Output stabilization at unobservable points: analysis via an example

J.P. Gauthier[a] M.A. Lagache[ab] U. Serres[b]

[a]Université de Toulon, France

[b]Université de Lyon, France

60th birthday of Jean-Michel Coron
IHP, June 2016

Table of contents

1. Introduction
2. Practical stability
3. Numerical simulations
4. Ongoing work: exact stabilization
5. Conclusion

1 Introduction

2 Practical stability

3 Numerical simulations

4 Ongoing work: exact stabilization

5 Conclusion

System under consideration

Consider the closed quantum system [1]

$$\begin{cases} \dot{x} = A(u)x = \begin{pmatrix} 0 & e & u_1 \\ -e & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{pmatrix} x \\ y = Cx = x_3 \end{cases} \qquad (\Sigma)$$

where

- $x = (x_1, x_2, x_3) \in S^2$ is the state variable
- $y \in \mathbb{R}$ is the measured output
- $u = (u_1, u_2) \in \mathbb{R}^2$ is the control variable

Aim:

Stabilize (Σ) to the target point $x_{\mathrm{t}} = (0, 0, -1)$ by mean of a smooth dynamic time invariant output feedback

[1] see e.g. [Boscain et al., 2015]

Problem

Problem:
The equilibrium point x_t corresponds to the null input, which makes the system unobservable

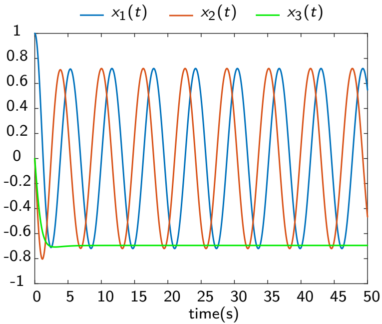

Figure: State variables of system (Σ) using a "naive" approach

Some results about output feedback:

- [Teel and Praly, 1994]
- [Coron, 1994]
- [Besancon and Hammouri, 2000]

1 Introduction

2 Practical stability

3 Numerical simulations

4 Ongoing work: exact stabilization

5 Conclusion

Stabilizing state feedback

Consider the state feedback

$$\lambda^{\mathrm{s}}(x) = r_1(x_1, x_2),$$

where r_1 is an arbitrary positive constant.

Proposition

The target point x_{t} is an asymptotically stable equilibrium for the closed-loop system resulting from applying the feedback control $u = \lambda^{\mathrm{s}}(x)$ to system (Σ). Moreover, its basin of attraction is $S^2 \setminus \{-x_{\mathrm{t}}\}$.

Sketch of proof.

A direct application of LaSalle's principle (see e.g. [LaSalle, 1968]) using

$$V(x) = x_3,$$

as a candidate Lyapunov function gives the desired result. □

The observer

The equations of the controller-observer system are

$$\begin{cases} \dot{\hat{x}} = A\left(\lambda^{\mathrm{s}}_{\delta}(\hat{x})\right)\hat{x} - r_2 C' C \varepsilon \\ \dot{\varepsilon} = \left(A\left(\lambda^{\mathrm{s}}_{\delta}(\hat{x})\right) - r_2 C' C\right)\varepsilon, \quad (\hat{x}, \varepsilon) \in \mathbb{R}^3 \times \mathbb{R}^3, \end{cases} \qquad \text{(CLO)}$$

where

$$\lambda^{\mathrm{s}}_{\delta}(\hat{x}) = \lambda^{\mathrm{s}}(\hat{x}) + (\delta, \delta),$$

and δ and r_2 are positive constants.

The observer

Lemma 1

All the inputs of the form $u = \lambda_\delta^{\mathrm{s}}(\hat{x})$ *applied to the full coupled system* (CLO) *make system* (Σ) *observable on any time interval* $[0, T]$, $T > 0$.

Sketch of proof.

- By contradiction: there exist a positive T and an input $\lambda_\delta^{\mathrm{s}}(\hat{x}(\cdot))$ that renders system (Σ) unobservable on $[0, T]$
- There exists a $\omega(\cdot) = (\omega_1, \omega_2, \omega_3) \not\equiv 0$ solution of (Σ) such that $\omega_3(\cdot) \equiv 0$
- Differentiating with respect to t and solving with respect to $\omega(\cdot)$, we get that $\omega(\cdot)$ vanishes identically □

Step 1 : Estimation errors goes to zero

Definition (from [Celle et al., 1989])

A persistent input (for bilinear systems) is a measurable bounded input u for which there exists a time interval $T > 0$, such that

$$\limsup_{\theta \to +\infty} \operatorname{ind}(u(\cdot + \theta), T) > 0,$$

where $\operatorname{ind}(u(\cdot), T)$ *is the index of universality of u on $[0, T]$, i.e. the smallest eigenvalue of the Gram-observability matrix.*

Corollary

All the inputs $\lambda^{\mathrm{s}}_{\delta}(\hat{x}(\cdot))$ are persistent.

Step 1 : Estimation error goes to zero

Sketch of proof.

- $x \in S^2$ and ε is decreasing (since $\frac{1}{2}\frac{d}{dt}\|\varepsilon\|^2 = -r_2(C\varepsilon)^2$)
  $\longrightarrow \{(\hat{x}(t), \varepsilon(t)) \mid t \geqslant 0\}$ lies in a compact K
- The mapping

$$\begin{array}{rccc} F: & \mathbb{R}^3 \times \mathbb{R}^3 & \to & \mathbb{R}_+ \\ & (\hat{x}_0, \varepsilon_0) & \mapsto & \mathrm{ind}(\lambda_\delta^{\mathrm{s}}(\hat{x}(\cdot)), T) \end{array}$$

  is continuous and nonnegative for all $T > 0$
- Since

$$\inf_K F \leqslant \limsup_{\theta \to +\infty} F(\hat{x}(\theta), \varepsilon(\theta)) = \limsup_{\theta \to +\infty} \mathrm{ind}(\lambda_\delta^{\mathrm{s}}(\hat{x}(\cdot + \theta)), T)$$

- By continuity, the infimum of F over K is reached, and is positive by the crucial Lemma 1 □

Step 1 : Estimation error goes to zero

Theorem ([Celle et al., 1989])

If $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^2)$ *is a persistent input, then the observation error tends to zero, i.e.*

$$\lim_{t \to +\infty} \varepsilon(t) = 0.$$

Corollary 2

For any trajectory of the coupled system (CLO)*, we have* $\lim_{t \to +\infty} \varepsilon(t) = 0$.

Step 2 : Asymptotic stability of an equilibrium point

Lemma 3

If δ is small enough, system (CLO) admits an asymptotically stable equilibrium point $(x_\delta^{\mathrm{s}}, 0)$ arbitrarily close to $(x_{\mathrm{t}}, 0)$ and an unstable equilibrium point $(x_\delta^{\mathrm{u}}, 0)$ arbitrarily close to $(-x_{\mathrm{t}}, 0)$.

Sketch of proof.

- Compute the two equilibrium points
- Rewrite the system using the constraint $\|x\| = \|\hat{x} - \varepsilon\| = 1$
- Linearize around the equilibrium points
- Perform the stability analysis on the linearized system □

Step 3 : The main result

Theorem

If δ is small enough, the point $(x^{\mathrm{s}}_{\delta}, 0) \in \mathbb{R}^6$ is asymptotically stable for system (CLO), and its region of attraction is $\mathbb{R}^6 \setminus \{(x^{\mathrm{u}}_{\delta}, 0)\}$.

Sketch of proof.

- From Corollary 2 the ω-limit points of system (CLO) are of the form $(\hat{x}, 0)$.
- Set $C^- = \{x \in S^2 \mid x_3 \leqslant 0\}$ and consider the function $L : C^- \to \mathbb{R}_+$ defined by $L(\hat{x}) = \frac{1}{2}\|\hat{x} - x^{\mathrm{s}}_{\delta}\|^2$.
- Using LaSalle's principle and Lemma 3 we get the desired result.

□

1 Introduction
2 Practical stability
3 Numerical simulations
4 Ongoing work: exact stabilization
5 Conclusion

Output feedback with perturbation

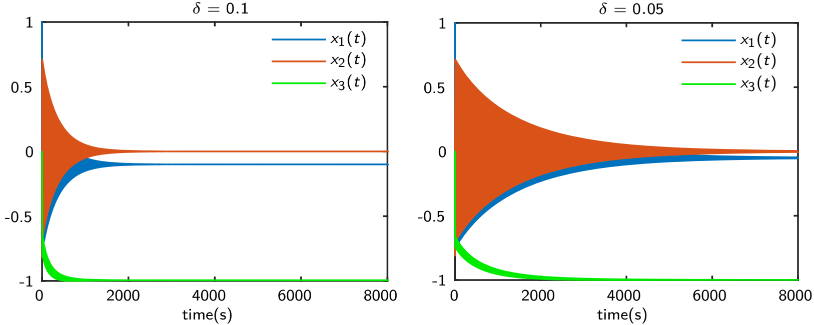

Figure: State variables of system (Σ) with $u = \lambda_\delta^{\mathrm{s}}(\hat{x})$ for $\delta = 0.1$ and $\delta = 0.05$.

Output feedback with perturbation

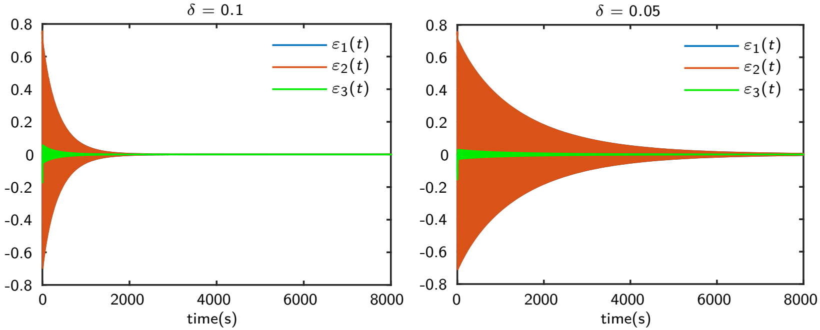

Figure: Observation errors of system (Σ) with $u = \lambda_\delta^{\mathrm{s}}(\hat{x})$ for $\delta = 0.1$ and $\delta = 0.05$.

1 Introduction

2 Practical stability

3 Numerical simulations

4 Ongoing work: exact stabilization

5 Conclusion

Decreasing perturbation

Consider the feedback

$$\lambda^{\mathrm{s}}_{\delta,\alpha}(\hat{x}) = r_1(\hat{x}_1, \hat{x}_2) + (|\hat{x}_3|^{\alpha} - 1)(\delta, \delta).$$

where r_1, δ and α are arbitrary positive constants.

The equations of the controller-observer system are

$$\begin{cases} \dot{\hat{x}} = A\left(\lambda^{\mathrm{s}}_{\delta,\alpha}(\hat{x})\right)\hat{x} - r_2 C' C\varepsilon \\ \dot{\varepsilon} = \left(A\left(\lambda^{\mathrm{s}}_{\delta,\alpha}(\hat{x})\right) - r_2 C' C\right)\varepsilon, \quad (\hat{x}, \varepsilon) \in \mathbb{R}^6. \end{cases} \tag{CLO2}$$

Main result

Lemma

The point $(x_t, 0)$ *is asymptotically stable for system* (CLO2) *and its basin of attraction is* $\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{-x_t, 0\}$.

Sketch of proof.

- Prove that $\lambda^s_{\delta,\alpha}$ is a stabilizing state feedback for system (Σ)
- Write system (CLO2) in $\mathbb{R}^5$ using $\|\hat{x} - \varepsilon\| = 1$ and prove that $0 \in \mathbb{R}^5$ is (locally) stable
- Prove that any trajectory of system (CLO2) converges to $(x_t, 0)$ using ω-limit arguments □

Numerical simulations

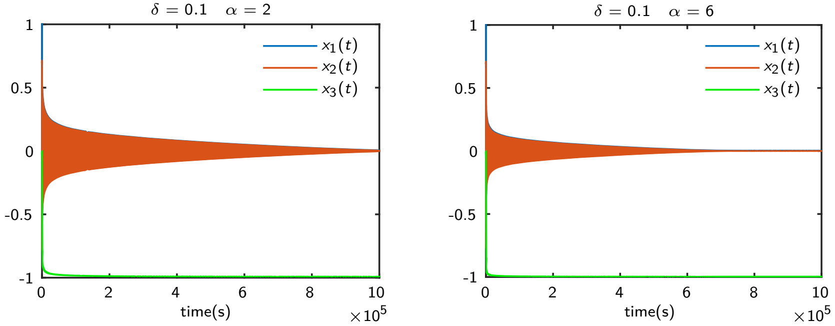

Figure: State variables of system (Σ) with $u = \lambda_\delta^{\mathrm{s}}(\hat{x})$ for $\alpha = 2$ and $\alpha = 6$ ($\delta = 0.1$ in both cases).

Numerical simulations

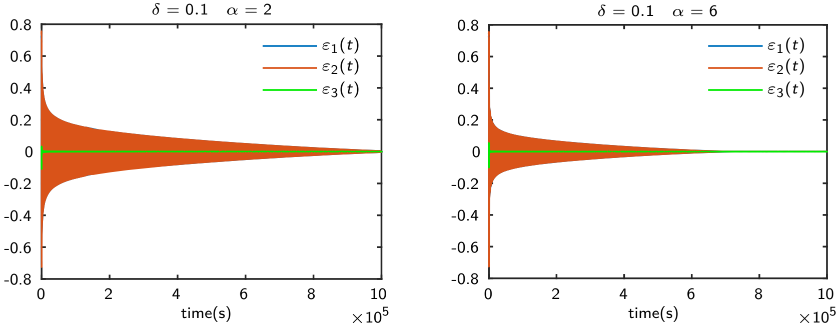

Figure: Observation errors of system (Σ) with $u = \lambda_{\delta}^{\mathrm{s}}(\hat{x})$ for $\alpha = 2$ and $\alpha = 6$ ($\delta = 0.1$ in both cases).

1 Introduction

2 Practical stability

3 Numerical simulations

4 Ongoing work: exact stabilization

5 Conclusion

Conclusion

For a particular example we saw:

- a method to stabilize the system arbitrarily close to the target point
- a method to stabilize the system to the target point
- in both cases the feedbacks are smooths and time invariant
- the second method is a lot slower than the first one

Perspectives:

- Find more qualitative proofs and extend to more general systems (of the same form)
- Extend to bilinear systems

Joyeux anniversaire Jean-Michel !

References

Besancon, G. and Hammouri, H. (2000).
Some remarks on dynamic output feedback control of non uniformly observable systems.
In *Decision and Control, 2000. Proceedings of the 39th IEEE Conference on*, volume 3, pages 2462–2465 vol.3.

Boscain, U., Gauthier, J.-P., Rossi, F., and Sigalotti, M. (2015).
Approximate controllability, exact controllability, and conical eigenvalue intersections for quantum mechanical systems.
Comm. Math. Phys., 333(3):1225–1239.

Celle, F., Gauthier, J.-P., Kazakos, D., and Sallet, G. (1989).
Synthesis of nonlinear observers: a harmonic-analysis approach.
Math. Systems Theory, 22(4):291–322.

Coron, J.-M. (1994).
On the stabilization of controllable and observable systems by an output feedback law.
Math. Control Signals Systems, 7(3):187–216.

Gauthier, J.-P. and Kupka, I. (2001).
Deterministic observation theory and applications.
Cambridge University Press, Cambridge.

LaSalle, J. P. (1968).
Stability theory for ordinary differential equations.
J. Differential Equations, 4:57–65.

Teel, A. and Praly, L. (1994).
Global stabilizability and observability imply semi-global stabilizability by output feedback.
Systems Control Lett., 22(5):313–325.