Pointwise second-order necessary optimality conditions and sensitivity relations

Nonlinear Partial Differential Equations and Applications

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joint work with Hélène Frankowska

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Problem:

$$\text{Minimize} \qquad \varphi(x(1))$$

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \forall\, t \\ u(t) \in U(t) & \forall\, t \\ x(0) \in K_0. \end{cases}$$

Optimal Control Problem

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Assumptions:

- f satisfies standard assumptions and is twice differentiable in x with Lipschitz derivative
- φ is differentiable
- K_0 and $U(t)$ are closed and nonempty

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Value function

$$V(t, x_0) = \inf \{\varphi(x(1)) \mid x \text{ solution of problem starting at } (t, x_0)\}.$$

Maximum Principle

Let $(\bar{x}, \bar{u})$ be an optimal solution. The solution $\bar{p}$ of

$$\begin{cases} -\dot{\bar{p}}(t) = f_x(t, \bar{x}(t), \bar{u}(t))^T \bar{p}(t) \\ -\bar{p}(1) = \nabla\varphi(\bar{x}(1)), \end{cases}$$

satisfies for all t

i) $\langle \bar{p}(0), k_0 \rangle \leq 0 \quad \forall k_0 \in T^{\flat}_{K_0}(\bar{x}(0))$

ii) $\langle \bar{p}(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle = \max_{u \in U(t)} \langle \bar{p}(t), f(t, \bar{x}(t), u) \rangle$.

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- Pointwise necessary condition

Maximum Principle

Let $(\bar{x}, \bar{u})$ be an optimal solution. The solution $\bar{p}$ of

$$\begin{cases} -\dot{\bar{p}}(t) = \mathcal{H}_x[t](t) \\ -\bar{p}(1) = \nabla\varphi(\bar{x}(1)), \end{cases}$$

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2. $\mathcal{H}[t] = \max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), \bar{p}(t), u)$.

- Pointwise necessary condition
- $\mathcal{H}(t, x, p, u) = \langle p, f(t, x, u) \rangle$
- $[t] = (t, \bar{x}(t), \bar{p}(t), \bar{u}(t))$

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Sensitivity Relations

Let $\bar{x}$ be an optimal solution. The solution $\bar{p}$ of

$$\begin{cases} -\dot{\bar{p}}(t) = \mathcal{H}_x[t](t) \\ -\bar{p}(1) = \nabla\varphi(\bar{x}(1)), \end{cases}$$

satisfies

$$-\bar{p}(t) \in \partial^+_x V(t, \bar{x}(t)) \quad \forall\, t.$$

- ∂^+_x is the superdifferential in x.

Introduction II

2nd-order pointwise conditions

- Goh conditions
  - U is time independent
  - Requires structural assumptions on U
- Jacobson conditions
  - continuous optimal control
  - optimal control in interior of U
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- Relationship with regularity of the value function

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- Investigated using matrix Riccati equations
- Relationship with regularity of the value function
- Link to higher order necessary conditions?

PRELIMINARIES

Tangent cone

$$T_K^\flat(x) = \operatorname*{Liminf}_{h\to 0+} \frac{K-x}{h}$$

2nd-order tangent

$$T_K^{\flat(2)}(x,u) = \operatorname*{Liminf}_{h\to 0+} \frac{K-hu-x}{h^2}$$

Normal cone

$$N_K^\flat(x) = \left\{ q \;\middle|\; \langle q,u\rangle \le 0 \quad \forall u \in T_K^\flat(x) \right\}$$

2nd-order normal

$$N_K^{\flat(2)}(x,q) = \left\{ Q \in \mathbf{S} \;\middle|\; \langle q,v\rangle + \frac{1}{2}\langle Qu,u\rangle \le 0 \right.$$
$$\left. \forall u \in T_K^\flat(x) \cap \{q\}^\perp, v \in T_K^{\flat(2)}(x,u) \right\}$$

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Superjet

$$J^{2,+}f(x) = \left\{ (q,Q) \;\middle|\; f(y) - f(x) \le \langle q, y-x\rangle + \frac{1}{2}\langle Q(y-x), y-x\rangle + o(|y-x|^2), \quad \forall y \right\}$$

Subjet

$$J^{2,-}f(x) = \left\{ (q,Q) \;\middle|\; \qquad \ge \qquad \right\}$$

Theorem.

Let $(\bar{x}, \bar{u}, \bar{p})$ be an optimal solution and adjoint state. Then for every $\Psi \in \mathbf{S}$ such that $(\nabla\varphi(\bar{x}(1)), \Psi) \in J^{2,+}\varphi(\bar{x}(1))$, the solution W of

$$\begin{cases} \dot{W}(t) = -\mathcal{H}_{px}[t]W(t) - W(t)\mathcal{H}_{xp}[t] - \mathcal{H}_{xx}[t], \\ W(1) = -\Psi, \end{cases}$$

satisfies

i) $W(0) \in N^{\flat(2)}_{K_0}(\bar{x}(0); \bar{p}(0))$

ii) $\max_{(v,M) \in \bar{F}(t)} \left\langle M^T \bar{p}(t) + W(t)v, v \right\rangle = 0, \quad$ a.e. in $[0, 1]$.

Notation

$\bar{F}(t) = \mathrm{co}\left\{(f(t, \bar{x}(t), u), f_x(t, \bar{x}(t), u)) - (f[t], f_x[t])) \mid u \in \bar{U}(t))\right\}$

$\bar{U}(t) = \left\{z \in U(t) \,\middle|\, z \in \arg\max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), \bar{p}(t), u)\right\}$

Theorem (Backward propagation).

Let $\bar{x}$ be optimal with $\bar{x}(t_0) = x_0$ and $\bar{p}$, W and Ψ be as in the maximum principles. Then

$$(-\bar{p}(t), -W(t)) \in J_x^{2,+} V(t, \bar{x}(t)), \ \forall\, t \in [t_0, 1].$$

Second-order sensitivity relations

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Theorem (Forward propagation).

Let $\bar{x}$ and $\bar{p}$ be as above. If for some $W_0 \in \mathbf{S}$ we have $(-\bar{p}(t_0), -W_0) \in J_x^{2,-} V(t_0, x_0)$. Then for

$$\begin{cases} \dot{W}(t) + \mathcal{H}_{px}[t] W(t) + W(t) \mathcal{H}_{xp}[t] + \mathcal{H}_{xx}[t] = 0, \\ W(t_0) = W_0, \end{cases}$$

the following sensitivity relation holds true:

$$(-\bar{p}(t), -W(t)) \in J_x^{2,-} V(t, \bar{x}(t)), \ \forall\, t \in [t_0, 1].$$

Jacobson type necessary conditions I

Relaxed Assumption:

- $f(t,\cdot,\cdot)$ is twice differentiable with Lipschitz derivatives
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Objectives:

1. Replace $\left\{ z \in U(t) \;\middle|\; z \in \arg\max_{u \in U(t)} \mathcal{H}(t, \bar{x}(t), \bar{p}(t), u) \right\}$ with more "explicit" set (tangents)
2. Pass to the limit in the expression

$$\max_{(v,M) \in \bar{F}(t)} \left\langle M^T \bar{p}(t) + W(t)v, v \right\rangle = 0, \quad \text{a.e. in } [0,1]$$

Theorem.

Let $\bar{x}$, $\bar{u}$, $\bar{p}$ and W be as in the second-order maximum principle with $\Psi := \varphi''(\bar{x}(1))$.

For a.e. $t \in [0,1]$ and for every $u \in T^{\flat}_{U(t)}(\bar{u}(t))$ such that either

(i) $\mathcal{H}_u[t] \neq 0$, $\mathcal{H}_u[t]u = 0$ and $\mathcal{H}_u[t]v + \frac{1}{2}u^T\mathcal{H}_{uu}[t]u = 0$ for some $v \in T^{\flat(2)}_{U(t)}(\bar{u}(t), u)$,

or

(ii) $\mathcal{H}_u[t] = 0$ and $u^T\mathcal{H}_{uu}[t]u = 0$,

we have

$$\left\langle f_u[t]^T \left(\mathcal{H}_{ux}[t] + W(t)f_u[t]\right) u, u \right\rangle \leq 0.$$

Inequality constraints

Assumption:

- $U(t) = \bigcap_{j=1}^{s} \{u \mid c^j(t,u) \leq 0\}$,
- c^j twice differentiable in u
- $\{\nabla_u c^j(t, \bar{u}(t))\}_{j=1}^{s}$ are linearly independent

INEQUALITY CONSTRAINTS

Assumption:

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Corollary.

Let $\bar{x}$, $\bar{u}$, $\bar{p}$ and W be as in the Jacobson conditions. Then there exist measurable, uniquely defined $\alpha_j \colon [0,1] \to \mathbb{R}_+$, such that for a.e. t,

(i) $\alpha_j(t) c^j(t, \bar{u}(t)) = 0$ for all $j \in \{1, \ldots, s\}$;

(ii) $\mathcal{H}_u[t] = \sum_{j=1}^{s} \alpha_j(t) \nabla_u c^j(t, \bar{u}(t))$;

(iii) $\max_{u \in U_0(t)} \langle f_u[t]^T (\mathcal{H}_{ux}[t] + W(t) f_u[t])\, u, u \rangle = 0$.

$$U_0(t) := \left\{ u \in T^{\flat}_{U(t)}(\bar{u}(t)) \;\middle|\; \mathcal{H}_u[t]u = 0 \text{ and } u^T \left(\mathcal{H}_{uu}[t] - \sum_{j=1}^{s} \alpha_j(t) c^j_{uu}(t, \bar{u}(t)) \right) u = 0 \right\}.$$

Example Part I

- $f(x,u) = \begin{pmatrix} u_1 & u_1 + u_2 & -x_1 x_2 + 9u_2^2 \end{pmatrix}^T$
- $\varphi(x) = x_3$
- $K_0 = \{(0,0,0)\}$
- $U = \left\{ u \in \mathbb{R}^2 \;\middle|\; 0 \leq u_2 \leq u_1 \leq 1 \right\}$

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Let $\bar{x}$, $\bar{u}$ be a candidate for optimality.

Observations I

- $\bar{x}_1$ and $\bar{x}_2$ are non negative and increasing
- $\bar{p}_3 \equiv -1$
- $\bar{p}_1$ and $\bar{p}_2$ are non negative and decreasing

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Hamiltonian

- $\mathcal{H}[t] = (\bar{p}_1(t) + \bar{p}_2(t))\bar{u}_1(t) + (\bar{p}_2(t) - 9\bar{u}_2(t))\,\bar{u}_2(t) + \bar{x}_1(t)\bar{x}_2(t)$

Example Part II

- $f(x,u) = \begin{pmatrix} u_1 & u_1 + u_2 & -x_1 x_2 + 9u_2^2 \end{pmatrix}^T$
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Case 1: $\bar{u}_1 \equiv 0$

- $(\bar{x}, \bar{u}) \equiv 0$, $(\bar{p}_1, \bar{p}_2) \equiv 0$, $\mathcal{H}_u[t] = 0$

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Critical Directions:

- $u^T \mathcal{H}_{uu}[t] u = 0$ for all $u \in \left\{ v \in T_U^\flat(0) \;\middle|\; v_2 = 0 \right\}$

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Second-order maximality condition:

- $0 \overset{!}{\ge} \langle W(t) f_u[t]u, f_u[t]u \rangle = 2u_1^2(1-t)$

Example Part III

- $f(x,u) = \begin{pmatrix} u_1 & u_1 + u_2 & -x_1x_2 + 9u_2^2 \end{pmatrix}^T$
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Case 2: $\bar{u}_2 > 0$ on set of positive measure

- For all t, only one tuple $(\bar{u}_1, \bar{u}_2)$ satisfies the (first-oder) maximality condition.

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Conclusion: Only candidate who satisfies first- and second order conditions is a global optimum.

Conclusions and Future Work

Main results

- Second-order maximum principle for general control constraints
- Jacobson type optimality conditions without structural assumptions on the optimal control
- Second-order sensitivity relations analogous the the first-order case

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Thank you