On the Stackelberg strategies in control theory

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Dedicated to Jean-Michel Coron in his 60th birthday

Outline

CONTROL PROBLEMS

What is usual: act to get good (or the best) results for

$$\begin{cases} E(U) = F \\ + \dots \end{cases}$$

What is easier? Solving? Controlling?

Two classical approaches:

- Optimal control
- Controllability

OPTIMAL CONTROL

A general optimal control problem

Minimize $J(v)$
Subject to $v \in \mathcal{V}_{ad}, \quad y \in \mathcal{Y}_{ad}, \ (v, y)$ satisfies

$$E(y) = F(v) \quad + \ \ldots \qquad (S)$$

Main questions: $\exists$, uniqueness/multiplicity, characterization, computation, . . .

We could also consider similar bi-objective optimal control:

"Minimize" $J_1(v), J_2(v)$
Subject to $v \in \mathcal{V}_{ad}, \ldots$

CONTROLLABILITY

A null controllability problem

$$\text{Find } (v, y)$$
$$\text{Such that } v \in \mathcal{V}_{ad},\ (v, y) \text{ satisfies } (ES),\ y(T) = 0$$

with $y : [0, T] \mapsto H$,

$$E(y) \equiv y_t + A(y) = F(v) \quad + \ \dots \qquad (ES)$$

Again many interesting questions: $\exists$, uniqueness/multiplicity, characterization, computation, . . .

A very rich subject for PDEs, see [Russell, J.-L. Lions, Coron, Zuazua, . . .]

Question: How can we adopt both viewpoints together?

Example: Optimal-control / controllability problem
A simplified model for the autonomous car driving problem

The system:

$$\dot{x} = f(x, u), \quad x(0) = x_0$$

Constraints:

$$\text{dist.}\,(x(t), Z(t)) \geq \varepsilon \quad \forall t$$
$$u \in \mathcal{U}_{ad} \quad (|u(t)| \leq C)$$

u determines direction and speed

Goals (prescribed x_T and $\hat{x}$):

- $x(T) = x_T$ (or $|x(T) - x_T| \leq \varepsilon \dots$)
- Minimize $\sup_t |x(t) - \hat{x}(t)|$

[Sontag, Sussman-Tang, . . .]

Optimal control + controllability
Automatic driving

Figure: The ICARE Project, INRIA, France. Autonomous car driving. Malis-Morin-Rives-Samson, 2004

The car in the street

Figure: Nissan ID. Autonomous car driving. 2015–2020

What is announced:

- Nissan ID 1.0 (2015), highways and traffic jams (no lane change)
- ID 2.0 (2018), overtaking and lane change
- ID 3.0 (2020), complete autonomous driving in town

http://reports.nissan-global.com/EN/?p=17295

Another way to connect optimal control and controllability:
HIERARCHICAL CONTROL (Stackelberg)

The main ideas in the context of Navier-Stokes:
Two controls - one leader, one follower

$$\begin{cases} y_t + (y \cdot \nabla) y - \Delta y + \nabla p = f 1_{\mathcal{O}} + v 1_{\omega}, & (x,t) \in \Omega \times (0,T) \\ \nabla \cdot y = 0, & (x,t) \in \Omega \times (0,T) \\ y = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y^0(x), & x \in \Omega \end{cases}$$

Different domains $\mathcal{O}$, ω

Two objectives:

- Get $y \approx y_d$ in $\mathcal{O}_d \times (0,T)$, with reasonable effort:

$$\text{Minimize } \alpha \iint_{\mathcal{O}_d \times (0,T)} |y - y_d|^2 + \mu \iint_{\omega \times (0,T)} |v|^2$$

  An optimal control problem
- Get $y(T) = 0$ - A null controllability problem

Before explaining what to do . . . let us complicate the situation!

BEYOND: A MORE COMPLEX CONTROL PROBLEM, NAVIER-STOKES (Stackelberg-Nash, Stackelberg-Pareto, . . .)

Three controls: one leader, two followers

$$\begin{cases} y_t + (y\cdot\nabla)y - \Delta y + \nabla p = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, & (x,t)\in\Omega\times(0,T) \\ \nabla\cdot y = 0, & (x,t)\in\Omega\times(0,T) \\ y = 0, & (x,t)\in\partial\Omega\times(0,T) \\ y(x,0) = y^0(x), & x\in\Omega \end{cases}$$

Different domains $\mathcal{O}$, $\mathcal{O}_i$, $(i = 1, 2)$

Three objectives:

- "Simultaneously", $y \approx y_{i,d}$ in $\mathcal{O}_{i,d}\times(0,T)$, $i = 1, 2$, reasonable effort:

  $$\text{Minimize } \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2, \quad i = 1, 2$$

  Bi-objective optimal control - The task of the followers
  In practice, an equilibrium $(v_1(f), v_2(f))$ for each f?
- Get $y(T) = 0$
  Null controllability - The task of the leader
  Can we find f such that $y(T) = 0$?

$$\begin{cases} y_t + (y\cdot\nabla)y - \Delta y + \nabla p = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, & (x,t)\in\Omega\times(0,T) \\ \nabla\cdot y = 0, & (x,t)\in\Omega\times(0,T) \\ y = 0, & (x,t)\in\partial\Omega\times(0,T) \\ y(x,0) = y^0(x), & x\in\Omega \end{cases}$$

Many applications:

- Heating: Controlling temperatures
  Heat sources at different locations - Heat PDE (linear, semilinear, etc.)
- Tumor growth: Controlling tumor cell densities
  Radiotherapy strategies - Reaction-diffusion PDEs
  bilinear control
- Fluid mechanics: Controlling fluid velocity fields
  Several mechanical actions - Stokes, Navier-Stokes or similar
- Finances: Controlling the price of an option
  Agents at different stock prices, etc. - Backwards in time heat-like PDE
  Degenerate coefficients

Contributions: Lions, Díaz-Lions, Glowinski-Periaux-Ramos, Guillén, . . .
Optimal control + AC

TOO DIFFICULT - A SIMPLIFIED PROBLEM

Again three controls: one leader, two followers

$$(H) \qquad \begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} + v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, & (x,t) \in (0,1)\times(0,T) \\ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \quad x \in (0,1) \end{cases}$$

Different intervals $\mathcal{O}$, $\mathcal{O}_i$

Again three objectives:

- Simultaneously, $y \approx y_{i,d}$ in $\mathcal{O}_{i,d} \times (0,T)$, $i = 1,2$, reasonable effort:

  $$\text{Minimize} \quad \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i \times (0,T)} |v_i|^2, \quad i = 1,2$$

  Bi-objective optimal control - Followers' task

- Get $y(T) = 0$
  Null controllability - Leader's task

What can we do?

THE STACKELBERG-NASH STRATEGY

Step 1: f is fixed

$$J_i(v_1, v_2) := \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2, \quad i = 1, 2$$

Find a Nash equilibrium $(v_1(f), v_2(f))$ with $v_i(f) \in L^2(\mathcal{O}_i \times (0,T))$:

$$J_1(v_1(f), v_2(f)) \le J_1(v_1, v_2(f)) \quad \forall v_1 \in L^2(\mathcal{O}_1 \times (0,T))$$
$$J_2(v_1(f), v_2(f)) \le J_2(v_1(f), v_2) \quad \forall v_2 \in L^2(\mathcal{O}_2 \times (0,T))$$

Equivalent to:

$$(HN) \quad \begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} - \dfrac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \dfrac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_i}, \quad i = 1, 2 \\ \phi_i(0,t) = \phi_i(1,t) = 0, \ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \ \phi_i(x,T) = 0, \quad x \in (0,1) \end{cases}$$

Then: $v_i(f) = -\frac{1}{\mu}\phi_i|_{\mathcal{O}_i\times(0,T)}$ (Pontryagin)

$\exists (v_1(f), v_2(f))$? Uniqueness?

THE STACKELBERG-NASH STRATEGY

Step 2: Find f such that

$$(HSN)_1 \quad \left\{ \begin{array}{l} y_t - y_{xx} = f 1_{\mathcal{O}} - \frac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \frac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_i}, \quad i = 1, 2 \\ \phi_i(0,t) = \phi_i(1,t) = 0, \ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \ \phi_i(x,T) = 0, \quad x \in (0,1) \end{array} \right.$$

$$(HSN)_2 \qquad y(x,T) = 0, \quad x \in (0,1)$$

with $\|f\|_{L^2(\mathcal{O}\times(0,T))} \le C\|y^0\|_{L^2}$

For instance, for $y_{i.d} \equiv 0$, equivalent to:
$R(L) \hookrightarrow R(M)$, with $Ly^0 := y(\cdot\,,T)$, $Mf := y(\cdot\,,T) \dots$
In turn, equivalent to: $\|L^*\psi^T\| \le \|M^*\psi^T\| \quad \forall \psi^T \in L^2(0,1)$
(classical, functional analysis; [Russell, 1973])

Theorem

Assume: $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$, $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, large μ
$\exists \hat\rho$ such that, if $\iint_{\mathcal{O}_d \times (0,T)} \hat\rho^2 |y_{i,d}|^2 \, dx\, dt < +\infty$, $i = 1, 2$, then:
$\forall y^0 \in L^2(\Omega)$ $\exists$ null controls $f \in L^2(\mathcal{O} \times (0,T))$ & Nash pairs $(v_1(f), v_2(f))$

Idea of the proof:

1 - Large $\mu \;\Rightarrow\; \forall f \in L^2(\mathcal{O} \times (0,T))$ $\exists!$ Nash equilibrium $(v_1(f), v_2(f))$

$$\begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} - \frac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \frac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_i}, \quad i = 1, 2 \\ \phi_i(0,t) = \phi_i(1,t) = 0, \; y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \; \phi_i(x,T) = 0, \quad x \in (0,1) \end{cases}$$

$$v_i(f) = -\frac{1}{\mu} \phi_i |_{\mathcal{O}_i \times (0,T)}$$

2 - $\|L^*\psi^T\| \leq \|M^*\psi^T\| \quad \forall \psi^T \in L^2(0,1)$ means observability:

$$\|\psi|_{t=0}\|^2 + \sum_{i=1}^{2} \iint_Q \hat{\rho}^{-2}|\gamma^i|^2 \,dx\,dt \leq C \iint_{\mathcal{O}\times(0,T)} |\psi|^2 \,dx\,dt$$

for all ψ^T, with

$$\begin{cases} -\psi_t - \psi_{xx} = \sum_{i=1}^2 \alpha_i \gamma^i 1_{\mathcal{O}_d}, \quad \gamma^i_t - \gamma^i_{xx} = -\frac{1}{\mu}\psi 1_{\mathcal{O}_i} \\ \psi|_{t=T} = \psi^T(x), \quad \gamma^i|_{t=0} = 0, \quad \text{etc.} \end{cases}$$

First remark: $\|\psi|_{t=t'}\|^2 \leq C\|\psi|_{t=t''}\|^2$ for $t' < t''$

Explanation: energy estimates, large μ

$$\begin{aligned} \|\psi|_{t=t'}\|^2 &\leq C\left(\|\psi|_{t=t''}\|^2 + \sum_{i=1}^2 \int_{t'}^{t''} \|\gamma^i|_{t=s}\|^2 \,ds\right) \\ &\leq C\left(\|\psi|_{t=t''}\|^2 + \sum_{i=1}^2 \frac{1}{\mu^2} \int_0^{t''} \|\psi|_{t=s}\|^2 \,ds\right) \end{aligned}$$

$$\int_0^{t''} \|\psi|_{t=s}\|^2 \,ds \leq C\|\psi|_{t=t''}\|^2$$

Consequence: $\|\psi|_{t=0}\|^2 + \sum_{i=1}^2 \iint_Q \hat{\rho}^{-2}|\gamma^i|^2 \,dx\,dt \leq C \iint_Q \rho^{-2}|\psi|^2 \,dx\,dt$

Second remark: $\iint_Q \rho^{-2}|\psi|^2\,dx\,dt \le C\iint_{\mathcal{O}\times(0,T)} \rho^{-2}|\psi|^2\,dx\,dt$

Explanation: Carleman estimates for ψ and $h := \sum_{i=1}^{2} \alpha_i\gamma^i$

$$\begin{cases} -\psi_t - \psi_{xx} = \sum_{i=1}^{2} \alpha_i\gamma^i 1_{\mathcal{O}_d}, \quad \gamma^i_t - \gamma^i_{xx} = -\frac{1}{\mu}\psi 1_{\mathcal{O}_i} \\ \psi|_{t=T} = \psi^T(x), \quad \gamma^i|_{t=0} = 0, \quad \text{etc.} \end{cases}$$

Non-empty $\omega \subset \mathcal{O} \cap \mathcal{O}_d$

$$\begin{aligned} I(\psi) + I_0(h) &\le C\left(I_{loc,\omega}(\psi) + I_{loc,\omega}(h) + \iint_Q \rho_s^{-2}|h|^2 + \iint_Q \rho_{0,s}^{-2}|\psi|^2\right) \\ &\le C\left(I_{loc,\omega}(\psi) + I_{loc,\omega}(h) + \iint_Q \rho_{0,s}^{-2}|\psi|^2\right) \\ &\le C\left(I_{loc,\omega}(\psi) + I_{loc,\omega}(h) + \varepsilon I(\psi)\right) \\ &\le C\left(I_{loc,\omega}(\psi) + \varepsilon I_0(h) + \varepsilon I(\psi)\right) \end{aligned}$$

EXTENSIONS

- Theorem holds also for different $\mathcal{O}_{i,d}$ if $\mathcal{O}_{1,d} \cap \mathcal{O} \neq \mathcal{O}_{i,d} \cap \mathcal{O}$

$$\begin{cases} -\psi_t - \psi_{xx} = \sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}}, \quad \gamma^i_t - \gamma^i_{xx} = -\frac{1}{\mu} \psi 1_{\mathcal{O}_i} \\ \psi|_{t=T} = \psi^T(x), \quad \gamma^i|_{t=0} = 0, \ \text{etc.} \end{cases}$$

Choose different (well chosen) weights - Introduce:

- $\mathcal{O}' \subset\subset \mathcal{O}$ and $\omega_i \subset\subset \mathcal{O}_{i,d} \cap \mathcal{O}'$, with $\omega_1 \neq \omega_2$
- Carleman weights for ω_1 and ω_2 that coincide outside $\mathcal{O}'$

Then:

Carleman estimates for ψ, γ^i $\Rightarrow$

$$I_0(\gamma^1) + I_0(\gamma^2) + \iint \rho^{-2} |\psi|^2 \leq C\, I_{loc,\mathcal{O}}(\psi)$$

EXTENSIONS (Cont.)

- **Boundary followers, distributed leader:**

$$\begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}}, & (x,t) \in (0,1) \times (0,T) \\ y(0,t) = v^1(t), \ \ y(1,t) = v^2(t), & t \in (0,T) \\ y(x,0) = y^0(x), & x \in (0,1) \end{cases}$$

Costs: $\alpha_i \iint_{\mathcal{O}_{i,d} \times (0,T)} |y - y_{i,d}|^2 + \mu \int_0^T |v_i|^2 \, dt, \ \ i = 1, 2$

Optimality system and adjoint:

$$\begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_{i,d}} \\ y(0,t) = -\frac{1}{\mu} \phi^1_x(0,t) \\ y(1,t) = \frac{1}{\mu} \phi^2_x(1,t) \\ \dots \end{cases} \qquad \begin{cases} -\psi_t - \psi_{xx} = \sum_{i=1}^2 \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}} \\ \gamma^i_t - \gamma^i_{xx} = 0 \\ \gamma^1(0,t) = -\frac{1}{\mu} \psi_x(0,t), \\ \gamma^2(1,t) = \frac{1}{\mu} \psi_x(1,t) \\ \dots \end{cases}$$

The observability estimate:

$$\| \psi|_{t=0} \|^2 + \sum_{i=1}^2 \iint_Q \hat{\rho}^{-2} |\gamma^i|^2 \leq C \, I_{loc,\mathcal{O}}(\psi)$$

OK under conditions above; for instance, $\mathcal{O}_{1,d} = \mathcal{O}_{2,d}$, $\mathcal{O}_{i,d} \cap \mathcal{O} \neq \emptyset$, large μ, $\iint_{\mathcal{O}_d \times (0,T)} \hat{\rho}^2 |y_{i,d}|^2 \, dx \, dt < +\infty, \ \ i = 1, 2$

EXTENSIONS (Cont.)

$$\begin{cases} -\psi_t - \psi_{xx} = \sum_{i=1}^{2} \alpha_i \gamma^i 1_{\mathcal{O}_{i,d}} \\ \gamma^i_t - \gamma^i_{xx} = 0 \\ \gamma^1(0,t) = -\frac{1}{\mu}\psi_x(0,t), \quad \gamma^2(1,t) = \frac{1}{\mu}\psi_x(1,t) \\ \dots \end{cases}$$

For the proof: Carleman estimates + nonzero Dirichlet conditions [Imanuvilov-Puel-Yamamoto] - For instance

$$I(\gamma^1) \le C\left(\frac{1}{\mu^2}\|\rho_*^{-1}\frac{\partial\psi}{\partial n}(0,\cdot)\|^2_{H^{1/4}(0,T)} + I_{loc,\omega}(\gamma^1)\right)$$

and so on ...

- Distributed followers, boundary leader:

$$\begin{cases} y_t - y_{xx} = v_1 1_{\mathcal{O}_1} + v_2 1_{\mathcal{O}_2}, \quad (x,t) \in (0,1)\times(0,T) \\ y(0,t) = f, \quad y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \quad x \in (0,1) \end{cases}$$

A similar result holds

- However: boundary followers + boundary leader is unknown!
We would need $\|\psi|_{t=0}\|^2 + \sum_{i=1}^{2} \iint_Q \hat\rho^{-2}|\gamma^i|^2 \le C\int_0^T \rho_*^{-2}|\psi_x(0,t)|^2\,dt$ for (ψ,γ^1,γ^2) as above ...

EXTENSIONS (Cont.)

- More followers, coefficients, non-scalar parabolic systems, other functionals, boundary controls, higher dimensions, etc.
- Semilinear systems:

$$\begin{cases} y_t - y_{xx} = F(x,t;y) + f 1_{\mathcal{O}} + \sum_{i=1}^m v_i 1_{\mathcal{O}_i} \\ y(0,t) = y(1,t) = 0, \quad t \in (0,T), \quad \text{etc.} \end{cases}$$

Nash quasi-equilibria

$$\begin{cases} y_t - y_{xx} = F(x,t;y) + f 1_{\mathcal{O}} - \dfrac{1}{\mu}\phi_1 1_{\mathcal{O}_1} - \dfrac{1}{\mu}\phi_2 1_{\mathcal{O}_2} \\ -\phi_{i,t} - \phi_{i,xx} = F'(x,t;y)\phi_i + \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_i}, \quad i = 1,2 \\ \phi_i(0,t) = \phi_i(1,t) = 0, \ y(0,t) = y(1,t) = 0, \quad t \in (0,T) \\ y(x,0) = y^0(x), \ \phi_i(x,T) = 0, \quad x \in (0,1) \end{cases}$$

$$v_i(f) = -\frac{1}{\mu}\phi_i|_{\mathcal{O}_i \times (0,T)}$$

NC: OK for Lipschitz-continuous F
Also: an equivalence result!

EXTENSIONS (Cont.)

- ECT: OK
- Constraints, for instance:

$$\begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} + \sum_{i=1}^m v_i 1_{\mathcal{O}_i} \\ y(0,t) = y(1,t) = 0, \quad t \in (0,T), \quad \text{etc.} \end{cases}$$

Find a constrained Nash equilibrium $(v_1(f), v_2(f))$ with $v_i(f) \in \mathcal{U}_{i,ad} \subset L^2(\mathcal{O}_i \times (0,T))$:

$$J_1(v_1(f), v_2(f)) \leq J_1(v_1, v_2(f)) \quad \forall v_1 \in \mathcal{U}_{1,ad}$$
$$J_2(v_1(f), v_2(f)) \leq J_2(v_1(f), v_2) \quad \forall v_2 \in \mathcal{U}_{2,ad}$$

Then, find f such that $y|_{t=T} = 0$
OK for local constraints, i.e. $\mathcal{U}_{i,ad} = \{ v_i \in L^2(\mathcal{O}_i \times (0,T)) : v_i(x,t) \in L_i \}$

AN INTERESTING QUESTION:

All this holds for large μ - What about small μ?

Recall: $J_i(v_1, v_2) := \alpha_i \iint_{\mathcal{O}_{i,d}\times(0,T)} |y - y_{i,d}|^2 + \mu \iint_{\mathcal{O}_i\times(0,T)} |v_i|^2, \quad i = 1, 2$

$$\left\{ \begin{array}{l} y_t - y_{xx} = f 1_{\mathcal{O}} - \frac{1}{\mu}(\phi_1 1_{\mathcal{O}_1} + \phi_2 1_{\mathcal{O}_2}) \\ -\phi_{i,t} - \phi_{i,xx} = \alpha_i (y - y_{i,d}) 1_{\mathcal{O}_i} \\ \text{etc.} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (\mathrm{Id.} - \frac{1}{\mu}\Lambda)(v^1, v^2) = (v_0^1, v_0^2) \\ v^i \in L^2(\mathcal{O}_i \times (0, T)) \end{array} \right.$$

for some compact, positive, self-adjoint Λ

Fredholm's alternative + Hilbert-Schmidt
$\Rightarrow \exists \mu_1 > \mu_2 > \dots$ (independent of f), with $\mu_n \to 0^+$ such that
$\exists!$ Nash equilibrium for all $\mu \neq \mu_n$ for all n

Do we have NC for these μ?

FINAL COMMENTS:

- The previous proof → a method to compute f and $(v_1(f), v_2(f))$
  Numerics?
- Other strategies? Stackelberg-Pareto controllability?

$$\beta J_1'(v^1, v^2) + (1-\beta) J_2'(v^1, v^2) = 0, \quad \beta \in (0,1)$$

For each f, we get a family of equilibria $(v_\beta^1(f), v_\beta^2(f))$, with $\beta \in (0,1)$

$$\begin{cases} y_t - y_{xx} = f 1_{\mathcal{O}} - \frac{1}{\mu}\left(\frac{1}{\beta}\phi 1_{\mathcal{O}_1} + \frac{1}{1-\beta}\phi 1_{\mathcal{O}_2}\right) \\ -\phi_t - \phi_{xx} = \alpha_1 \beta (y - y_{1,d}) 1_{\mathcal{O}_{1,d}} + \alpha_2 (1-\beta)(y - y_{2,d}) 1_{\mathcal{O}_{2,d}} \\ \dots \end{cases}$$

$$\begin{cases} -\psi_t - \psi_{xx} = \alpha_1 \beta \gamma 1_{\mathcal{O}_{1,d}} + \alpha_2 (1-\beta) \gamma 1_{\mathcal{O}_{2,d}} \\ \gamma_t^i - \gamma_{xx}^i = -\frac{1}{\mu}\left(\frac{1}{\beta}\psi 1_{\mathcal{O}_1} + \frac{1}{1-\beta}\psi 1_{\mathcal{O}_2}\right) \\ \dots \end{cases}$$

$\exists$ "common" null controls, i.e. f such that $y(T) = 0$ for several β?
$\exists$ average null controls, i.e. f such that $\left(\int_0^1 y \, d\beta\right)(T) = 0$?

- Navier-Stokes? OPEN Locally (for small y_0)? ALSO OPEN
  [Guerrero, Carreño, Gueye, . . .]

In progress . . .

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