Partial controllability of parabolic systems

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Paris, Institut Henri Poincaré

Example

Let be $T > 0$, $\omega \subset \Omega \subset \mathbb{R}^N$ and the parabolic system

$$\begin{cases} \partial_t y_1 = \Delta y_1 + y_2 + \mathbb{1}_\omega u & \text{in } Q_T := \Omega \times (0,T), \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T := \partial\Omega \times (0,T), \\ y(0) = y_0 & \text{in } \Omega. \end{cases}$$

Can we find, for all initial condition y_0, a control u such that

$$y_1(T; y_0, u) = 0 \ ?$$

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Plan

1. **Kalman conditions**
2. **Partial controllability of parabolic systems**
3. **Null controllability VS optimal control**

Differential equations

Let $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$

$$\begin{cases} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}^n. \end{cases} \qquad \text{(ODE)}$$

Système (ODE) is said :

- **approximatively controllable** if

$$\forall y_0,\ y_T \text{ and } \varepsilon > 0\ \exists u \text{ s.t. } \|y(T) - y_T\| \leqslant \varepsilon$$

- **exactly controllable** if

$$\forall y_0,\ y_T\ \exists u \text{ s.t. } y(T) = y_T$$

- **null controllable** if

$$\forall y_0\ \exists u \text{ s.t. } y(T) = 0$$

Proposition

Those notions of controllability are equivalent.

Differential equations

Let $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$

$$\begin{cases} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}^n. \end{cases} \quad \text{(ODE)} \qquad P: \begin{array}{ccc} \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ (y_1, y_2)^* & \mapsto & y_1. \end{array}$$

Système (ODE) is said :

- **partially approximatively controllable** if

$$\forall y_0,\ y_T \text{ and } \varepsilon > 0\ \exists u \text{ s.t. } \|Py(T) - Py_T\| \leqslant \varepsilon$$

- **partially exactly controllable** if

$$\forall y_0,\ y_T\ \exists u \text{ s.t. } Py(T) = Py_T$$

- **partially null controllable** if

$$\forall y_0\ \exists u \text{ s.t. } Py(T) = 0$$

Proposition

Those notions of partial controllability are equivalent.

Differential equations

Let $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$

$$\left\{\begin{array}{ll} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}^n. \end{array}\right. \qquad \text{(EDO)}$$

Theorem (Kalman, 1969)

System (EDO) is **controllable** in $(0,T)$ if and only if

$$\operatorname{rank}(B|AB|...|A^{n-1}B) = n.$$

Example :

$$\left\{\begin{array}{l} \partial_t y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \quad \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}^n. \end{array}\right.$$

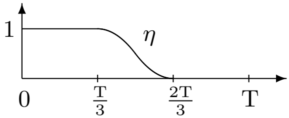
$$\Rightarrow \quad \operatorname{rank}(B|AB) = \operatorname{rank}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2.$$

Proof : 3 equations, 1 control

⇐ • Let $z := y - \eta\overline{y}$, where

$$\begin{cases} \partial_t \overline{y} = A\overline{y} & \text{in } [0,T], \\ \overline{y}(0) = y_0 \in \mathbb{R} \end{cases}$$

and $\eta \in \mathcal{C}^\infty[0,T]$ s.t. :



Thus, z is solution to

$$\begin{cases} \partial_t z = Az + Bu + h & \text{in } [0,T], \\ z(0) = 0. \end{cases}$$

where $h := -\partial_t \eta \overline{y}$.

Proof : 3 equations, 1 control

$$\begin{cases} \partial_t z = Az + Bu + h & \text{in } [0,T], \\ z(0) = 0. \end{cases}$$

- If $K := (B|AB|A^2B)$ is invertible :

$$w := K^{-1}z,$$

where w satisfies

$$K\partial_t w = AKw + Bu + h.$$

$$\text{Cayley Hamilton} \Rightarrow A^3 := \alpha_0 I + \alpha_1 A + \alpha_2 A^2.$$

Then

$$\begin{cases} AK = (AB|A^2B|A^3B) = KC, \\ B = Ke_1, \end{cases} \quad \text{with } C = \begin{pmatrix} 0 & 0 & \alpha_0 \\ 1 & 0 & \alpha_1 \\ 0 & 1 & \alpha_2 \end{pmatrix}.$$

Proof : 3 equations, 1 control

Thus w is solution to

$$\begin{cases} \partial_t w_1 &= & & \alpha_0 w_3 + g_1 + u, \\ \partial_t w_2 &= w_1 + & & \alpha_1 w_3 + g_2, \\ \partial_t w_3 &= & w_2 + & \alpha_2 w_3 + g_3, \\ w(0) &= 0, & & \end{cases}$$

where $g := K^{-1} h$.

Choose :

$$\begin{cases} w_3 &= 0, \\ w_2 &= -g_3, \\ w_1 &= \partial_t w_2 - g_2, \\ u &= \partial_t w_1 - g_1. \end{cases}$$

We recall : $h := -\partial_t \eta \overline{y}$.

So we have

$$w(0) = w(T) = 0.$$

□

Let $A \in \mathcal{L}(\mathbb{R}^n)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$

$$\left\{ \begin{array}{ll} \partial_t y = Ay + Bu & \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}^n. \end{array} \right. \quad \text{(EDO)} \qquad P: \begin{array}{ccc} \mathbb{R}^p \times \mathbb{R}^{n-p} & \to & \mathbb{R}^p, \\ (y_1, y_2)^* & \mapsto & y_1. \end{array}$$

Theorem

System (ODE) is **partially controllable** in $(0,T)$ iff

$$\operatorname{rank}(PB|PAB|...|PA^{n-1}B) = p.$$

Example :

$$\left\{ \begin{array}{l} \partial_t y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u \quad \text{in } (0,T), \\ y(0) = y_0 \in \mathbb{R}^n. \end{array} \right.$$

$$\Rightarrow \quad \operatorname{rank}(PB|PAB) = \operatorname{rank}(1\ 0) = 1.$$

PDE : constant matrices

Let $T>0$, $\Omega\in\mathbb{R}^N$ (bounded), $\omega\subset\Omega$, $A\in\mathcal{L}(\mathbb{R}^n)$ and $B\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$.

$$\left\{\begin{array}{ll}\partial_t y=\Delta y+Ay+B\mathbb{1}_\omega u & \text{in } Q_T,\\ y=0 & \text{on } \Sigma_T,\\ y(0)=y_0 & \text{in } \Omega.\end{array}\right. \qquad \text{(S)}$$

Theorem (Ammar-Khodja et al 2009)

System (S) is **approximatively/null controllable** in $(0,T)$ iff

$$\operatorname{rank}(B|AB|...|A^{n-1}B)=n.$$

Theorem ([1] Ammar-Khodja, Chouly, D. 2016)

System (S) is **partially approximatively/null controllable** in $(0,T)$ iff

$$\operatorname{rank}(PB|PAB|...|PA^{n-1}B)=p.$$

[1] F. Ammar-Khodja, F. Chouly and M. Duprez. *Partial null controllability of parabolic linear systems*, Math. Control Relat. Fields, 2 (2016), 185-216.

Space dependant coefficient

Remark

Parabolic systems are not (EC) and (PEC).
Moreover,

$$\text{(NC)}\Rightarrow\text{(AC)}$$

$$\text{(PNC)}\Rightarrow\text{(PAC)}$$

Let $\alpha \in L^\infty(\Omega)$.

$$\begin{cases} \partial_t y_1 = \Delta y_1 + \alpha(x) y_2 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_t y_2 = \Delta y_2 & \text{in } Q_T, \\ y = 0 & \text{on } \Sigma_T, \\ y(\cdot,0) = y^0 & \text{in } \Omega. \end{cases} \tag{2×2}$$

Theorem (F. Ammar Khodja, F. Chouly, M. Duprez, 2016)

The system is **partially approximatively controllable**.

Space dependant coefficient

Theorem (F. Ammar Khodja, F. Chouly, M. Duprez, 2016)

Let $\Omega :=]0,2\pi[$.

(i) If $\omega \subset]\pi, 2\pi[$ and for all $x \in]0,2\pi[$

$$\alpha(x) := \sum_{j=1}^{\infty} \frac{1}{j^2} \cos(15jx),$$

then the system is **not partially null controllable**.

(ii) If $\alpha = \sum \alpha_p \cos(px)$ and there exists $C_1 > 0$ and $C_2 > 2\pi$ such that

$$|\alpha_p| \leqslant C_1 e^{-C_2 p} \text{ for all } \ p \in \mathbb{N},$$

then the system is **partially null controllable**.

Dual system :

$$\begin{cases} -\partial_t \varphi = \Delta\varphi + A^*\varphi & \text{in } Q_T, \\ \varphi = 0 & \text{on } \Sigma_T, \\ \varphi(T,\cdot) = P^*\varphi_T & \text{in } \Omega. \end{cases} \quad \text{with} \quad P^* : \begin{array}{ccc} \mathbb{R}^p & \to & \mathbb{R}^p \times \mathbb{R}^{n-p} \\ \varphi_T & \mapsto & (\varphi_T, 0_{n-p})^*. \end{array}$$

System (2×2) is **partially null controllable** on $(0,T)$, if and only if there exists $C_{obs} > 0$ such that for all $\varphi_T \in L^2(\Omega)^p$

$$\|\varphi(0)\|^2_{L^2(\Omega)^n} \leqslant C_{obs} \int_0^T \|B^*\varphi\|^2_{L^2(\omega)^m}$$

Remark :
We can now take the eigenvectors of the operator $\Delta + A^*$ as initial condition in the dual system.

Consider the HUM functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\omega\times(0,T)} u^2 dxdt + \frac{1}{2\epsilon}\int_\Omega y_1(T;y_0,u)^2 dx,$$

where (y,u) satisfies $S_{2\times 2}$.

Theorem

(i) System $(S_{2\times 2})$ is partially approximatively controllable from y_0 iff

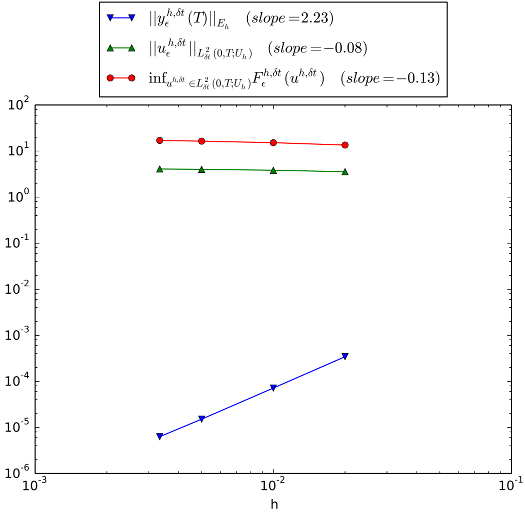
$$y_1(T;y_0,u_\varepsilon)\underset{\varepsilon\to 0}{\longrightarrow}0.$$

(ii) System $(S_{2\times 2})$ is partially null controllable from y_0 iff

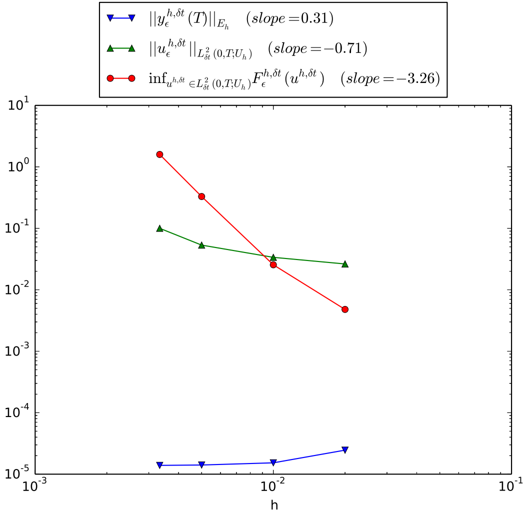
$$M^2_{y_0,T} := 2\sup_{\varepsilon>0}\left(\inf_{L^2(Q_T)} J_\varepsilon\right) < \infty.$$

In this case, $\|y_1(T;y_0,u_\varepsilon)\|_\Omega \leqslant M_{y_0,T}\sqrt{\varepsilon}$.

[1] Boyer F. : On the penalised HUM approach and its applications to the numerical approximation of null-controls for parabolic problems (2013).



Partially null controllable case : $\alpha = 1$.



Non-partially null controllable case : $\alpha = \sum_{p\geqslant 1} \frac{1}{p^2}\cos(15px)$.

Perspectives : semilinear case

Example : Description of a brain tumor

$$\begin{cases} \partial_t y_1 = d_1 \Delta y_1 + a_1(1 - y_1/k_1)y_1 - (\alpha_{1,2}y_2 + \kappa_{1,3}y_3)y_1 & \text{in } Q_T, \\ \partial_t y_2 = d_2 \Delta y_2 + a_2(1 - y_2/k_2)y_2 - (\alpha_{2,1}y_1 + \kappa_{2,3}y_3)y_2 & \text{in } Q_T, \\ \partial_t y_3 = d_3 \Delta y_3 - a_3 y_3 + \mathbb{1}_\omega u & \text{in } Q_T, \\ \partial_n y_i := \nabla y_i \cdot \vec{n} = 0 \quad \forall\, 1 \leqslant i \leqslant 3 & \text{on } \Sigma_T, \\ y(x,0) = y_0 & \text{in } \Omega, \end{cases}$$

where

1. y_1 is the density of tumor cells,
2. y_2 is the density of normal cells,
3. y_3 is the drug concentration,
4. u is the rate at which the drug is being injected (*control*),
5. d_i, a_i, k_i, $\alpha_{i,j}$, $\kappa_{i,j}$ are known constants.

[1] Chakrabarty, Hanson : Distributed parameters deterministic model for treatment of brain tumors using Galerkin finite element method, 2009.

Thank you for your attention !

And Happy Birthday Jean-Michel !