Global stabilization of a Korteweg-de Vries equation with saturating distributed control

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KdV equation with a distributed control

The Korteweg-de Vries describes approximately long waves in water of relatively shallow depth.
For all $L > 0$, it is described as follows

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + u = 0 \\ y(t,0) = y(t,L) = y_x(t,L) = 0 \\ y(0,x) = y_0 \end{cases} \qquad \text{(KdV-u)}$$

Stabilization

References : [Perla Menzala et al., 2002], [Rosier and Zhang, 2006], [Pazoto, 2005]

In the following, we will focus on (KdV-u).

KdV equation with a distributed control

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In the following, we will focus on (KdV-u).

The paper [Cerpa, 2014] is a good introduction to the control of this equation.

Case without control : critical length phenomenon

$$\begin{cases} y_t + y_x + y_{xxx} = 0 \\ y(t,0) = y(t,L) = y_x(t,L) = 0 \\ y(0,x) = y_0(x) \end{cases} \qquad \text{(LKDV)}$$

Critical length set for the linear KdV equation [Rosier, 1997]

If $L \in \left\{ 2\pi\sqrt{\frac{k^2+kl+l^2}{3}} \middle/ k, l \in \mathbb{N}^* \right\}$, there exist solutions of (LKDV) for which the energy does not decay to zero.

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With $L = 2\pi$, $y^e = 1 - \cos(x)$ is an equilibrium solution. Indeed

$$y^e_t + y^e_x + y^e_{xxx} = 0$$

Thus, with $y_0(x) = 1 - \cos(x)$, the solution does not decay to zero.

Case without control : critical length phenomenon

Critical length set for the linear KdV equation [Rosier, 1997]

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Local asymptotic stability of 0 with $L = 2\pi$ [Chu, Coron and Shang, 2015]

Let us assume that $L = 2\pi$ and $u = 0$. Then $0 \in L^2(0, L)$ is (locally) asymptotically stable for (KdV-u).

Thus the nonlinearity yy_x improves the stability. Note that the stability is *local*.

Global stabilization of $y = 0$ with a distributed control without constraint : general case

In [Pazoto, 2005] and [Rosier and Zhang, 2006], the authors use a control $u(t,x) = a(x)y(t,x)$ with a defined as follows

$$a = \begin{cases} 0 < a_0 \leq a(x) \leq a_1, \quad \forall x \in \omega, \\ \text{where } \omega \text{ is a nonempty open subset of } (0, L). \end{cases} \tag{loc-control}$$

They prove that the origin of (KdV-u) is globally asymptotically stabilized with such a control.

Saturation function : finite dimension

Usual saturation

For all $s \in \mathbb{R}$, the function $\mathtt{sat}$ satisfies

$$\mathtt{sat}(s) = \begin{cases} -u_0 & \text{if } s \leq -u_0 \\ s & \text{if } -u_0 \leq s \leq u_0, \\ u_0 & \text{if } s \geq u_0. \end{cases}$$

where u_0 denotes the saturation level.

Saturating a controller can lead to catastrophic behavior for the stability of the system.

Saturation in infinite dimension

Saturation operator

For any function s and all $x \in [0, L]$, the operator $\mathfrak{sat}$ satisfies

$$\mathfrak{sat}(s)(x) = \texttt{sat}(s(x)) \qquad \text{(SAT-}\texttt{loc}\text{)}$$

Illustration of the saturation

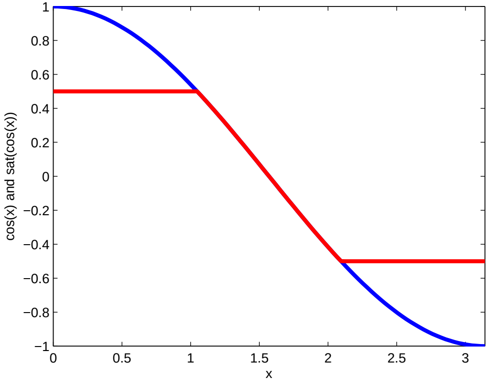

FIGURE: $x \in [0, \pi]$. Red : $\mathfrak{sat}(\cos(x))$ and $u_0 = 0.5$, Blue : $\cos(x)$.

Distributed control saturated

System under consideration

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x + \mathfrak{sat}(ay) = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = 0, \\ y(0,x) = y_0(x). \end{cases} \qquad \text{(KdV-sat)}$$

Remark : A similar work has been done on the wave equation [Prieur, Tarbouriech and Gomes da Silva Jr, 2016] and the linear KdV equation [SM, Cerpa, Prieur and Andrieu, 2015].

Well-posedness theorem

Theorem (Well posedness (SM-Cerpa-Prieur-Andrieu))

For any initial conditions $y_0 \in L^2(0, L)$*, there exists a unique* mild *solution* $y \in \mathcal{B}(T) := C(0, T; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$ *to* (KdV-sat)*.*

$\mathcal{B}(T)$ is endowed with the following norm

$$\|y\|_{\mathcal{B}(T)} := \max_{t\in[0,T]} \|y(t)\|_{L^2(0,L)} + \left(\int_0^T \|y(t)\|^2_{H^1(0,L)} dt\right)^{1/2}.$$

Recall : $L^2(0, L) = \left\{f,\ \|f\|_{L^2(0,L)} := \sqrt{\int_0^L f(x)^2 dx} < \infty\right\}$

$H^1(0, L) = \left\{f,\ \|f\|_{H^1(0,L)} := \|f\|_{L^2(0,L)} + \|f'\|_{L^2(0,L)} < \infty\right\}$

Global asymptotic stability theorem

Theorem (Global asymptotic stability(SM-Cerpa-Prieur-Andrieu))

There exist

- *a positive value* $\mu^\star$,
- *a class* $\mathcal{K}_\infty$ *function* α,

such that, for any initial condition $y_0 \in L^2(0, L)$, *every solution* y *to* (KdV-sat) *satisfies, for all* $t \geq 0$

$$\|y(t,.)\|_{L^2(0,L)} \leq \alpha(\|y_0\|_{L^2(0,L)})e^{-\mu^\star t},$$

Recall : α is said to be a class $\mathcal{K}_\infty$ function if

- it is nonnegative,
- it is strictly increasing,
- $\lim_{r\to+\infty} \alpha(r) = +\infty$
- $\alpha(0) = 0$.

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Example : The function

$$r \mapsto \alpha(r) = r$$

is a class $\mathcal{K}_\infty$ function.

Global asymptotic stability theorem

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$$\|y(t,.)\|_{L^2(0,L)} \leq \alpha(\|y_0\|_{L^2(0,L)})e^{-\mu^\star t},$$

We have in fact
Globally asymptotically stable + Semi-globally exponentially stable

Semi-global exponential stability

Semi-global exponential stability

The origin for the system (KdV-sat) is said to be *semi-globally exponentially stable* in $L^2(0, L)$ if for any $r > 0$ there exist two constants $C = C(r) > 0$ and $\mu = \mu(r) > 0$ such that for any $y_0 \in L^2(0, L)$ such that $\|y_0\|_{L^2(0,L)} \leq r$ the weak solution $y = y(t, x)$ to (KdV-sat) satisfies

$$\|y(t, .)\|_{L^2(0,L)} \leq C\|y_0\|_{L^2(0,L)} e^{-\mu t} \qquad \forall t \geq 0.$$

Sector condition

Sector condition

Let r be a positive value. Let a be defined by

$$a = \begin{cases} 0 < a_0 \leq a(x) \leq a_1, \quad \forall x \in \omega, \\ \text{where } \omega \text{ is a nonempty open subset of } (0, L). \end{cases}$$

Given $s \in L^\infty(0, L)$ satisfying, for all $x \in [0, L]$, $|s(x)| \leq r$, we have

$$\Big(\mathfrak{sat}(a(x)s(x)) - k(r)a(x)s(x)\Big)s(x) \geq 0, \quad \forall x \in [0, L], \qquad \text{(sec-cond-}\texttt{loc}\text{)}$$

with

$$k(r) = \min\left\{\frac{u_0}{a_1 r}, 1\right\}.$$

Some estimates

Bounded initial conditions

Let r be a positive value such that

$$\|y_0\|_{L^2(0,L)} \leq r.$$

Some estimates

$$\|y(T,.)\|^2_{L^2(0,L)} = \|y_0\|^2_{L^2(0,L)} - \int_0^T |y_x(\sigma,0)|^2 dt - 2\int_0^T \int_0^L \mathfrak{sat}(ay)y dx dt$$

(Stability)

$$\|y\|^2_{L^2(0,T;H^1(0,L))} \leq \frac{8T+2L}{3}\|y_0\|^2_{L^2(0,L)} + \frac{TK}{27}\|y_0\|^4_{L^2(0,L)}.$$

(Regularity)

Claim

Claim

For any $T > 0$ and any $r > 0$ there exists a positive constant $C = C(r, T) > 1$ such that for any solution y to (KdV-sat) with an initial condition $y_0 \in L^2(0, L)$ such that $\|y_0\|_{L^2(0,L)} \leq r$, it holds that

$$\|y_0\|^2_{L^2(0,L)} \leq C \left(\int_0^T |y_x(t,0)|^2 dt + 2 \int_0^T \int_0^L \mathfrak{sat}(ay(t,x)) y(t,x) dtdx \right). \tag{Claim}$$

Proving this Claim allows to prove the semi-global exponential stability of the origin of the equation when using $u = \mathfrak{sat}(ay)$.

Assuming the Claim

Indeed, if this claim holds, we obtain with (Stability)

$$\|y(kT,.)\|^2_{L^2(0,L)} \leq \gamma^k \|y_0\|^2_{L^2(0,L)} \qquad \forall k \geq 0$$

where $1 - \frac{1}{C} := \gamma \in (0,1)$.
Once again with (Stability), we have

$$\|y(t,.)\|_{L^2(0,L)} \leq \|y(kT,.)\|_{L^2(0,L)} \text{ for } kT \leq t \leq (k+1)T.$$

Therefore, **if we assume the Claim**, we have, for all $t \geq 0$,

$$\|y(t,.)\|^2_{L^2(0,L)} \leq \frac{1}{\gamma} \|y_0\|_{L^2(0,L)} e^{\frac{\log \gamma}{T} t}$$

Proof of the Claim

Another useful estimate

$$T\|y_0\|^2_{L^2(0,L)} \leq \int_0^T \int_0^L |y(t,x)|^2 dxdt + \int_0^T (T-t)|y_x(t,0)|^2 dt$$
$$+2\int_0^T (T-t)\int_0^L \mathfrak{sat}(ay)ydxdt.$$

(Stability-2)

We prove also the Claim with

$$\|y\|^2_{L^2(0,T;L^2(0,L))} \leq C_1 \left(\int_0^T |y_x(t,0)|^2 dt \right.$$
$$\left. +2\int_0^T \int_0^L \mathfrak{sat}(ay(t,x))y(t,x)dtdx \right),$$

Proof of the Claim

We proceed by contradiction. Suppose the claim fails to be true. Then there exists a sequence of solution $y^n \in \mathcal{B}(T)$ of (KdV-sat) such that

Bounded initial conditions

$$\|y^n(0,.)\|_{L^2(0,L)} \leq r \qquad \text{(IC-bounded)}$$

Contradiction argument

$$\lim_{n\to+\infty} \frac{\|y^n\|^2_{L^2(0,T;L^2(0,L))}}{\int_0^T |y^n_x(t,0)|^2 dt + 2\int_0^T \int_0^L \mathfrak{sat}(ay^n) y^n dxdt} = +\infty. \qquad \text{(Contra-argu)}$$

Using the sector condition

Note that in a first hand we have, from the fact that the solution is bounded in $L^2(0, L)$ and from the hidden regularity (Regularity)

$$\|y^n\|^2_{L^2(0,T;H^1(0,L))} \leq \beta := \frac{8T + 2L}{3} r^2 + \frac{TK}{27} r^4.$$

Therefore,

$L^\infty(0, L)$-regularity

$$\forall x \in [0, L], \int_0^T |y^n(t, x)|^2 dt \leq L\|y^n\|^2_{L^2(0,T;H^1(0,L))} \leq L\beta. \tag{L^∞-reg}$$

Using the sector condition

Now let us consider $\Omega_i \subset [0, T]$ defined as follows

$$\Omega_i = \left\{ t \in [0, T],\ \sup_{x \in [0,L]} |y(t,x)| > i \right\}.$$

In the following, we will denote by Ω_i^c its complement. It is defined by

$$\Omega_i^c = \left\{ t \in [0, T],\ \sup_{x \in [0,L]} |y(t,x)| \leq i \right\}. \qquad \text{(Space-sec-cond)}$$

Using the sector condition

Since the function $t \mapsto \sup_{x\in[0,L]} |y^n(t,x)|^2$ is a nonnegative function, we have

$$\int_0^T \sup_{x\in[0,L]} |y^n(t,x)|^2 dt \geq \int_{\Omega_i} \sup_{x\in[0,L]} |y^n(t,x)|^2 dt \geq i^2 \nu(\Omega_i),$$

where $\nu(\Omega_i)$ denotes the Lebesgue measure of Ω_i. Therefore, with $\int_0^T |y^n(t,x)|^2 dt \leq L\beta$, we obtain

$$\nu(\Omega_i) \leq \frac{L\beta}{i^2}.$$

We deduce from the previous equation that

$$\max\left(T - \frac{L\beta}{i^2}, 0\right) \leq \nu(\Omega_i^c) \leq T. \qquad \text{(Lebesgue-measure)}$$

Using the sector condition

Moreover, since $|y(t,x)| \leq i$ in Ω_i^c, then we can use the sector condition

$$\Big(\mathfrak{sat}(a(x)s(x)) - k(r)a(x)s(x)\Big)s(x) \geq 0, \quad \forall x \in [0, L],$$

and we obtain, for all $i \in \mathbb{N}$

$$\int_0^T \int_0^L \mathfrak{sat}(ay^n)y^n dtdx \geq \int_{\Omega_i^c} \int_0^L ak(i)(y^n)^2 dtdx.$$

(Stability-sec-cond)

Using the sector condition

Let $\lambda^n := \|y^n\|_{L^2(0,T;L^2(0,L))}$ and $v^n(t,x) = \frac{y^n(t,x)}{\lambda^n}$. Notice that λ^n is bounded. Hence, up to extracting a subsequence, we may assume that

$$\lambda^n \to \lambda \geq 0.$$

Then v^n fullfills

$$\begin{cases} v_t^n + v_x^n + v_{xxx}^n + \lambda^n v^n v_x^n + \dfrac{\mathfrak{sat}(a\lambda^n v^n)}{\lambda^n} = 0, \\ v^n(t,0) = v^n(t,L) = v_x^n(t,L) = 0, \\ \|v^n\|_{L^2(0,T;L^2(0,L))} = 1. \end{cases}$$

Using the sector condition

Due to the contradiction argument, that is

$$\lim_{n\to+\infty} \frac{\|y^n\|^2_{L^2(0,T;L^2(0,L))}}{\int_0^T |y_x^n(t,0)|^2 dt + 2\int_0^T \int_0^L \mathfrak{sat}(ay^n)y^n dxdt} = +\infty,$$

then we have

$$\int_0^T |v_x^n(t,0)|^2 dt + 2\int_0^T \int_0^L \frac{\mathfrak{sat}(a\lambda^n v^n)}{\lambda^n} v^n dtdx \to 0.$$

Using the sector condition

And since we have

$$\int_0^T \int_0^L \frac{\mathfrak{sat}(a\lambda^n v^n)}{\lambda^n} v^n dtdx \geq \int_{\Omega_i^c} \int_0^L ak(i)(v^n)^2 dtdx.$$

then we obtain that

$$\int_0^T |v_x^n(t,0)|^2 dt + 2 \int_{\Omega_i^c} \int_0^L k(i)a(v^n)^2 dtdx \to 0$$

Conclusion

Using a result of Aubin Lions, we prove that v^n converges strongly to $v \in L^2(0, T; L^2(0, L))$. Moreover, we have, for all $i \in \mathbb{N}$

$$v(t,x) = 0,\ \forall x \in \omega, \forall t \in \bigcup_{i\in\mathbb{N}} \Omega_i^c, \text{ and } v_x(t,0) = 0,\ \forall t \in (0,T).$$

With

$$\max\left(T - \frac{L\beta}{i^2}, 0\right) \leq \nu(\Omega_i^c) \leq T,$$

we know that

$$\nu\left(\bigcup_{i\in\mathbb{N}} \Omega_i^c\right) = T.$$

Thus, for almost every $t \in [0, T]$

$$v(t,x) = 0,\ \forall x \in \omega, \text{ and } v_x(t,0) = 0.$$

Conclusion

Moreover, v solves

$$\begin{cases} v_t + v_x + v_{xxx} + \lambda v v_x = 0, \\ v(t,0) = v(t,L) = v_x(t,L) = 0, \\ \|v\|_{L^2(0,T;L^2(0,L))} = 1. \end{cases}$$

Thus $v \in \mathcal{B}(T)$ and in particular v is continuous. Thus

$$v(t,x) = 0,\ \forall x \in \omega, \forall t \in [0,T], \text{ and } v_x(t,0) = 0,\ \forall t \in (0,T).$$

Conclusion

With a unique continuation argument given by [Saut and Scheurer, 1987], we prove that

$$v(t,x) = 0, \forall x \in [0, L], \forall t \in [0, T]$$

It is in contradiction with $\|v\|_{L^2(0,T;L^2(0,L))} = 1$. Thus, the claim is true and we have

Semi-global exponential stability

Let y be a solution to the KdV equation with a saturated control. Thus, for every initial condition y_0 satisfying

$$\|y_0\|_{L^2(0,L)} \leq r,$$

there exist positive values $\mu := \mu(r)$ and $K := K(r)$ such that

$$\|y\|_{L^2(0,L)} \leq K e^{-\mu t} \|y_0\|_{L^2(0,L)}, \quad \forall t \geq 0$$

$y_0(x) = 1 - \cos(x)$, $L = 2\pi$ and $\omega = \left[\frac{1}{3}L, \frac{2}{3}L\right]$

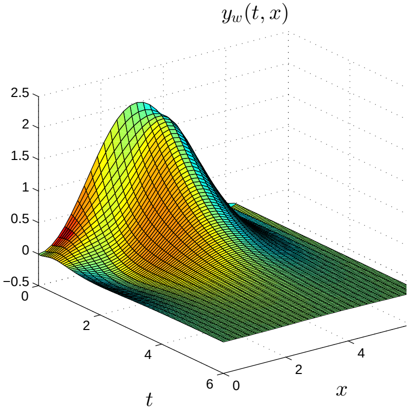

FIGURE: Solution y(t,x) with a localized feedback law without saturation

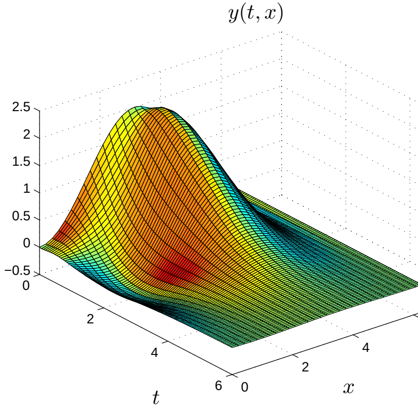

FIGURE: Solution y(t,x) with a localized feedback law saturated ; $u_0 = 0.5$

$y_0(x) = 1 - \cos(x)$, $L = 2\pi$ and $\omega = \left[\frac{1}{3}L, \frac{2}{3}L\right]$

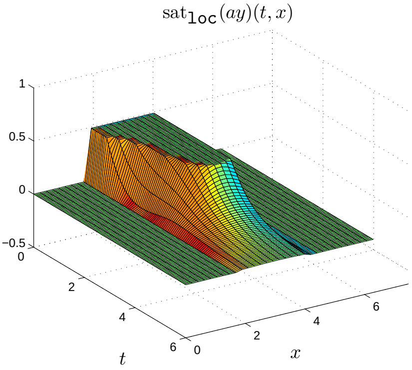

FIGURE: Control $f = \mathfrak{sat}(ay)$ where $\omega = \left[\frac{1}{3}L, \frac{2}{3}L\right]$, $u_0 = 0.5$

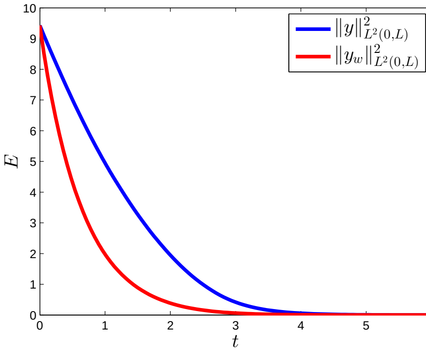

FIGURE: Energy functions of the solutions

Perspectives

Very few papers deal with stabilization of PDEs with bounded boundary controls (see [Lasiecka and Seidman, 2003]). For instance, is the origin for the following linear hyperbolic equation

$$\begin{cases} z_t + \Lambda z_x = 0, \\ z(t,0) = Hz(t,1) + B\mathfrak{sat}(Kz(t,1)), \end{cases}$$

where $\lambda(\Lambda) > 0$, stable ?

THANK YOU FOR YOUR ATTENTION !
HAPPY BIRTHDAY JEAN-MICHEL !

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