Analysis of a system modelling the motion of a piston in a viscous gas

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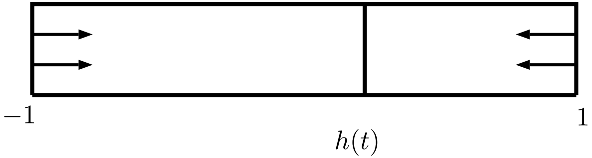
Joint work with Takéo Takahashi and Marius Tucsnak.

Outline

Setting up the problem

- We consider a one dimensional model for the motion of a particle (piston) in a cylinder filled with a viscous compressible gas.



- Gas-piston system evolves in the interval $(-1, 1)$ and $h : [0, \infty) \mapsto (-1, 1)$ denotes the position of the particle.
- The extremities of the cylinder are fixed, but the gas is allowed to penetrate inside the cylinder.
- The gas is modelled by the 1D compressible Navier-Stokes equations, whereas the piston obeys Newton's second law.

Governing equations

Motion of the gas:
described in the Eulerian coordinate system by its density $\rho = \rho(t, x)$ and the velocity $u = u(t, x)$, which satisfy the one dimensional compressible Navier-Stokes system in

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \qquad t \geq 0, x \neq h(t) \\ &\rho(\partial_t u + u\partial_x u) - \partial_{xx} u + \partial_x(\rho^\gamma) = 0, \qquad t \geqslant 0, x \neq h(t) \end{aligned} \tag{1.1}$$

where $\gamma \geqslant 1$.

Motion of the Piston:

$$\begin{aligned} &u(t, h(t)) = \dot{h}(t) && (t \geqslant 0), \\ &m\ddot{h}(t) = [\partial_x u - \rho^\gamma](t, h(t)) && (t \geqslant 0), \end{aligned}$$

where m is the mass of the piston and the symbol $[f](t, x)$ stands for the jump at instant t of f at x, i.e.,

$$[f](t, x) = f(t, x^+) - f(t, x^-).$$

The position of the piston (and, consequently, the domain occupied by the gas) is one of the unknowns of the problem, we have a free boundary value problem.

Initial Condition:

$$\begin{cases} h(0) = h_0, \quad \dot{h}(0) = \ell_0, & \\ u(0,x) = u_0(x), \quad \rho(0,x) = \rho_0(x) & (x \in [-1,1] \setminus \{h_0\}). \end{cases} \tag{1.2}$$

Boundary Condition:

$$u(t,-1) = 0, \quad u(t,1) = 0 \qquad (t \geqslant 0), \tag{1.3}$$

$$\begin{cases} u(t,-1) = u_{-1}(t) > 0, \quad u(t,1) = 0 & (t \geqslant 0), \\ \rho(t,-1) = \rho_{-1}(t) & (t \geqslant 0), \end{cases} \tag{1.4}$$

or

$$\begin{cases} u(t,-1) = u_{-1}(t), \quad u(t,1) = -u_1(t) & (t \geqslant 0), \\ u_{-1}(t) > 0, \quad u_1(t) > 0 & (t \geqslant 0), \\ \rho(t,-1) = \rho_{-1}(t) & (t \geqslant 0), \\ \rho(t,1) = \rho_1(t) & (t \geqslant 0). \end{cases} \tag{1.5}$$

Outline

Known Results in 1D

- **Shelukhin**, , 1978 :
  - Global in time existence of classical solutions, with initial conditions : $u_0 \in C^{2+\alpha}$, $\rho_0 \in C^{1+\alpha}$, $0 < \alpha < 1$ and homogeneous boundary condition.
- **Shelukhin**, 1982:
  - Similar result as above when gas and the piston are supposed to be heat conducting, and homogeneous boundary conditions.
- **Antman and Wilber,** 2007 :
  - asymptotic behavior of solutions as the ratio of the mass of the gas and of the mass of the piston tends to zero.

Known results in higher dimension:

Rigid Structure immersed in Compressible fluid :

- **Desjardins and Esteban,** 2000 :
  - Global existence of a weak solution for $\gamma \geq 2$ and upto collision.
- **Feireisl,** 2003 :
  - Global existence of a weak solution for $\gamma > N/2$ and regardless of possible collisions of two or more rigid bodies and/or a contact of the bodies with the boundary
- **Boulakia and Guerrero,**2009 :
  - Global in time strong solution for small initial data.
- **Hieber and Murata** 2015 :
  - Local in time strong solution in $L^p - L^q$ setting.

Goal

Existence and uniqueness of global in time strong solutions of the initial and boundary value problem.

- non homogeneous boundary conditions.
- less regular initial data.

Theorem (D.M, Takéo Takahashi and Marius Tucsnak)

- *Let* $T > 0$ *and assume that* $h_0 \in (-1, 1)$, $\ell_0 \in \mathbb{R}$
- $u_0 \in H^1(-1, 1)$ *and* $u_0(h_0) = \ell_0$
- $\rho_0 \in H^1(-1, h_0) \cap H^1(h_0, 1)$ *and* $\rho_0(x) > 0 \ \forall \ x \in [-1, 1] \setminus \{h_0\}$.
- $u_{-1},\ u_1 \in H^1(0, T)$, $\rho_{-1},\ \rho_1 \in H^1(0, T)$ *and*
  $\rho_{-1}(t) > 0, \quad \rho_1(t) > 0$

Then, the initial and boundary value problem formed by (1.1), (1.2) *and* (1.5) *admits a unique strong solution on* $[0, T]$.

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Strategy:

We follow a classical strategy:

- Existence and uniqueness of local in time strong solution.
  - To use fixed point argument.
  - We first rewrite the system in a fixed domain.
  - We rewrite the system in *Lagrangian mass co-ordinate.*
- Derive a priori estimates to show we do not have "contact," "vaccum" or "blow-up" in finite time.

Outline

Lagrangian-mass transformation

Let $\xi = \Psi(t,x)$, where $\Psi(t,x)$ is the signed mass of the gas filling the domain between $h(t)$ and x at instant t. More precisely, we set

$$\xi = \Psi(t,x) = \int_{h(t)}^{x} \rho(t,\eta)\,\mathrm{d}\eta \qquad (t \geqslant 0, \quad -1 \leqslant x \leqslant 1). \quad (3.1)$$

Assume (ρ, u) is a smooth enough solution of (1.1), (1.2) and (1.5) (this means, in particular, the function ρ is positive and bounded away from zero). Then, for every $t \geqslant 0$ the map $x \mapsto \Psi(t,\cdot)$, is a C^1 diffeomorphism which is strictly increasing from $(-1,1)$ to $(-\xi_{-1}(t), \xi_1(t))$, where

$$\xi_{-1}(t) = \int_{-1}^{h_0} \rho_0(\eta)\,\mathrm{d}\eta + \int_0^t \rho_{-1}(s)u_{-1}(s)\,\mathrm{d}s,$$

$$\xi_1(t) = \int_{h_0}^{1} \rho_0(\eta)\;\mathrm{d}\eta + \int_0^t \rho_1(s)u_1(s)\,\mathrm{d}s,$$

for every $t \in [0, T]$. Moreover,

$$\Psi(t, h(t)) = 0 \qquad (t \in [0, T]).$$

System in moving known domain

Let $\Phi(t,\cdot) = \Psi^{-1}(t,\cdot)$ and we set

$$\widetilde{\rho}(t,\xi) = \rho(t,\Phi(t,\xi)), \quad \widetilde{u}(t,\xi) = u(t,\Phi(t,\xi)),$$

We have the following system in $(0,T) \times (-\xi_{-1}(t), \xi_1(t)), \xi \neq 0$

$$\begin{aligned}
&\partial_t \widetilde{\rho} + \widetilde{\rho}^2 \partial_\xi \widetilde{u} = 0\\
&\partial_t \widetilde{u} - \partial_\xi \left(\widetilde{\rho}(\partial_\xi \widetilde{u})\right) + \partial_\xi (\widetilde{\rho}^\gamma) = 0,\\
&\widetilde{u}(t,0) = \dot{h}(t),\\
&m\ddot{h}(t) = \left[\widetilde{\rho}\,(\partial_\xi \widetilde{u}) - \widetilde{\rho}^\gamma\right](t,0),\\
&\widetilde{u}(t,-\xi_{-1}(t)) = u_{-1}(t),\ \ \widetilde{u}(t,\xi_1(t)) = -u_1(t),\\
&\widetilde{\rho}(t,-\xi_{-1}(t)) = \rho_{-1}(t), \quad \widetilde{\rho}(t,\xi_1(t)) = \rho_1(t)\\
&\widetilde{\rho}(0,\xi) := \widetilde{\rho}_0(\xi) = \rho_0(\Phi(0,\xi))\\
&\widetilde{u}(0,\xi) := \widetilde{u}_0(\xi) = u_0(\Phi(0,\xi))\\
&h(0) = h_0, \quad \dot{h}(0) = \ell_0.
\end{aligned}$$

System in fixed known domain:

we define

$$y = \Gamma(t,\xi) = \begin{cases} \frac{\xi}{\xi_{-1}(t)} & \text{for} \quad \xi \in [-\xi_{-1}(t), 0], \\ \frac{\xi}{\xi_1(t)} & \text{for} \quad \xi \in [0, \xi_1(t)]. \end{cases}$$

and by setting

$$\zeta(t,y) = \left[\widetilde{\rho}(t,\Gamma^{-1}(t,y))\right]^{-1}, \ \overline{u}(t,y) = \widetilde{u}(t,\Gamma^{-1}(t,y)),$$

we have the following system in $(0,T) \times (-1,1), y \neq 0$

$$\partial_t \zeta + \beta \partial_y \zeta - \alpha \partial_y \overline{u} = 0$$

$$\partial_t \overline{u} + \beta \partial_y \overline{u} - \alpha \partial_y \left(\frac{\alpha}{\zeta} \partial_y \overline{u} \right) + \alpha \partial_y \left(\frac{1}{\zeta^\gamma} \right) = 0$$

$$\overline{u}(t,0) = \dot{h}(t),$$

$$m\ddot{h} = \left[\frac{\alpha}{\zeta} \partial_y \overline{u} - \frac{1}{\zeta^\gamma} \right](t,0)$$

$$\overline{u}(t,-1) = u_{-1}(t), \quad \overline{u}(t,1) = -u_1(t),$$

$$\zeta(t,-1) = \frac{1}{\rho_{-1}(t)}, \quad \zeta(t,1) = \frac{1}{\rho_1(t)} + \text{initial conditions}$$

where

$$\alpha(t,y) = \begin{cases} \dfrac{1}{\xi_{-1}(t)} & \text{for} \quad y \in [-1,0), \\ \dfrac{1}{\xi_{1}(t)} & \text{for} \quad y \in (0,1] \end{cases}$$

$$\beta(t,y) = \begin{cases} -\dfrac{y\dot{\xi}_{-1}(t)}{\xi_{-1}(t)} & \text{for} \quad y \in [-1,0), \\ -\dfrac{y\dot{\xi}_{1}(t)}{\xi_{1}(t)} & \text{for} \quad y \in (0,1], \end{cases}$$

- $\beta = 0$, in case of homogeneous boundary condition.

Construction of Fixed point map

Let $f_1 \in L^2(0, T; L^2(-1, 1))$ and $f_2 \in L^2[0, T]$ and consider the following system

$$
\begin{aligned}
&\partial_t \zeta + \beta \partial_y \zeta - \alpha \partial_y \overline{u} = 0, \\
&\partial_t \overline{u} - \alpha_0 \partial_y \left(\frac{\alpha_0}{\zeta_0} \partial_y \overline{u} \right) = f_1, \\
&\overline{u}(t, \pm 0) = \dot{h}, \\
&m\ddot{h} = \left[\frac{\alpha_0}{\zeta_0} \partial_y \overline{u} \right](t, 0) + f_2, \\
&\ + \ \text{initial conditions and boundary conditions}
\end{aligned}
$$

- Linearization preserves the coupling of the equations of the fluid and of the structure.
- essential in our approach for obtaining the local existence result for initial data less regular than other works

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$$\mathcal{B}_T = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in L^2(Q_{0,T}) \times L^2[0,T] \;\middle|\; \|f_1\|_{L^2(Q_{0,T_*})} + \|f_2\|_{L^2[0,T_*]} \leqslant 1 \right\},$$

We consider the map

$$\begin{cases} \mathcal{N} : \mathcal{B}_T & \rightarrow & L^2(Q_{0,T}) \times L^2[0,T], \\ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} & \mapsto & \begin{bmatrix} \mathcal{F}_1(\zeta, \overline{u}) \\ \mathcal{F}_2(\zeta, \overline{u}) \end{bmatrix}. \end{cases}$$

where

$$\mathcal{F}_1(\zeta, \overline{u}) = \alpha \partial_y \left(\frac{\alpha}{\zeta} \partial_y \overline{u} \right) - \alpha_0 \partial_y \left(\frac{\alpha_0}{\zeta_0} \partial_y \overline{u} \right) - \alpha \partial_y \left(\frac{1}{\zeta^\gamma} \right) - \beta \partial_y \overline{u},$$

$$\mathcal{F}_2(\zeta, \overline{u}) = \left[\left(\frac{\alpha}{\zeta} - \frac{\alpha_0}{\zeta_0} \right) (\partial_y \overline{u}) \right] (t,0) - \left[\frac{1}{\zeta^\gamma} \right] (t,0).$$

- To show, there exists T_* small enough such that $\mathcal{N}$ is a strict contraction in $\mathcal{B}_{T_*}$

Outline

Global in time Existence

Main steps to prove global existence:

- Assume all the hypothesis and (h, ρ, u) is a strong solution solution defined on $[0, \tilde{\tau}]$ for every $\tilde{\tau} \in (0, \tau)$.
- To prove global in time existence of solution, enough to show the following :
  - **No contact:** $-1 < h(t) < 1,$ for all $t \in [0, \tau)$
  - **No Vacuum:**
$$\inf_{\substack{t \in [0,\tau) \\ x \in [-1,1] \setminus \{h(t)\}}} \rho(t, x) \geqslant C.$$
  - **No blow-up:**
$$\sup_{t \in [0,\tau)} \big(\| u(t, \cdot) |_{H^1(-1,1)} + \| \rho(t, \cdot) \|_{H^1(-1,h(t))} + \| \rho(t, \cdot) \|_{H^1(h(t),1)} \big) \leqslant C$$

Mass and energy estimates

- For every $t \in [0, \tau)$ we have

$$\int_{-1}^{h(t)} \rho(t,x)\,\mathrm{d}x \leqslant C, \quad \int_{h(t)}^{1} \rho(t,x)\,\mathrm{d}x \leqslant C.$$

  - Integrating the density equation.

- **Energy estimate:** There exists a strictly positive constant C such that

$$\int_{-1}^{1} \rho(t,x)u^2(t,x)\,\mathrm{d}x + \int_{-1}^{1} \rho^{\gamma}(t,x)\,\mathrm{d}x + \dot{h}^2(t) \leqslant C \qquad (t \in [0,\tau)).$$

  - Multiplying the velocity equation by u and integrating by parts.

bound of ρ

We define the auxiliary function $B(t,x)$ in $[-1,1]\setminus\{h(t)\}$ satisfying the following properties

$$\partial_x B = \rho u, \quad \partial_t B = \partial_x u - \rho^\gamma - \rho u^2,$$
$$B(0,x) = B_0(x) = \int_{h_0}^{x} \rho_0(y) u_0(y) dy.$$

- for every $t \in [0,\tau)$ we have

$$\sup_{x\in[-1,1]\setminus\{h(t)\}} |B| \leqslant C\Big(1 + \Big(\int_{-1}^{1} \rho(t,x)u^2(t,x)\,\mathrm{d}x\Big)^{1/2} + \int_0^t \int_{-1}^{1} (\rho^\gamma(\sigma,x) + \rho(\sigma,x)u^2(\sigma,x))\,\mathrm{d}x\,\mathrm{d}\sigma\Big).$$

- In particular

$$\sup_{x\in[-1,1]\setminus\{h(t)\}} |B| \leqslant C$$

Upper Bound of density and no contact

- $\dfrac{\partial}{\partial t}(\rho e^B) + u\dfrac{\partial}{\partial x}(\rho e^B) + \rho^{\gamma+1}e^B = 0,$
- We first get

$$\sup_{-1\leqslant x<h(t)} \rho(t,x) \leqslant M\exp\left(2\sup_{\substack{\sigma\in[0,\tau)\\ -1\leqslant x<h(\sigma)}} |B(\sigma,x)|\right) \leq C,$$

- From the relation:

$$\int_{-1}^{h(t)} \rho(t,x)\,\mathrm{d}x = \int_{-1}^{h_0} \rho_0(x)\,\mathrm{d}x + \int_0^t \rho_{-1}(\sigma)u_{-1}(\sigma)\,\mathrm{d}\sigma \qquad (t\in[0,\tau))$$

  - we get

$$1 + h(t) \geq C(1+h_0) > 0$$

- $h(t) - 1 < 0.$

Lower Bound of density

- $\frac{\partial}{\partial t}(\frac{1}{\rho}e^{-B}) + u\frac{\partial}{\partial x}(\frac{1}{\rho}e^{-B}) - \rho^{\gamma-1}e^{-B} = 0.$

$$\sup_{-1\leqslant x\leqslant h(t)} \frac{1}{\rho(t,x)} \leqslant M\exp\left(2\sup_{\substack{\sigma\in[0,\tau)\\ -1\leqslant x<h(\sigma)}} |B(\sigma,x)|\right) + C\exp\left(4\sup_{\substack{\sigma\in[0,\tau)\\ -1\leqslant x<h(\sigma)}} |B(\sigma,x)|\right)$$

Norm estimates

For every $t \in [0, \tau)$ the following estimates hold:

$$\int_{-1}^{h(t)} (\partial_x u)^2 \,\mathrm{d}x + \int_0^t \int_{-1}^{h(s)} \left[(\partial_t u)^2 + (\partial_{xx} u)^2 \right] \,\mathrm{d}x \,\mathrm{d}s \leqslant C \left(1 + \int_0^t \int_{-1}^{h(s)} (\partial_x \rho)^2 \,\mathrm{d}x \,\mathrm{d}s \right),$$

$$\sup_{t \in [0,\tau)} \int_{-1}^{h(t)} (\partial_x \rho)^2 (t, x) \,\mathrm{d}x \leqslant C,$$

Outline

Future direction of work and open questions

- Global in time strong solution for higher dimensional models.
- Study of Fluid-Structure system involving temperature equation.
- Contollability and stabilizabilty of compressible fluid-structure models.

Thank you.