$ are different.

Relative controllability

Relative controllability criterion

Theorem (M.)

The following statements are equivalent:

- Σ_{contr} *is relatively controllable in time* T*;*
- $$\operatorname{Span}\left\{\Xi_{\mathbf{n}} B w \mid \mathbf{n} \in \mathbb{N}^N,\ \Lambda \cdot \mathbf{n} \leq T,\ w \in \mathbb{C}^m\right\} = \mathbb{C}^d;$$

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- $\exists \varepsilon_0 > 0$ *such that, for every* $p \in [1, +\infty]$, $\varepsilon \in (0, \varepsilon_0)$, $x_0 \in L^p((-\Lambda_{\max}, 0), \mathbb{C}^d)$, *and* $x_1 \in L^p((0, \varepsilon), \mathbb{C}^d)$, *there exists* $u \in L^p((0, T + \varepsilon), \mathbb{C}^m)$ *such that the solution* x *of* Σ_{contr} *with initial condition* x_0 *and control* u *satisfies* $x \in L^p((-\Lambda_{\max}, T + \varepsilon), \mathbb{C}^d)$ *and* $x(T + \cdot)|_{[0,\varepsilon]} = x_1$.

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- *Σ_{contr} is relatively controllable in some time $T > 0$ if and only if*

$$\operatorname{Span}\left\{\Xi_{\mathbf{n}} B e_j \mid \mathbf{n} \in \mathbb{N}^N,\ \lvert\mathbf{n}\rvert_1 \leq d-1,\ j \in [\![1, m]\!]\right\} = \mathbb{C}^d.$$

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