Control of the Grushin equation

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Plan de l'exposé

Control for hypoelliptic diffusions

This talk is about
a **small** contribution to a **big** problem :

Controllability of degenerate parabolic equation $\partial_t - L$

- $L = \sum_{i=1}^{N} X_i^2$, type I
- $L = \sum_{i=1}^{N} X_i^2 + X_0$, type II

where the $X_i's$ are real vector fields which generate with there iterated brackets a Lie algebra equal to the full tangent space at any point.

Apart some particular results, this problem is essentially open

1. Type I. Parabolic degenerate operators :
   Grushin type equations, Heat equation on the Heisenberg group :
   Alabau, Beauchard, Cannarsa, Guglielmi, Pravda-Starov, ...
2. Type II. Kolmogorov and Fokker-Planck equations :
   Beauchard-Helffer-Henry-Robbiano, LeRousseau- Moyano, ...

Plan de l'exposé

The Grushin control problem

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ be the one dimensional torus, and $M =]-1,1[_x \times \mathbb{T}_y$. Let $P = (\partial_x)^2 + x^2\partial_y^2$ be the Grushin operator and A the unbounded operator on $L^2(M)$

$$D(A) = \{f \in L^2(M),\ P(f) \in L^2(M),\ f|_{\partial M} = 0\}, \quad A(f) = -P(f).$$

The exact controllability to 0 in time T by an open subset ω of M for the parabolic equation $\partial_t + A$ is equivalent to the existence of a constant C such that for all $g \in L^2(M)$ the following observability inequality holds true

$$\int_M |e^{-TA}(g)(x,y)|^2 dxdy \leq C \int_{]0,T[\times\omega} |e^{-tA}(g)(x,y)|^2\, dt\, dxdy \quad (2.1)$$

The Beauchard-Cannarsa-Guglielmi result

The following theorem is due to K. Beauchard, P. Cannarsa and R. Guglielmi (2013) (K. Beauchard, P. Cannarsa, R. Guglielmi. *Null controllability of Grushin-type operators in dimension two*, JEMS)

Theorem

Let ω be a vertical strip $\omega =]a, b[\times \mathbb{T}$, with $0 < a < b < 1$. Then there exists $T^* \geq a^2/2$ such that the following holds true :

- *For $T > T^*$, the observability inequality* (2.1) *holds true.*
- *For $T < T^*$, the observability inequality* (2.1) *is untrue.*

The computation of the critical time T^* has been recently performed in the symmetric case $\omega = (]-b, -a[\cup]a, b[) \times \mathbb{T}$ by K. Beauchard, L. Miller and M. Morancey in *2D Grushin-type equations : minimal time and null controllable data* (preprint). They find $T^* = a^2/2$, and that for $T = T^*$, the system is not null controllable.

Main result

The following theorem is due to Armand Koenig (june 2016 and soon on arXiv !)

Theorem

Assume that the control set ω is disjoint from a horizontal strip $]-1,1[_x \times]c,d[_y$ $(c \neq d)$. Then for any time $T > 0$, the exact controllability to 0 by ω in time T is untrue.

Plan de l'exposé

1. Introduction
2. The Grushin equation
3. A simplified model and the Runge theorem
4. The Grushin case
   - semiclassical reduction
   - Properties of $\lambda_{n,0}$

The model on the disc

Let H be the Hilbert space $H = \{\sum_{n\geq 1} a_n e^{in\theta}, \sum |a_n|^2 < \infty\}$ and $A = |\partial_\theta|$ the positive unbounded self-adjoint operator on H

$$A(\sum_{n\geq 1} a_n e^{in\theta}) = \sum_{n\geq 1} n a_n e^{in\theta}.$$

Then the controllability to 0 in time T by an open subset ω of $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is equivalent to the existence of a constant C such that

$$\sum_{n\geq 1} e^{-2nT} |a_n|^2 = \int_{\mathbb{T}} |f(T,\theta)|^2 \, d\theta \leq C \int_{]0,T[\times\omega} |f(t,\theta)|^2 \, dt \, d\theta \quad (3.1)$$

with $f(t,\theta) = e^{-tA}(f(0,.)) = \sum_{n\geq 1} a_n e^{-nt+in\theta}$.

Theorem

Let $\omega \neq \mathbb{T}$. Then for any $T > 0$ the observability inequality (3.1) is never true.

proof of theorem 3.1

For the proof theorem 3.1, introduce the complex variable $z(t,\theta) = e^{-t+i\theta}$. Then $]0,T[\times\mathbb{T}$ is map on the annulus $A_T = \{e^{-T} < |z| < 1\}$, and one has

$$e^{-tA}(f(0,.)) = \sum_{n\geq 1} z^n = f(z), \quad \text{holomorphic in the annulus} \quad A_T.$$

We may assume $\omega = \{|\theta| > a\}$ with $a > 0$ small. Let

$$\mathcal{D} = \{z = e^{-t+i\theta},\ t \in]0,T[,\ \theta \in \omega\}$$

Then the observability inequality (3.1) implies that there exists C such that for all holomorphic function f on $\mathbb{C}$ with $f(0) = 0$

$$\int_{|z|<e^{-T}} |f(z)|^2 dxdy \leq C \int_{\mathcal{D}} |f(z)|^2 dxdy \tag{3.2}$$

This is untrue thanks to the Runge approximation theorem.

Plan de l'exposé

semiclassical reduction

Recall $M =]-1,1[\times\mathbb{T}$ and $P = \partial_x^2 + x^2\partial_y^2$. Set for $n \in \mathbb{Z}$

$$A_n = -\partial_x^2 + x^2 n^2, \quad \text{Dirichlet on the boundary} \quad x = \pm 1$$

Let $e_{n,k}(x), k \geq 0$ an orthonormal basis of eigenfunctions of A_n in $L^2(]-1,1[)$

$$A_n(e_{n,k}) = \lambda_{n,k} e_{n,k}$$

The contradiction to the observability inequality is done by taking a special Cauchy data at time $t = 0$. It will be concentrated for any n on the first eigenfunction of A_n.

$$g(x,y) = \sum_{n \geq 0} a_n e_{n,0}(x) e^{in\theta}$$

$$e^{-tA}(g)(x,y) = \sum_{n \geq 0} a_n e_{n,0}(x) e^{-t\lambda_{n,0} + in\theta}$$

One has to prove that for any $\varepsilon > 0$, $T > 0$ there exists a sequence $(a_n^\varepsilon) \in l^2(\mathbb{N})$ such that

$$\sum_{n\geq 0} |a_n^\varepsilon|^2 e^{-2T\lambda_{n,0}} \geq 1/\varepsilon$$

$$\sum_{n,m\geq 0} a_n^\varepsilon \overline{a}_m^\varepsilon \left(\int_{-1}^{1} e_{n,0}(x) e_{m,0}(x) dx\right) \int_0^T \int_{\theta\in\omega} e^{-t(\lambda_{n,0}+\lambda_{m,0})+i(n-m)\theta} \, d\theta dt = 1$$

Observe that if we simplify (a little bit strongly !) by taking

$$\lambda_{n,0} = n \quad \text{and} \quad \int_{-1}^{1} e_{n,0}(x) e_{m,0}(x) dx = 1$$

this is exactly the previous model case.

The following result is one of the key points which allows to prove that the simplification $\lambda_{n,0} = n$ is not too bad. The proof relies on a precise study of the 1-d Dirichlet problem

$$(-\partial_x^2 + x^2 n^2)(e_{n,0}) = \lambda_{n,0} e_{n,0}, \quad e_{n,0}(\pm 1) = 0$$

Theorem

There exists $\gamma \in S^{3/2}$ such that

$$\lambda_{n,0} = n + e^{-n}\gamma(n-1)$$

$$U_{\theta,r(\theta)} = \{|z| > r(\theta), |\arg(z)| < \theta\}, \quad \theta \in]0, \pi/2[$$

The set S^d is the set of holomorphic functions defined on $\bigcup_{0<\theta<\pi/2} U_{\theta,r(\theta)}$ such that for any $\theta \in]0, \pi/2[$, there exists $C_\theta > 0$ such that

$$|\gamma(z)| \leq C_\theta |z|^d \quad \text{on} \quad U_{\theta,r(\theta)}.$$

The following proposition is classical.

Proposition

Let $\gamma \in S^d$. Then the Fourier transform

$$\hat{\gamma}(\xi) = \int_{r(0)}^{+\infty} \gamma(x) e^{-ix\xi}\, dx$$

which is defined for $\{\Im(\xi) < 0\}$ extends homomorphically to $\mathbb{C} \setminus i[0, \infty[$.

As a consequence, one has the following proposition

Proposition

Let $\gamma \in S^d,\ d > 0$. Then the kernel K_γ defined for $|\zeta| < 1$ by

$$K_\gamma(\zeta) = \sum_{n > r(0)} \gamma(n) z^n$$

extends homomorphically to the set $\mathbb{C} \setminus [1, +\infty[$.

Main result

The following theorem is one of the key point of the proof.

Theorem

Let $\gamma \in S^d$. Let H_γ the operator defined on entire functions of $z \in \mathbb{C}$ by the formula

$$H_\gamma(\sum_{n>r(0)} a_n z^n) = \sum_{n>r(0)} \gamma(n) a_n z^n$$

.

Let U be a domain in $\mathbb{C}$, star shaped with respect to $z = 0$.
Then for all $\delta > 0$, there exists $C_\delta > 0$ independant of γ and $\theta_\delta \in]0, \pi/2[$ such that, with $U^\delta = \{z, dist(z, U) < \delta\}$, one has

$$\|H_\gamma(f)\|_{L^\infty(U)} \leq C_\delta \|f\|_{L^\infty(U^\delta)}$$

Theorem

Let $\gamma \in S^d$. *Let* $\rho(n) = e^{-n}\gamma(n)$ *and* $\lambda_n = n + \rho(n-1)$. *Let* A_ρ *be the operator with domain* $\{\sum_{n>r(0)} a_n e^{in\theta}, \sum |\lambda_n a_n|^2 < +\infty\}$ *defined by*

$$A_\rho((\sum a_n e^{in\theta}) = \sum \lambda_n a_n e^{in\theta}.$$

Let $\omega \subset \mathbb{T}$ *be a strict open subset of* $\mathbb{T}$, *and* $T > 0$.
Then the parabolic equation

$$\partial_t f + A_\rho f = 0$$

is not controllable by $\omega \times]0, T[$.

As a consequence, using explicit functions related to the Runge theorem, one can construct for any $\varepsilon > 0$ a sequence (a_n^ε) such that

$$\sum_{n\geq 0} |a_n^\varepsilon|^2 e^{-2T\lambda_{n,0}} \geq 1/\varepsilon$$

$$\sum_{n,m\geq 0} a_n^\varepsilon \overline{a}_m^\varepsilon \int_0^T \int_{\theta\in\omega} e^{-t(\lambda_{n,0}+\lambda_{m,0})+i(n-m)\theta} \, d\theta dt = 1$$

It remains to to add the extra factor $(\int_{-1}^1 e_{n,0}(x) e_{m,0}(x) dx)$ in front of $a_n^\varepsilon \overline{a}_m^\varepsilon$. this relies as above on a precise microlocal symbolic calculus.

Bon anniversaire
Jean-Michel !