Nonlinear flows and optimality for functional inequalities

Maria J. Esteban

CEREMADE

CNRS & Université Paris-Dauphine

IN COLLABORATION WITH

J. DOLBEAULT, M. LOSS

Dedicated to Jean-Michel Coron, on his 60th birthday

OUTLINE

- Use of linear and nonlinear flows to prove Sobolev-like inequalities on manifolds
- Generalities of functional inequalities
- The Caffarelli-Kohn-Nirenberg inequalities
- Symmetry and symmetry breaking for extremals of Caffarelli-Kohn-Nirenberg inequalities

Sobolev-like inequalities on the sphere

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(S^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(S^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(S^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\,, \qquad (1)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \ne 2$, is such that $p \le 2^* := \frac{2\,d}{d-2}$ if $d \ge 3$.

The case $p = \frac{2\,d}{d-2}$ corresponds to the Sobolev inequality (equivalent via the stereographic projection).

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \ge S \left(\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx \right)^{\frac{d-2}{d}} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)\,,$$

PROOFS OF (1) + MINIMIZERS ARE CONSTANTS BY: Bidaut-Véron – Véron (PDE, rigidity methods, 1991); Beckner (harmonic analysis methods, 1993); Bakry-Ledoux et al (“carré du champ" method, linked to a flow method, 1996 +; only for $2 < p \le 2^\# := \frac{2\,d^2+1}{(d-1)^2} < 2^*$).

Linear flow method

Let us define $\rho = |u|^p$. The two inequalities below are equivalent

$$\|\nabla u\|^2_{\mathrm{L}^2(S^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(S^d)} \geq \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(S^d)}\,.$$

$$\int_{S^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\omega \geq \frac{d}{p-2}\left[\left(\int_{S^d} \rho\,d\omega\right)^{\frac{2}{p}} - \int_{S^d} \rho^{\frac{2}{p}}\,d\omega\right].$$

If we define the functionals $\mathcal{E}_p$ and $\mathcal{I}_p$ respectively by

$$\mathcal{I}_p[\rho] := \int_{S^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\omega\ , \quad \mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\left(\int_{S^d} \rho\,d\omega\right)^{\frac{2}{p}} - \int_{S^d} \rho^{\frac{2}{p}}\,d\omega\right] \quad \text{if} \quad p \neq 2\,,$$

then the above inequalities amount to $\mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$. To establish such inequalities, one can use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

where Δ denotes the Laplace-Beltrami operator on $\mathbb{S}^d$. We have $\frac{d}{dt}\left(\int_{S^d} \rho\,d\omega\right) = 0$

$$\text{If } p \leq 2^{\#}, \qquad \frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho] \quad \text{and} \quad \frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]\,.$$

Nonlinear versus linear flow

We want to prove $\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho] \geq 0$. For $p \leq 2^\#$,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - \, d\,\mathcal{E}_p[\rho]\Big) \leq (-d + d)\,\mathcal{I}_p[\rho] = 0\,.$$

Not difficult to prove that ρ converges to a constant as $t \to +\infty$ and

$$\lim_{t\to+\infty}\Big(\mathcal{I}_p[\rho] - \, d\,\mathcal{E}_p[\rho]\Big) = 0\,.$$

What if $2^\# < p < 2^*$?

LEMMA [Dolbeault, E., Loss]. When $2^\# < p < 2^*$, we can find a function ρ_0 such that ρ solution of $\frac{\partial\rho}{\partial t} = \Delta\rho\,,\ \ \rho(t=0) = \rho_0$, and

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - \, d\,\mathcal{E}_p[\rho]\Big)\Big|_{t=0} > 0\,.$$

Then, we can get the same result by considering the flow $\frac{d\,\rho}{dt} = \Delta\rho^m$, for a well-chosen $m \neq 1$.

The computations are much more involved, but the idea is “more or less" the same. And it covers also the case $p \in (1,2)$.

Prove rigidity directly, "without the flow": heuristics

1) AIM: On $\mathbb{R}^d$, show that minimizers of $E[v]$ does not depend on the angles ω, only on r.

2) Define flow (linear, nonlinear), $\frac{d}{dt} v = H[v]$. show that it is well defined for all times.

3) $\frac{d}{dt} E[v(t)] = -|A[v(t)]| - |C| \, |\nabla_\omega v(t)|^2 \leq 0$.

4) If E bounded below, for any initial value v_0, when $t \to +\infty$, $v(t) \to w$, minimizer. And $|\nabla_\omega w| = 0$.

To carry out this program, we need to prove a lot of things about the flow, and this not always easy for nonlinear flows... or very technical at least. Way out?

ALTERNATIVE: Consider any minimizer of E or even any critical point of E, that is, a function w that satisfies $E'(w) = 0$.

Consider the same flow as above, with initial datum $v_0 = w$.

$$\frac{d}{dt} E[v(t)]_{|t=0} = -|A[w]| - |C| \, |\nabla_\omega w|^2 = E'[w] \cdot H[w] = 0. \quad \text{So,} \quad \nabla_\omega w \equiv 0 \,.$$

Attainability and value of best constants in functional inequalities

$$F(Dv, v, x) \le C\ G(D^2v, Dv, v, x) \qquad \forall v \in X\ .$$

Functional inequalities play an important role in obtaining a priori estimates for solutions of PDEs, in analyzing the long time behavior of solutions of evolution problems, in describing the blow-up profile for finite time blow-up phenomena, etc

Important questions :

- Is C attained in X? What is its value??
- If yes, how do the optimal functions v look like?

If we know a priori that the optimal solutions have some symmetry properties, for instance, that they are radially symmetric, then it might be easier to compute the value of C.

Caffarelli-Kohn-Nirenberg (CKN) inequalities

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \qquad \forall\ v \in \mathcal{D}_{a,b}$$

with $a \le b \le a+1$ if $d \ge 3\,,\qquad a < b \le a+1$ if $d = 2\,,\qquad a \ne \frac{d-2}{2}$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$$b-a\ \to\ 0 \iff p \to \tfrac{2d}{d-2}$$

$$b-(a+1)\ \to\ 0 \iff p\ \to\ 2_+$$

$$\frac{1}{C_{a,b}} = \inf_{\mathcal{D}_{a,b}} \quad \frac{\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx}{\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}}$$

The symmetry issue

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \qquad \forall\, v \in \mathcal{D}_{a,b}$$

$\mathsf{C}_{a,b}$ = best constant for general functions v
$\mathsf{C}^*_{a,b}$ = best constant for radially symmetric functions v

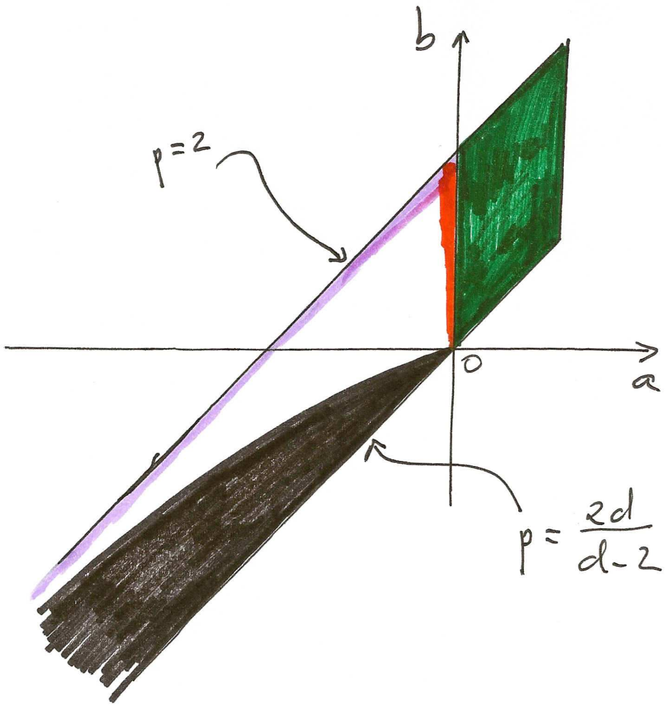
$$\mathsf{C}^*_{a,b} \le C_{a,b}$$

Up to scalar multiplication and dilation, the optimal radial function is

$$v^*_{a,b}(x) = \left(1 + |x|^{-\frac{2a\,(1+a-b)}{b-a}}\right)^{-\frac{b-a}{1+a-b}}$$

Questions: is optimality (equality) achieved ? do we have $v_{a,b} = v^*_{a,b}$?

Symmetry and symmetry breaking ($d \geq 3$)



Case $a > 0$, Existence and symmetry : Th. Aubin, G. Talenti, E. Lieb, Chou-Chu, P.L. Lions, Horiuchi,...

Case $a < 0$, symmetry breaking: Catrina-Wang, Felli-Schneider.

Case $a < 0$: Lin, Wang; Dolbeault, E., Tarantello (d=2); Betta-Brock-Mercaldo-Posteraro ($b > 0$)

Case $a < 0$: Dolbeault, E., Loss, Tarantello

$$b^{FS}(a) = \frac{d(d-2-2a)}{2\sqrt{(d-2-2a)^2+4(d-1)}} - \frac{1}{2}(d-2-2a)$$

Generalized Caffarelli-Kohn-Nirenberg inequalities (CKN)

Let $d \geq 3$. For any $p \in [2, p(\theta, d) := \frac{2d}{d-2\theta}]$, there exists a positive constant $\mathsf{C}(\theta, p, a)$ such that

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta, p, a)\left(\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a - a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\mathrm{CKN}}(\theta, p, a)$ and (see [Del Pino, Dolbeault, Filippas, Tertikas]):

$$\mathsf{C}_{\mathrm{CKN}}(\theta, p, a) \geq \mathsf{C}^*_{\mathrm{CKN}}(\theta, p, a) = \mathsf{C}^*_{\mathrm{CKN}}(\theta, p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

$$\mathsf{C}^*_{\mathrm{CKN}}(\theta, p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}}\left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}}\left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta}\left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}}\left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}$$

and for θ small, we have proved that there is symmetry breaking for certain values of (Λ, p) such that $u^*_{\Lambda,p}$ is stable! In principle in all cases where we have observed this phenomenum, $\theta \leq 0.7$ approx.

(Dolbeault, E., Tarantello, Tertikas (2011))

An optimal symmetry result

With the change of variables : $r \mapsto r^\alpha, \quad v(r,\omega) = w(r^\alpha,\omega)$, and with

$$n = \frac{d - b\,p}{\alpha} = \frac{d - 2\,a - 2}{\alpha} + 2\,, \qquad \mathsf{D}w = \left(\alpha\,\frac{\partial w}{\partial r}, \frac{1}{r}\,\nabla_\omega w\right)$$

$p = \frac{2n}{n-2}$ and the CKN the inequality becomes

$$\alpha^{1-\frac{2}{p}}\left(\int_{\mathbb{R}^d} |w|^p\,d\mu\right)^{\frac{2}{p}} \le \mathsf{C}_{a,b}\int_{\mathbb{R}^d} |\mathsf{D}w|^2\,d\mu\,, \quad d\mu := r^{n-1}\,dr\,d\omega \text{ ``} = dx\,(\mathbb{R}^n)\text{''}$$

This inequality scales like a “critical Sobolev inequality" in $\mathbb{R}^n$, but n does not need to be an integer...

The parameters α and n vary in the ranges $0 < \alpha < \infty$ and $d < n < \infty$ and the *Felli-Schneider curve* in the (α, n) variables is given by

$$\alpha = \sqrt{\frac{d-1}{n-1}} =: \alpha_{\,\mathrm{FS}}$$

THEOREM [2015].- If $\alpha \le \alpha_{\,\mathrm{FS}}$ and $d \ge 2$, optimality is achieved by radial functions.

Notations

If ∇_ω denotes the gradient with respect to the angular variables $\omega \in S^{d-1}$ and Δ_ω is the Laplace-Beltrami operator on S^{d-1}, we define

$$\mathsf{D}w = \left(\alpha \, \frac{\partial w}{\partial r}, \frac{1}{r} \, \nabla_\omega w \right) \, ,$$

we define the self-adjoint operator L by

$$\mathrm{L}w := - \, \mathsf{D}^* \, \mathsf{D} \, w = \alpha^2 \, w'' + \alpha^2 \, \frac{n-1}{r} \, w' + \frac{\Delta_\omega \, w}{r^2}$$

The fundamental property of L is the fact that

$$\int_{\mathbb{R}^d} w_1 \, \mathrm{L}w_2 \, d\mu = - \int_{\mathbb{R}^d} \mathsf{D}w_1 \cdot \mathsf{D}w_2 \, d\mu \quad \forall \, w_1 , \, w_2 \in \mathcal{D}(\mathbb{R}^d)$$

$\triangleright$ Heuristics: we look for a monotonicity formula along a well chosen nonlinear flow, based on the analogy with the decay of the Fisher information along the fast diffusion flow in $\mathbb{R}^d$

Fisher information decay and a fast diffusion equation

Let $u = |w|^p, \qquad p = \frac{2\,n}{n-2}$.

Up to multiplicative constants, $\int_{\mathbb{R}^d} |w|^p\,d\mu = \int_{\mathbb{R}^d} u\,d\mu$, and $\int_{\mathbb{R}^d} |\mathsf{D}w|^2\,d\mu = \mathcal{I}[u]$, with

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|\mathsf{Dp}|^2\,d\mu\,, \quad \mathsf{p} = \frac{m}{1-m}\,u^{m-1} \quad \text{and} \quad m = 1 - \frac{1}{n}$$

Here $\mathcal{I}$ is the *Fisher information* and p is the *pressure function.*

Next, define the fast diffusion equation (flow)

$$\frac{\partial u}{\partial t} = \mathrm{L}u^m\,, \quad m = 1 - \frac{1}{n}$$

▷ STRATEGY: Assume that $\alpha \le \alpha_{\mathrm{FS}}$,

1) prove that for all $t \ge 0$, $\frac{d}{dt}\int_{\mathbb{R}^d} u(t)\,d\mu = 0$ and $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] \le 0$,

2) prove that $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = 0$ means, in particular, that u is radially symmetric.

Mass conservation and Fisher information decay along the fast diffusion flow

Easy to see: the mass $\int_{\mathbb{R}^d} u\,d\mu$ is conserved along the flow.

$$\text{With}\quad \mathsf{p} = \frac{m}{1-m}\,u^{m-1}\;, \qquad \mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|\mathsf{Dp}|^2\,d\mu\;,$$

Proposition 1.- Let $u_0 \geq 0$. Up to estimates near the origin and near infinity,

$$\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = -\,2\,(n-1)^{n-1}\,\mathcal{K}[\mathsf{p}]\,,$$

with

$$\mathcal{K}[\mathsf{p}] := \int_{\mathbb{R}^d}\left(\frac{1}{2}\,\mathrm{L}\,|\mathsf{Dp}|^2 - \mathsf{Dp}\cdot\mathsf{DLp} - \frac{1}{n}\,(\mathrm{L}\mathsf{p})^2\right)\mathsf{p}^{1-n}\,d\mu$$

Proposition 2.- If $\alpha \leq \alpha_{\,\mathrm{FS}}$, $\mathcal{K}[\mathsf{p}] \geq 0$.

Proposition 3.- If $u_0 \geq 0$ is a critical point, then $0 = I'[u_0]\cdot \mathrm{L}u_0^m = \frac{d}{dt}\mathcal{I}[u(t,\cdot)]\Big|_{t=0} = C\,\mathcal{K}[\mathsf{p}_0]$.

Proposition 4.- If $\alpha \leq \alpha_{\,\mathrm{FS}}$, $\mathcal{K}[\mathsf{p}_0] = 0$ implies that p_0 is independent of ω and $\mathsf{p}_0 = a + b\,r^2$.

Proving decay (1/2)

$$\mathcal{K}[\mathsf{p}] := \int_{\mathbb{R}^d} \mathsf{k}[\mathsf{p}]\, \mathsf{p}^{1-n}\, d\mu\,,$$

$$\mathsf{k}[\mathsf{p}] := \mathcal{Q}(\mathsf{p}) - \frac{1}{n}\,(\mathcal{L}\,\mathsf{p})^2 = \frac{1}{2}\,\mathcal{L}\,|\mathsf{Dp}|^2 - \mathsf{Dp}\cdot\mathsf{D}\,\mathcal{L}\,\mathsf{p} - \frac{1}{n}\,(\mathcal{L}\,\mathsf{p})^2$$

Lemma.- Let $n \neq 1$ be any real number, $d \in \mathbb{N}$, $d \geq 2$, and consider a function $\mathsf{p} \in C^3((0,\infty)\times\mathcal{M})$, where $(\mathcal{M}, g)$ is a smooth, compact Riemannian manifold. Then we have

$$\mathsf{k}[\mathsf{p}] = \alpha^4 \left(1 - \frac{1}{n}\right) \left[\mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta_\omega\,\mathsf{p}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla_\omega \mathsf{p}' - \frac{\nabla_\omega \mathsf{p}}{r}\right|^2 + \frac{1}{r^4}\,\mathsf{k}_{\mathcal{M}}[\mathsf{p}]\,,$$

with

$$\mathsf{k}_{\mathcal{M}}[\mathsf{p}] := \frac{1}{2}\,\Delta_\omega\,|\nabla_\omega \mathsf{p}|^2 - \nabla_\omega \mathsf{p}\cdot\nabla_\omega \Delta_\omega\,\mathsf{p} - \tfrac{1}{n-1}\,(\Delta_\omega\,\mathsf{p})^2 - (n-2)\,\alpha^2\,|\nabla_\omega \mathsf{p}|^2$$

Proving decay (2/2)

Proposition.- Assume that $d \geq 3$, $n > d$ and $\mathcal{M} = S^{d-1}$. There is a positive constant $\zeta_\star$ such that

$$\int_{S^d} \mathsf{k}_\mathcal{M}[\mathsf{p}]\, \mathsf{p}^{1-n}\, d\omega \geq (n-2)\left(\alpha_{\mathrm{FS}}^2 - \alpha^2\right) \int_{S^d} |\nabla_\omega \mathsf{p}|^2\, \mathsf{p}^{1-n}\, d\omega + \zeta_\star\,(n-d) \int_{S^d} |\nabla_\omega \mathsf{p}|^4\, \mathsf{p}^{1-n}\, d\omega$$

Proof based on the Bochner-Lichnerowicz-Weitzenböck formula.

So, we have shown that

$$\mathcal{K}[\mathsf{p}] \geq \text{sum of squares} + (n-2)\left(\alpha_{\mathrm{FS}}^2 - \alpha^2\right) \int_{S^d} |\nabla_\omega \mathsf{p}|^2\, \mathsf{p}^{1-n}\, d\omega$$

and therefore, if $\alpha \leq \alpha_{\mathrm{FS}}$, $\mathcal{K}[\mathsf{p}] \geq 0$.

End of the proof

If $\alpha \leq \alpha_{\rm FS}$ (resp. $\Lambda \leq \Lambda_{\rm FS}$), and if p_0 is a critical point of the E-L equations for CKN, written in the good variables, then

$$0 \;=\; \mathcal{K}[\mathsf{p}_0] \;\geq \int_{\mathbb{R}^d} \alpha^4 \left(1-\frac{1}{n}\right) \left[\mathsf{p}_0'' - \frac{\mathsf{p}_0'}{r} - \frac{\Delta_\omega\,\mathsf{p}_0}{\alpha^2\,(n-1)\,r^2}\right]^2 \mathsf{p}_0^{1-n}\,d\mu$$

$$+ \int_{\mathbb{R}^d} (n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right) |\nabla_\omega \mathsf{p}_0|^2\,\mathsf{p}_0^{1-n}\,d\mu + \int_{\mathbb{R}^d} \zeta_\star\,(n-d)\,|\nabla_\omega \mathsf{p}_0|^4\,\mathsf{p}_0^{1-n}\,d\mu\,,$$

where $\zeta_\star > 0$ and $n > d$.

So, $\nabla_\omega \mathsf{p}_0 \equiv 0$, that is, p_0 does not depend on ω, which means radial symmetry.

Moreover, $\mathsf{p}_0'' - \frac{\mathsf{p}_0'}{r} - \frac{\Delta_\omega\,\mathsf{p}_0}{\alpha^2\,(n-1)\,r^2} \equiv 0$, which implies that for some $a, b > 0$, $\mathsf{p}_0 = a + b\,r^2$.

Extensions : The robustness of the method

With the definitions

$$\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q}\,, \quad \|w\|_{\mathrm{L}^q(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}\,,$$

consider the family of (subcritical) *Caffarelli-Kohn-Nirenberg interpolation inequalities* given by

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le C_{\beta,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\,\|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \quad \forall\,w \in \mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)\,. \tag{1}$$

Here the parameters β, γ and p are subject to the restrictions

$$d \ge 2\,, \quad \gamma - 2 < \beta < \frac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with} \quad p_\star := \frac{d-\gamma}{d-\beta-2} \tag{2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}\,.$$

REMARK. The “critical" case $p = p_\star$ corresponds to the kind of inequalities discussed above.

REMARK. In a recent paper with J. Dolbeault, M. Loss and M. Muratori we have extended our methodology, but several important changes have to be made in order to fit the sub-criticality of the inequalities: Renyi entropies instead of Fisher information, other regularity results, etc.)