Small time global exact null controllability of the incompressible Navier-Stokes equation with Navier slip-with-friction boundary condition

Joint work with Jean-Michel Coron and Franck Sueur

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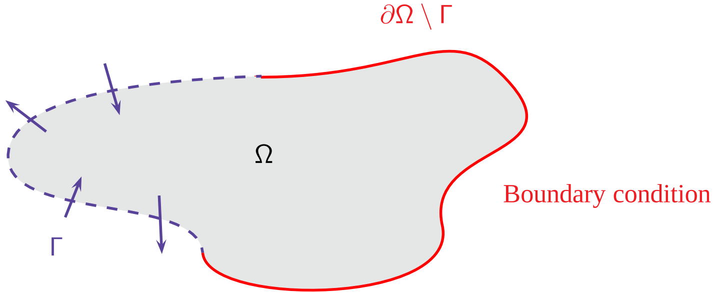
Small time global results in fluid mechanics

implies to study boundary layers

Small time: $T \ll 1$
Global: large states $|u(t,\cdot)|_{L^2(\Omega)} \gg 1$



Our goal: null controllability $u(T,\cdot) = 0.+$

Boundary layer profiles

When $\mathrm{Re} \gg 1$, the fluid behaves like the solution of an inviscid equation inside the domain. However, near the boundary, viscous effects prevail.

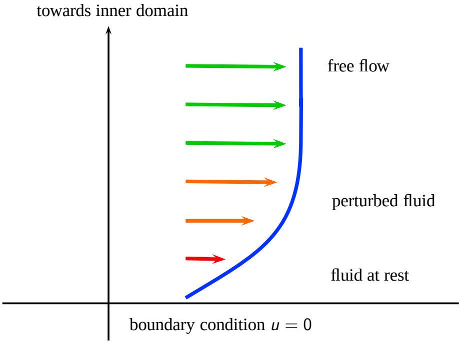


Figure: Blasius speed profile for the Dirichlet boundary condition

Incompressible Navier-Stokes equation

inside the domain Ω

$u : \Omega \to \mathbb{R}^d$ is the velocity,
$p : \Omega \to \mathbb{R}$ is the pressure,

$$\begin{cases} u_t + (u \cdot \nabla)u - \Delta u + \nabla p = 0 & \text{on } \Omega, \\ \mathrm{div}(u) = 0 & \text{on } \Omega, \\ \mathrm{BC} & \text{on } \partial\Omega \setminus \Gamma, \\ u(0, \cdot) = u^* & \text{on } \Omega. \end{cases} \tag{NS}$$

No boundary condition on Γ (control region): under-determined system.

Navier slip-with-friction boundary condition

on the uncontrolled part of the boundary

On $\partial\Omega \setminus \Gamma$, we assume:

$$u \cdot n = 0 \quad \text{and} \quad [D(u)n + Au]_{\tan} = 0, \qquad \text{(Navier)}$$

where

$$D_{ij}(f) := \frac{1}{2}\left(\partial_i f_j + \partial_j f_i\right),$$
$$[f]_{\tan} := f - (f \cdot n)n$$

and $A : \partial\Omega \to \mathcal{M}_d(\mathbb{R})$ is smooth (but not necessarily constant, neither signed / coercive).

Special cases

of boundary condition $u \cdot n = 0$ and $[D(u)n + Au]_{\mathrm{tan}} = 0$

- **perfect slip**, when A is the Weingarten map:

$$\begin{aligned} & \mathrm{curl} u = 0 \quad \text{in 2D}, \\ u \cdot n = 0 \quad & \mathrm{curl} u \wedge n = 0 \quad \text{in 3D}. \end{aligned}$$

- **slip condition** when $A = 0$:

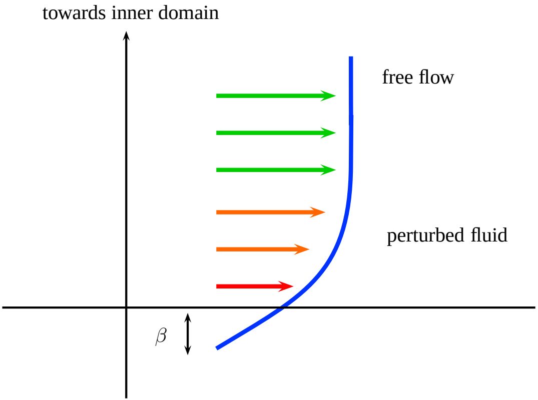
$$u \cdot n = 0 \quad \text{and} \quad [D(u)n]_{\mathrm{tan}} = 0.$$

- **scalar case** $A = \frac{1}{\beta}\mathrm{Id}$ **for flat boundaries**:

$$u \cdot n = 0 \quad \text{and} \quad u_{\mathrm{tan}} = \beta \partial_n u_{\mathrm{tan}}.$$

Slip length β

Imagine that the profile is displaced by a length β inside the boundary.



Our result

Theorem (Coron, M., Sueur)

Let Ω be a smooth connected bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$.
Let $\Gamma \subset \partial\Omega$ intersecting all connected components of $\partial\Omega$.
Let A be a smooth matrix-valued function on $\partial\Omega$.
Let $u^* \in L^2(\Omega)$ be divergence free, tangent to the boundary.

For any $T > 0$, there exists a trajectory u (in an appropriate functional space) solution to (NS) *and* (Navier) *such that:*

$$u(T, \cdot) = 0.$$

Key ideas of the proof

1. Controllability of the Euler equation by means of the return method [Coron 1993], [Coron 1996], [Glass 1997], [Glass 2000].

2. Vanishing viscosity asymptotic boundary layer expansion for the Navier boundary condition [Iftimie, Sueur 2011].

3. Well prepared dissipation of the boundary layer [M. 2014].

I. Controllability of Euler

Usual scaling argument

We let $T = \varepsilon \ll 1$ be a small time and u^* be a large initial data. We introduce:

$$u^\varepsilon(t,x) := \varepsilon u(\varepsilon t, x),$$
$$p^\varepsilon(t,x) := \varepsilon^2 p(\varepsilon t, x).$$

These are now defined for $t \in (0,1)$. Moreover, the initial data is now small: $u^\varepsilon(0,\cdot) = \varepsilon u^*$. Expand:

$$u^\varepsilon(t,x) = u^0(t,x) + \varepsilon u^1(t,x) + \dots$$

I. Controllability of Euler

Choice of a reference trajectory

Build a *return method like* reference trajectory.

$$
\begin{cases}
\partial_t u^0 + \left(u^0 \cdot \nabla\right) u^0 + \nabla p^0 = 0 & [0,1] \times \Omega, \\
\operatorname{div} u^0 = 0 & [0,1] \times \Omega, \\
u^0 \cdot n = 0 & [0,1] \times \partial\Omega \setminus \Gamma, \\
u^0(0,\cdot) = 0 & \Omega, \\
u^0(T,\cdot) = 0 & \Omega.
\end{cases} \tag{1}
$$

You can choose $u^0(t,x) = \nabla\theta^0(t,x)$.
In 2D, you can even choose $u^0(t,x) = \alpha(t) \times \nabla\theta(x)$.

I. Controllability of Euler

Flushing of the initial data

The initial data u^* is flushed by the reference trajectory.

$$
\begin{cases}
\partial_t u^1 + \left(u^0 \cdot \nabla\right) u^1 + \left(u^1 \cdot \nabla\right) u^0 + \nabla p^1 = 0 & [0,1] \times \Omega, \\
\mathrm{div}\ u^1 = 0 & [0,1] \times \Omega, \\
u^1 \cdot n = 0 & [0,1] \times \partial\Omega \setminus \Gamma, \\
u^1(0,\cdot) = u^* & \Omega.
\end{cases}
\tag{2}
$$

At the final time $u^1 = 0$.

Is it enough?

for Navier-Stokes

In, [Coron 1996], Jean-Michel tried to apply this method to the Navier-Stokes equation in 2D with Navier boundary conditions. With the same scaling:

$$\begin{cases} u_t^\varepsilon + (u^\varepsilon \cdot \nabla)u^\varepsilon - \varepsilon\Delta u + \nabla p^\varepsilon = 0, \\ \operatorname{div}(u^\varepsilon) = 0, \\ u^\varepsilon(0,\cdot) = \varepsilon u^*. \end{cases}$$

However, it is not sufficient to conclude. Indeed, although this method yields good controllability inside the domain, we only have weak estimates of the final state in $W^{-1,\infty}(\Omega)$ near the boundaries.
$\Rightarrow$ We need to compute what happens near the boundaries.

II. Vanishing viscosity expansion of Navier-Stokes

Convergence of Navier-Stokes to Euler?

Do the solutions u^ε of:

$$\begin{cases} u_t^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \varepsilon \Delta u + \nabla p^\varepsilon = 0 & (0, T) \times \Omega, \\ \operatorname{div}(u^\varepsilon) = 0 & (0, T) \times \Omega, \\ u^\varepsilon \cdot n = 0 & (0, T) \times \partial\Omega, \\ [D(u^\varepsilon) n + A u^\varepsilon]_{\mathrm{tan}} = 0 & (0, T) \times \partial\Omega, \\ u^\varepsilon(0, \cdot) = u^* & \Omega \end{cases}$$

on $(0, T) \times \Omega$ converge to the corresponding solution of Euler?
Can we write an asymptotic expansion?

The answer is yes! The expansion looks like:

$$u^\varepsilon(t,x) = u^0_{\mathrm{Euler}}(t,x) + \sqrt{\varepsilon} v\left(t, x, \frac{\varphi(x)}{\sqrt{\varepsilon}}\right) + \ldots,$$

where $\varphi(x)$ is the distance to the boundary $\partial\Omega$. We see the thickness of the boundary layer and we introduce the fast variable $z = \varphi(x)/\sqrt{\varepsilon}$. In this expansion, v is only tangential: $v \cdot n \equiv 0$.

II. Vanishing viscosity expansion of Navier-Stokes

PDE for the boundary layer profile

$$\begin{cases} \partial_t v + \left[(u^0 \cdot \nabla) v + (v \cdot \nabla) u^0 \right]_{\tan} + u^0_\flat z \partial_z v - \partial_{zz} v = 0, & \\ \partial_z v(\cdot,\cdot,0) = g^0 & \text{at } z = 0, \\ v(0,\cdot,\cdot) = 0 & \text{at } t = 0, \end{cases}$$

where we introduce the following definitions:

$$u^0_\flat(t,x) = \frac{u^0(t,x) \cdot n(x)}{\varphi(x)}, \qquad \text{in } [0,T] \times \Omega,$$

$$g^0(t,x) = 2\chi(x) \left[D\left(u^0(t,x)\right) n(x) + A u^0(t,x) \right]_{\tan} \quad \text{in } [0,T] \times \Omega.$$

Well-posedness and estimates can be proven.

Is it *small* enough. . .

to be able to conclude with a local result?

At the final time, we have:

$$\|u^\varepsilon(t=1,\cdot)\|_{L^2(\Omega)} \approx \left\|\sqrt{\varepsilon} v\left(t=1,\cdot,\frac{\varphi(\cdot)}{\sqrt{\varepsilon}}\right)\right\|_{L^2(\Omega)} \approx \varepsilon^{3/4}.$$

But this is not enough... scaling back, this yields:

$$\|u(\varepsilon,\cdot)\|_{L^2(\Omega)} \approx \varepsilon^{-1/4}.$$

III. Well-prepared dissipation of the boundary layer

A 1D model with the Burgers equation

Here $\Omega = (0,1)$, $u^* \in L^2(0,1)$ and $T > 0$.

$$\begin{cases} u_t + uu_x - u_{xx} = q_{\mathrm{int}}(t) & \text{on} \quad (0,T) \times (0,1), \\ u(t,0) = q_{\mathrm{bc}}(t) & \text{on} \quad (0,T), \\ u(t,1) = 0 & \text{on} \quad (0,T), \\ u(0,x) = u^*(x) & \text{on} \quad (0,1) \end{cases} \tag{3}$$

using two scalar controls q_{int} and q_{bc}.

III. Well-prepared dissipation of the boundary layer

Use the left boundary control to crush the initial data

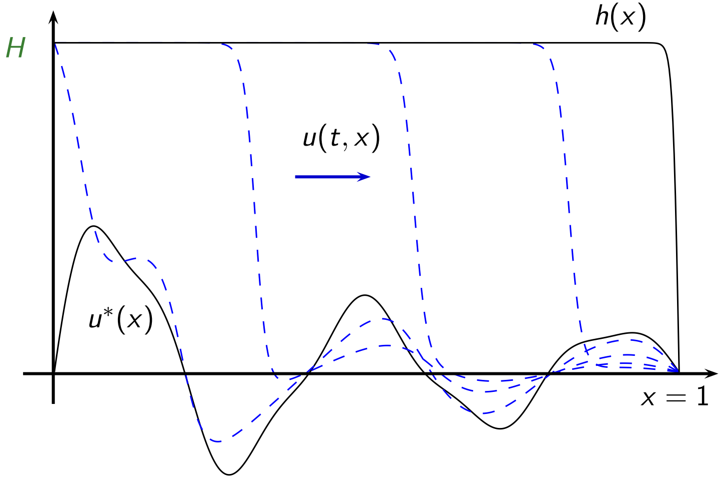


Figure: After a time of order $1/H$, we almost reach a steady state $u(t,x) \approx h(x)$.

III. Well-prepared dissipation of the boundary layer

Go back down using the inner scalar control

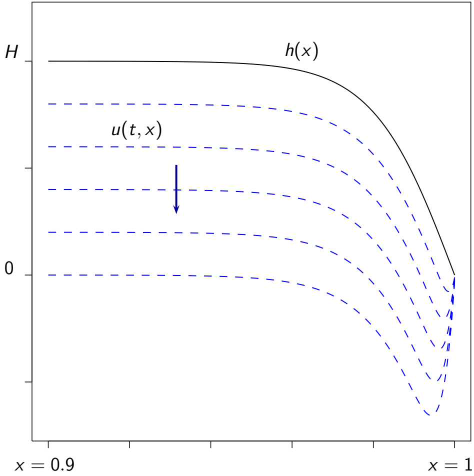


Figure: ... but this create a boundary layer residue near $x = 1$ ...

III. Well-prepared dissipation of the boundary layer

Natural lazy-control smoothing effect

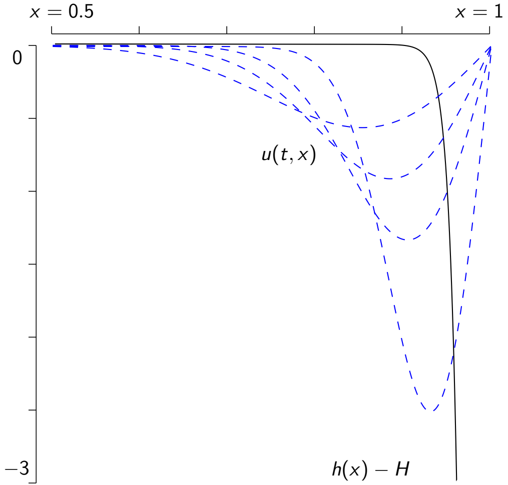


Figure: Relax. Choose null controls. Watch the diffusion kill the residue.

Proof of the main theorem

Two time scales

- Let T be the (possibly small) time in which we want to prove global exact null controllability.
- Introduce $\varepsilon \ll 1$.
- Consider two time intervals: $[0, \varepsilon T]$ and $[\varepsilon T, T]$.
- After usual scaling, these will be $[0, T]$ and $[T, T/\varepsilon]$.

Proof of the main theorem

Dissipation estimates for the boundary layer

In the second phase $[T, T/\varepsilon]$, $u_0 = 0$. Thus

$$\begin{cases} \partial_t v + \left[(u^0 \cdot \nabla)v + (v \cdot \nabla)u^0\right]_{\tan} + u_\flat^0 z \partial_z v - \partial_{zz} v = 0, \\ \qquad \partial_z v(\cdot,\cdot,0) = g^0 \quad \text{at } z = 0, \\ \qquad v(0,\cdot,\cdot) = 0 \quad \text{at } t = 0, \end{cases}$$

So we have a heat equation on the half line $z \geq 0$, where $x \in \Omega$ is merely a parameter. Decay properties depend on vanishing moments of the initial data (at time T). If you assume enough vanishing moments, you can prove any polynomial decay in $L^2(\mathbb{R}^+)$ (or stronger spaces).

Proof of the main theorem

How to ensure vanishing moments?

Use the transport term like in this model system:

$$\begin{cases} \partial_t v + (u^0 \cdot \nabla) v - \partial_{zz} v = 0, \\ \partial_z v(\cdot, \cdot, 0) = g^0 \quad \text{at } z = 0, \\ v(0, \cdot, \cdot) = 0 \quad \text{at } t = 0, \end{cases}$$

to make sure the moments vanish at $t = T$.

Perspectives

- Possible improvements: controllability to the trajectories, remove the assumption on Γ, strong solutions, ...
- The main perspective is to attempt to apply this method to the Dirichlet boundary condition $u = 0$ on $\partial\Omega \setminus \Gamma$ in order to prove the same result in this more difficult case[1].

[1] note: try to finish before Jean-Michel's birthday; could make a great gift.