Singular Optimal Control : a Degenerate Parabolic-Hyperbolic example

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Nonlinear Partial Differential Equations and Applications

A conference in the honor of Jean-Michel Coron for his 60th birthday

Introduction

Let $(\varepsilon, T, L, M) \in (0, +\infty)^3 \times \mathbb{R}$. We consider

$$\begin{cases} y_t - \varepsilon y_{xx} + M y_x = 0 & \text{in } (0, L) \times (0, L), \\ y(0,t) = u(t),\ y(L,t) = 0 & \text{on } (0, T), \\ y(x, 0) = y^0(x) & \text{in } (0, L). \end{cases} \tag{TD}$$

For $y^0 \in H^{-1}(0, L)$ we denote by $U(\varepsilon, T, L, M, y^0)$ the set of controls $u \in L^2(0, T)$ such that the corresponding solution of (TD) satisfies $y(\cdot, T) \equiv 0$.

We can define the quantity which measures the cost of the null controllability of (TD):

$$K(\varepsilon, T, L, M) := \sup_{\|y^0\|_{H^{-1}(0,L)} \leq 1} \left\{ \min\{\|u\|_{L^2(0,T)} : u \in U(\varepsilon, T, L, M, y^0)\} \right\}.$$

The underlying transport equation is controllable if and only if $T > L/|M|$.

J.-M. Coron and S. Guerrero Singular optimal control: A linear 1-D parabolic-hyperbolic example, 2005.

State of the art

- ✓ O. Glass 2009, Uniform controllability
- ✓ P. Lissy 2012, Link with cost of controllability in small time
- ✓ S. Guerrero, G. Lebeau 2007, Higher dimension and Lipschitz transport coefficient
- ✓ O. Glass, S Guerrero 2007, Uniform controllability of the Burgers equation
- ✓ M. Léautaud 2010, Uniform controllability of scalar conservation laws

Degenerate transport diffusion equations

Let $(\varepsilon, T, L, M, \alpha) \in (0,+\infty)^4 \times (0,1)$. We consider

$$\begin{cases} y_t - \varepsilon(x^{\alpha+1} y_x)_x + M x^\alpha y_x = 0 & \text{in } (0,L)\times(0,L), \\ y(0,t) = u(t),\ y(L,t) = 0 & \text{on } (0,T), \\ y(x,0) = y^0(x) & \text{in } (0,L). \end{cases} \tag{TD}$$

Assume that (TD) is null controllable in some space H, we denote by $U(\varepsilon, T, L, M, \alpha, y^0)$ the set of controls $u \in L^2(0,T)$ such that the corresponding solution of (TD) satisfies $y(\cdot, T) \equiv 0$.

- ✓ We are interested in the behaviour of the cost of the null controllability:

$$\mathcal{K}(\varepsilon, T, L, M, \alpha) := \sup_{\|y^0\|_H \le 1} \left\{ \min\{\|u\|_{L^2(0,T)} : u \in U(\varepsilon, T, L, M, \alpha, y^0)\} \right\}.$$

- ✓ For every $(\varepsilon, T, L, M, \alpha) \in (0,+\infty)^4 \times (0,1)$ such that $M/\varepsilon > \alpha$ and any $y^0 \in L^2((0,L); x^{-M/\varepsilon} \mathrm{d}x)$, there exits a control $u \in L^2(0,T)$ sush that the associated solution to (TD) satisfies $y(\cdot, T) \equiv 0$.

A singular Sturm-Liouville Problem

Consider the differential expression defined by

$$\mathcal{A}[y](x) := -\varepsilon(x^{\alpha+1}y')' + Mx^{\alpha}y' = \lambda y(x), \quad x \in (0, L),\ \lambda \in \mathbb{R}.$$

✓ Particular case of Bessel differential equation

$$x^2y'' + axy' + (bx^{\ell} + c)y = 0, \quad x \in (0, \infty), \quad \ell \neq 0.$$

The solutions can be written in terms of Bessel functions:

$$b \neq 0: \quad y(x) = x^{\frac{1}{2}(1-a)} Z_{\nu}(\kappa^{-1}\sqrt{b}x^{\kappa}), \quad \nu := \frac{1}{\ell}\sqrt{(1-a)^2 - 4c}, \quad \kappa := \frac{\ell}{2},$$

✓ Then, under the structural assumption $M/\varepsilon > \alpha$, we can prove that $(\mathcal{A}, D(\mathcal{A}))$ is self-adjoint on $L^2((0, L); x^{-M/\varepsilon}\mathrm{d}x)$ and generates an analytic semigroup of bounded linear operators $S(t)_{t\geq 0}$. Let $\nu := (M/\varepsilon - \alpha)/(1 - \alpha)$ and $\kappa := \frac{1}{2}(1 - \alpha)$, we have

$$\Phi^n(x) := \frac{(2\kappa)^{\frac{1}{2}}}{|J'_{\nu}(j_{\nu,n})|} x^{\frac{1}{2}(M/\varepsilon - \alpha)} J_{\nu}\left(j_{\nu,n}\, x^{\kappa}\right), \quad \lambda_n := \varepsilon(\kappa j_{\nu,n})^2 \quad x \in (0, 1).$$

Uniform Controllablity

There exist $\mathcal{Q}(T,L,M,\alpha) > 0$ and $\mathcal{C}(T,L,M,\alpha) > 0$ such that, for every $(\varepsilon,T,L,M,\alpha) \in (0,+\infty)^4 \times (0,1)$ such that $M/\varepsilon > \alpha$, we have

$$\mathcal{K}(\varepsilon,\alpha,T,L,M) \leq \exp\left(-\frac{\mathcal{Q}(T,L,M,\alpha)}{\varepsilon}\right) \quad \text{if} \quad T > \frac{(2\sqrt{6})L^{1-\alpha}}{M(1-\alpha)}$$

and

$$\mathcal{K}(\varepsilon,\alpha,T,L,M) \geq \exp\left(\frac{\mathcal{C}(T,L,M,\alpha)}{\varepsilon}\right) \quad \text{if} \quad T < \frac{(0,98)L^{1-\alpha}}{M(1-\alpha)}.$$

✓ G., P. Lissy, Singular optimal control of a $1-D$ Parabolic-Hyperbolic Degenerate equation, accepted in ESAIM- Control Optim. Calc. Var.

The moment method

✓ Let $y^0 \in L^2((0,L); x^{-M/\varepsilon}\mathrm{d}x) := H$. Then $u \in U(\varepsilon, T, L, M, \alpha, y^0)$ if and only if

$$\left\langle y^0, \Phi^n \right\rangle_H = -\frac{(M/\varepsilon - \alpha)(2j_{\nu,n})^\nu}{2^\nu \Gamma(\nu+1)} \int_0^T u(t) \exp\left(-\lambda_n (T-t)\right) \mathrm{d}t, \quad \forall n \in \mathbb{N}\backslash\{0\}.$$

✓ Find a biorthogonal family $\{\Psi_k(t)\}_{k\in\mathbb{N}\backslash\{0\}}$

$$\int_0^T \Psi_k(t) e^{-\lambda_\ell (T-t)} dt = \delta_{k\ell}, \quad k, \ell \in \mathbb{N}\backslash\{0\}.$$

✓ Construct a family $\{J_k(z)\}_{k\in\mathbb{N}\backslash\{0\}}$ of entire functions of exponential type satisfying

$$J_k(-i\lambda_\ell) = \delta_{k\ell}, \quad k, \ell \in \mathbb{N}\backslash\{0\}.$$

✓ Then using Paley-Wiener theorem to construct the biorthogonal family by inverse Fourier transform.

Some elements of proof I

An entire function having $\{-i\varepsilon(\kappa j_{\nu,k})^2,\ k \in \mathbb{N}\backslash\{0\}\}$ as simple zeros is

$$\Lambda(z) := \prod_{k=1}^{+\infty}\left(1 - \frac{iz}{\varepsilon(\kappa j_{\nu,k})^2}\right) = \Gamma(\nu+1)\left(\frac{2\sqrt{\varepsilon}\kappa}{\sqrt{iz}}\right)^{\nu} J_\nu\left(\frac{\sqrt{iz}}{\sqrt{\varepsilon}\kappa}\right).$$

Moreover, $\Lambda(\cdot)$ is of exponential type and

$$|\Lambda(z)| \leq \exp\left(\frac{\sqrt{|z|}}{\kappa\sqrt{\varepsilon}}\right) \quad \text{as} \quad |z| \to +\infty.$$

✓ Now, we consider

$$\tilde{J}_k(z) := \frac{\Lambda(z)}{\Lambda'(-i\lambda_k)(z+i\lambda_k)},$$

one easily deduces that

$$\tilde{J}_k(-i\lambda_l) = \delta_{kl}.$$

✓ $\tilde{J}_k(z)$ cannot be bounded on the real line.

Some elements of proof I

We must use a multiplier to make the functions $\tilde{J}_k(\cdot)$ bounded on the real line and of relevant exponential type. Let us set

$$J_k(z) := \tilde{J}_k(z)\frac{H(z)}{H(-i\lambda_k)}.$$

✓ Where, H is constructed to satisfy

$$\begin{cases} H(-i\lambda_k) \geq C_\beta & \forall\, k \geq 1, \\ |H(z)| \leq e^{\beta|\Im(z)|} & \forall\, z \in \mathbb{C}, \\ H(ix) \geq Ce^{\gamma|x|} & \forall\, x \in \mathbb{R}. \end{cases} \qquad \text{(Mult)}$$

C, β and γ to be chosen in terms of $\varepsilon, T, M, \alpha$. We follow G. Tenenbaum and M. Tucsnak.

✓ Precise asymptotics estimates for Bessel functions and their zeros.

Some elements of proof II

- Let u the optimal control assiociated to the first eigenvalue. Let us introduce the function $f : \mathbb{C} \to \mathbb{C}$ define by

$$f(s) := \int_{-T/2}^{-T/2} u\left(t + \frac{T}{2}\right) e^{-it\left(\frac{s-i\delta}{\varepsilon}\right)} \mathrm{d}t \quad s \in \mathbb{C}.$$

  for a $\delta > 0$, that will be choosen later. Then, f is an entire function satifying

$$f(a_k) = 0, \quad k \in \mathbb{N}\backslash\{0,1\}, \quad \text{with} \quad a_k := i\left((\varepsilon\kappa j_{\nu,k})^2 + \delta\right), \quad k \in \mathbb{N}\backslash\{0\}.$$

- Moreover, f satisfies

$$\log|f(s)| \leq \frac{T|\Im(s) - \delta|}{2\varepsilon} + \log\left(\mathcal{K}T^{1/2}\frac{|J'_\nu(j_{\nu,1})|}{\sqrt{2\kappa}}\right).$$

- Classical representation of entire functions of exponential type $\mathcal{A}$ in $\mathbb{C}^+$

$$\log|f(z)| = \mathcal{A}\Im(z) + \sum_{\ell=1}^{\infty} \log\left|\frac{z - a_\ell}{z - \overline{a}_\ell}\right| + \frac{\Im(z)}{\pi}\int_{-\infty}^{+\infty} \frac{\log|f(s)|}{|s - z|^2}\mathrm{d}s.$$

Some open problems

- ✓ What to do if $M < 0$.
- ✓ Diffusion with constant coefficient.
- ✓ BV coefficients.
- ✓ Higher dimension

Happy Birthday Jean-Michel