Control of nonholonomic or underactuated mechanical systems: the examples of the unicycle and the slider

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Systems not stabilizable by means of continuous state feedback laws (Brockett condition)

Similarities concerning Controllability, Stabilizability, Flatness properties

Two different approaches for trajectory tracking of nonsingular reference trajectories and for fixed point stabilization
Modeling

\[
\begin{align*}
\dot{x} &= v_1 \cos(\psi) \\
\dot{y} &= v_1 \sin(\psi) \\
\dot{\psi} &= \Omega
\end{align*}
\]

\[
\begin{align*}
mx &= \tau_1 \cos(\psi) \\
m\dot{y} &= \tau_1 \sin(\psi) \\
I\dot{\psi} &= \tau_2
\end{align*}
\]
Trajectory tracking problem

The objective is to find a control law \((v_1, \Omega)^\top = k(x, x_r, y, y_r, \psi, \psi_r, v_{1r}, \Omega_r)\) such that the errors \(e_x = x - x_r, e_y = y - y_r\) and \(e_\psi = \psi - \psi_r\) asymptotically converge to 0 when \(t\) tends to infinity.

Idem for the slider:

The objective is to find a control law \((\tau_1, \tau_2)^\top = k(x, x_r, y, y_r, \dot{x}, \dot{x}_r, \dot{y}, \dot{y}_r, \psi, \psi_r, \dot{\psi}, \psi_{1r}, \dot{\psi}_{1r}, \tau_{2r})\) such that the error vector \((e_x, e_y, e_x, e_y, e_\psi, e_{\psi})^\top\) asymptotically converges to 0 when \(t\) tends to infinity.
Tangent linearized systems

The Unicycle: the TL system is controllable if

\[
\begin{pmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3
\end{pmatrix} = \begin{pmatrix}
0 & \psi_r & 0 \\
-\psi_r & 0 & 0 \\
0 & 0 & v_{1r}
\end{pmatrix} \begin{pmatrix}
e_1 \\
e_2 \\
e_3
\end{pmatrix} + \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
u_1 \\
\dot{w}_2
\end{pmatrix}
\]

\[(v_{1r}, \psi_r) \neq (0, 0)\]

The slider: the TL system is controllable if

\[
\begin{pmatrix}
\dot{e}_1 \\
\dot{e}_2 \\
\dot{e}_3 \\
\dot{e}_4 \\
\dot{e}_5 \\
\dot{e}_6
\end{pmatrix} = \begin{pmatrix}
0 & \psi_r & 1 & 0 & 0 & 0 \\
-\psi_r & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & \psi_r & 0 & 0 & 0 \\
0 & 0 & -\psi_r & 0 & u_{1r} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
e_4 \\
e_5 \\
e_6
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
u'_1 \\
u_2
\end{pmatrix}
\]

\[(u_{1r}, \psi_r) \neq (0, 0)\]
The non controllability of the tangent linearized system at a fixed point does not necessarily imply that the original nonlinear system does not satisfy the STLC property. In fact, it can be shown that the LARC is satisfied for both systems, as well as the STLC property.

STLC is satisfied for the two systems at each trajectory respectively

\[(v_1, \psi) \in \mathbb{R}^2, \quad (u_1, \psi) \in \mathbb{R}^2\]

**BUT**

Let us consider a neighborhood of the origin. It cannot be in the image of the unicycle dynamics at the origin.

The same holds for the slider. Therefore, due to Brockett’s theorem, these two systems cannot be stabilized at fixed equilibrium points by means of continuous state feedback laws.
Controllability of Time-Varying linear systems

Proposition 3.1. Let $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ be of class $C^1$ and let $B$ a real matrix of size $n \times m$. We assume that for every $s \in \mathbb{R}^*$, the pair $(A(s), B)$ is controllable. Let $T > 0$, $M > 0$ and $\varepsilon > 0$. Then there exist $\delta_1 > 0$, $\delta_2 > 0$ such that for every $T$-periodic Lipschitz function $\phi(t)$ satisfying:

- i) $|\phi(t)| \leq M$ for all $t \geq 0$;
- ii) $|\phi(t) - \phi(t')| \leq \delta_2 |t - t'|$ for all $t, t' \geq 0$,
- iii) $\forall t \geq 0, \exists s : t - \delta_1 \leq s \leq t$ such that $|\phi(t)| \geq \varepsilon$

then the system $\Sigma$ is controllable on every time interval of length at least equal to $2T$, where $\Sigma$ is the following linear time-varying system:

$$\Sigma : \quad X = A(\phi(t))X + Bu$$

The condition iii) on the function $\phi(t)$ is known as “persistently exciting condition”. This result will be used in the context of point stabilization of the slider:

[G. Kern, « Uniform controllability of a class of linear time-varying systems », IEEE TAC, 1982]
Definition found in [18] and [19]:

**Definition** The system defined by $\dot{x} = f(x, u)$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, is said to be flat if there exist a function $h : \mathbb{R}^n \times (\mathbb{R}^m)^{r+1} \to \mathbb{R}^m$, a function $\phi : (\mathbb{R}^m)^r \to \mathbb{R}^n$ and $\psi : (\mathbb{R}^m)^{r+1} \to \mathbb{R}^m$ such that:

$$y = h(x, u, \dot{u}, \ldots, u^{(r)})$$
$$x = \phi(y, \dot{y}, \ldots, y^{(r-1)})$$
$$u = \psi(y, \dot{y}, \ldots, y^{(r-1)}, y^{(r)})$$

$y$ is called the flat output of the system.

The unicycle robot is flat with flat outputs $Y_1 = x$ and $Y_2 = y$:

$$\psi = \arctan \left( \frac{\dot{y}}{\dot{x}} \right) \text{ and } v_1 = \pm \sqrt{x^2 + y^2}$$

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{pmatrix}
\cos(\psi) & -\sin(\psi) \chi_1 \\
\sin(\psi) & \cos(\psi) \chi_1
\end{pmatrix}
\begin{pmatrix}
U_1 \\
\Omega
\end{pmatrix} = \Delta
\begin{pmatrix}
U_1 \\
\Omega
\end{pmatrix}
\]

The system is dynamic feedback linearizable, the decoupling matrix being singular when the longitudinal velocity is zero. The extended state $\chi_1$ is the longitudinal velocity $v_1$ which has to be delayed.
The unicycle robot is flat with flat outputs $Y_1 = x$ and $Y_2 = y$:

$$\psi = \arctan\left(\frac{\dot{y}}{\dot{x}}\right) \text{ and } v_1 = \pm \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \cos(\psi) & -\sin(\psi) \chi_1 \\ \sin(\psi) & \cos(\psi) \chi_1 \end{pmatrix} \begin{pmatrix} U_1 \\ \Omega \end{pmatrix} = \Delta \begin{pmatrix} U_1 \\ \Omega \end{pmatrix}$$

The system is dynamic feedback linearizable, the decoupling matrix being singular when the longitudinal velocity is zero.
Flatness-based control of the unicycle

Tracking non singular reference trajectories for the unicycle robot using dynamic feedback linearization
The slider is flat with flat outputs $Y_1 = x$ and $Y_2 = y$:

$$\psi = \arctan \left( \frac{\dot{y}}{\dot{x}} \right) \text{ and } \tau_1 = \pm m \sqrt{\dot{x}^2 + \dot{y}^2}$$

$$\begin{pmatrix} x^{(4)} \\ y^{(4)} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} -2\chi_1 \psi \sin \psi - \psi^2 \chi_1 \cos \psi \\ 2\chi_1 \psi \cos \psi - \psi^2 \chi_1 \sin \psi \end{pmatrix} + \Delta \begin{pmatrix} U_1 \\ \tau_2 \end{pmatrix} \text{ with } \Delta = \frac{1}{m} \begin{pmatrix} \cos \psi & -\frac{\chi_1 \sin \psi}{I} \\ \sin \psi & \frac{\chi_1 \cos \psi}{I} \end{pmatrix}$$

The system is dynamic feedback linearizable, the decoupling matrix being singular when the longitudinal acceleration is zero, the extended state $\chi_1$ is the longitudinal acceleration which has to be delayed.
Flatness-based control of the slider

Posture tracking of a circle by a slider using dynamic linearizing feedback
Context: control of hovercrafts

- Autonomous Indoor exploration for wheeled mobile robots
  - SLAM
  - Trajectory generation and tracking control laws
  - 3D reconstruction
  - Object recognition...
- Adapt these technologies to a hybrid terrestrial and aerial quadrotor prototype
State of Art: Nonlinear control for hovercraft

- Slider dynamic behavior similar to hovercraft
- Tilting thrust

- Hovercraft model proposed in [8]
  - Simplified model derived from an underactuated surface vessel modeling
  - Kinematic and dynamic equations

Hovercrafts belong to a more general class of marine vehicles which are known to be not asymptotically stabilizable at equilibrium points by continuous state feedback laws ([9]).

See also [7], [8], [10]


State of Art: Non linear control for marine vehicles

- Trajectory tracking
  - Non linear control laws based on a Lyapunov analysis
    - Surface vessel/ Position tracking/ constraint : longitudinal speed ≠ 0 ([11])
    - Surface vessel/ Posture tracking/ Exciting reference trajectory ([12])
  - Flatness [13]
    - Hovercraft stabilization/ Constraint on the reference trajectory
  - Sliding mode [14]
    - Hovercraft stabilization

State of Art: Non linear control for marine vehicles

- Point stabilization
  - Time-varying control laws
    - Smooth feedback but slow convergence \([11]\)
    - Homogeneous continuous feedback, fast convergence, low robustness \([9], [16]\) (Surface vessel stabilization)
  - Discontinuous control laws
    Lyapunov based analysis \([8]\) (Hovercraft Stabilization)

- Practical stabilization
  - Hovercraft Stabilization/ Position tracking/ \(C^3\) reference trajectory \([17]\)
  - Transverse functions \([18]\)

Dynamic modeling

- **Model**

\[
\begin{align*}
\ddot{x} &= C_{\psi_1} S_{\theta_2} u_1 \\
\ddot{y} &= S_{\psi_1} S_{\theta_2} u_1 \\
\ddot{\psi}_1 &= \frac{\sin(\theta_2)}{C(\theta_2) + J_1} u_2 + \frac{\cos(\theta_2)}{C(\theta_2) + J_1} u_3 + \psi_1 \dot{\theta}_2 \frac{\sin(2\theta_2)(J_2'' - I_2'')}{C(\theta_2) + J_1} \\
\ddot{\theta}_2 &= u_4 - \frac{B(\theta_2)}{I_2''} \dot{\psi}_1^2
\end{align*}
\]

- **Commands**

\[
\begin{align*}
u_1 &= \frac{a}{M} \sum_{i=1}^{4} \omega_i^2 \\
u_2 &= \frac{al}{\sqrt{2}} \left( \omega_1^2 - \omega_2^2 - \omega_3^2 + \omega_4^2 \right) \\
u_3 &= -b(\omega_1^2 - \omega_2^2 + \omega_3^2 - \omega_4^2) \\
u_4 &= \frac{al}{\sqrt{2I_2''}} \left( \omega_3^2 + \omega_4^2 - \omega_1^2 - \omega_2^2 \right) + \Gamma_{\text{servo}}
\end{align*}
\]
Dynamic modeling

\[
\begin{align*}
\dot{x} &= \cos(\psi)p \\
\dot{y} &= \sin(\psi)p \\
\dot{\psi} &= v_2 \\
\dot{\theta} &= v_3 \\
p &= \sin(\theta)u_1
\end{align*}
\]

the dynamical equations of this quadrotor are quite similar to the slider dynamics
Outputs $x$, $y$, and $\theta$ are flat outputs:

\[
\begin{cases}
    x = x \\
    y = y \\
    \eta = \sqrt{\frac{x^2 + y^2}{\sin^2(\theta_2)}} \\
    \psi_1 = \arctan\left(\frac{y}{x}\right) \\
    \theta_2 = \theta_2
\end{cases}
\]

\[
\begin{align*}
    y &= h(x, u, \dot{u}, \ldots, u^{(r)}) \\
    x &= \phi(y, \dot{y}, \ldots, y^{(r-1)}) \\
    u &= \psi(y, \dot{y}, \ldots, y^{(r-1)}, y^{(r)})
\end{align*}
\]

\[
\begin{pmatrix}
    \ddot{x} \\
    \ddot{y} \\
    \dot{\theta}_2
\end{pmatrix} = \begin{pmatrix}
    \cos(\psi_1)\sin(\theta_2) & -\eta\sin(\psi_1)\sin(\theta_2) & \eta\cos(\psi_1)\cos(\theta_2) \\
    \sin(\psi_1)\sin(\theta_2) & \eta\cos(\psi_1)\sin(\theta_2) & \eta\sin(\psi_1)\cos(\theta_2) \\
    0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
    v_1 \\
    v_2 \\
    v_3
\end{pmatrix}
\]

\[\det = \eta (\sin \theta_2)^2\]
Control law: flatness

- Control law:

\[
\begin{align*}
    u_1 &= \frac{\cos(\arctan(\frac{\dot{y}}{\dot{x}}))}{\sin(\theta_2)} V_\dot{x} + \frac{\sin(\arctan(\frac{\dot{y}}{\dot{x}}))}{\sin(\theta_2)} V_\dot{y} - \sqrt{\dot{x}^2 + \dot{y}^2} \frac{\cos(\theta_2)}{\sin^2(\theta_2)} V_\dot{\theta_2} \\
    u_2 &= -\frac{\sin(\arctan(\frac{\dot{y}}{\dot{x}}))}{\sqrt{\dot{x}^2 + \dot{y}^2}} V_\ddot{x} + \cos(\arctan(\frac{\dot{y}}{\dot{x}})) \frac{\cos(\theta_2)}{\sqrt{\dot{x}^2 + \dot{y}^2}} V_\ddot{y} \\
    u_3 &= V_\dot{\theta_2}
\end{align*}
\]

- Stability and convergence are assured for the closed loop system with:

\[
\begin{align*}
    V_\ddot{x} &= \ddot{x}_{ref} + k_{2x}(x_{ref} - \ddot{x}) + k_{1x}(x_{ref} - \dot{x}) + k_{0x}(x_{ref} - x) + k_{-1x}(\int x_{ref} dt - \int x dt) \\
    V_\ddot{y} &= \ddot{y}_{ref} + k_{2y}(y_{ref} - \ddot{y}) + k_{1y}(y_{ref} - \dot{y}) + k_{0y}(y_{ref} - y) + k_{-1y}(\int y_{ref} dt - \int y dt) \\
    V_\dot{\theta_2} &= \dot{\theta}_{2ref} + k_{0\theta_2}(\theta_{2ref} - \theta_2) + k_{-1\theta_2}(\int \theta_{2ref} dt - \int \theta_2 dt)
\end{align*}
\]
Experimental platform
- Motion Capture system
- Drone

Identification process
- Aerodynamic forces and moments
- Friction effects (static and kinetic)
- Grey box identification

Trajectory tracking results
- software MOTIVE
  - Streaming VRPN
  - Remote control
- Embedded computer
  - Microcontroller MikroKopter
- Infrared cameras s250e Optitrack
- Communication ZigBee
- Drone
  - Serial port
Experimental room
Non singular trajectory tracking

- Trajectory tracking realized by the flatness control law
  - Reference trajectory constraints: $\eta (\sin \theta_2)^2 \neq 0$
  - Derivatives until the second order for the state and third order for the reference

- Experimental conditions
  - Circular trajectory tracking with radius 1.1m
  - Initial position: 50cm from the reference trajectory
  - Ground: parquet
Trajectory tracking results

![Graph showing trajectory tracking results](image-url)
Trajectory tracking results
Fixed-point stabilization

Theorem 4.1. [11] Let us consider the following control inputs:

\[
\begin{align*}
\dot{u}(X,t) &= w(X)g(t) \\
\dot{v}(X,t) &= -q(X,u(X,t))
\end{align*}
\]

where:

(1)

\[
g \in C^\infty(\mathbb{R}, [-1, 1])
\]

\[
\forall t \in \mathbb{R}, \quad g(t + T) = g(t)
\]

\[
\forall t \in \mathbb{R}, \quad \exists \varepsilon \in \mathbb{N} - \{0\}, \quad g^{(\varepsilon)}(t) \neq 0
\]

for example, \( g(t) = \sin(\omega t) \)

(2)

\[
w : \mathbb{R}^n \rightarrow \mathbb{R}, \quad w \in C^\infty(\mathbb{R}^n, \mathbb{R})
\]

\[
w(0) = 0
\]

\[
\forall X \neq 0, \quad w(X) = 0 \Rightarrow q(X, 0) \neq 0
\]

\[
\forall X \quad \text{s.t.} \quad w(X) \neq 0, \quad \exists \varepsilon \in \mathbb{N} \quad \text{s.t.}
\]

\[
\forall u \in [-|w(X)|, |w(X)|], \quad \frac{\partial q}{\partial u} \neq 0
\]

(3)

\[
q : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}
\]

\[
q(X, u) = \sum_{i=1}^{n} \frac{\partial q}{\partial x_i}(x)f_i(X, u)
\]

where \( V \) is a Lyapunov function candidate such that:

\[
V : \mathbb{R}^n \rightarrow \mathbb{R}
\]

\[
V(X) \rightarrow +\infty \quad \text{when} \quad |X| \rightarrow +\infty
\]

\[
V(0) = 0
\]

\[
\forall X \in \mathbb{R}^n - \{0\}, \quad V(X) > 0
\]

If the control laws are chosen as in equation (55), then the origin 0 ∈ \( \mathbb{R}^n \) is globally asymptotically stable for

\[
\dot{X} = v(X, t) f(X, u(X, t))
\]

The theorem applies for the unicycle robot with

\[ \mathbf{X} = (x, y)^\top \text{, the controls } v = v_1, u = \psi, f(X, u) = (\cos(\psi), \sin(\psi))^\top \]

\[ q(X, u) = x \cos(\psi) + y \sin(\psi), \quad w(X) = y, \quad g(t) = \sin(\omega t) \]

and \[ V(X) = \frac{x^2 + y^2}{2}. \]
Fixed point stabilization for the unicycle

\textbf{FIGURE 10.} Time-periodic feedback for unicycle stabilization at the rest point $x_{ref} = 0$, $y_{ref} = 0$ and $\psi_{ref} = 0$. The initial state is $x_0 = 0$, $y_0 = 3$, $\psi = 0$ and gains are $K_1 = 5$, $K_2 = 1$ and $\omega = 3$. 
Hybrid control for the unicycle

Stabilization of the slider at a zero velocity

- The theorem applies for the slider with

\[ X = (\dot{x}, \dot{y})^T, \text{ the controls } v = \frac{\bar{F}_1}{m}, u = \psi, f(X,u) = (\cos(\psi), \sin(\psi))^T \]

\[ q(X,u) = \dot{x}\cos(\psi) + \dot{y}\sin(\psi), \ w(X) = \dot{y}, \ g(t) = \sin(\omega t) \]

and \[ V(X) = \frac{\dot{x}^2 + \dot{y}^2}{2}. \]
Stabilization of the slider at a zero velocity
We can consider the error tracking system at an equilibrium point with

\[ \tau_{1r} = \tau_{2r} = 0 \]

\[
\Sigma_1 : \begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{pmatrix} = \begin{pmatrix} 0 & \psi & 1 & 0 \\ -\psi & 0 & 0 & 1 \\ 0 & 0 & -\frac{d_1}{m} & \psi \\ 0 & 0 & -\psi & -\frac{d_2}{m} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_1
\]

\[ \psi = -\frac{d_3}{I} \psi + u_2 \]

A first idea, inspired from [27], is to define \( u_1 \) ensuring the stabilization of the time-varying system \( \Sigma_1 \).

The second step is then to use \( u_2 \), to make the yaw rate \( \psi \) persistently exciting.

Controllability of Time-Varying linear systems

**Proposition 3.1.** Let $A: \mathbb{R} \to \mathbb{R}^{n \times n}$ be of class $C^1$ and let $B$ a real matrix of size $n \times m$. We assume that for every $s \in \mathbb{R}^*$, the pair $(A(s), B)$ is controllable. Let $T > 0$, $M > 0$ and $\varepsilon > 0$. Then there exist $\delta_1 > 0$, $\delta_2 > 0$ such that for every $T$-periodic Lipschitz function $\phi(t)$ satisfying:

- i) $|\phi(t)| \leq M$ for all $t \geq 0$;
- ii) $|\phi(t) - \phi(t')| \leq \delta_2 |t - t'|$ for all $t, t' \geq 0$;
- iii) $\forall t \geq 0$, $\exists s : t - \delta_1 \leq s \leq t$ such that $|\phi(t)| \geq \varepsilon$

then the system $\Sigma$ is controllable on every time interval of length at least equal to $2T$, where $\Sigma$ is the following linear time-varying system:

$$
\Sigma : \quad X = A(\phi(t))X + Bu
$$

The condition iii) on the function $\phi(t)$ is known as “persistently exciting condition”. This result will be used in the context of point stabilization of the slider.

[G. Kern, « Uniform controllability of a class of linear time-varying systems », IEEE TAC, 1982]
Fixed point stabilization of the slider

Proposition 4.4. Let us consider $\Sigma_1$ in closed-loop with $u_1$ defined as follows:

$$u_1(t) = \frac{1}{m} (-k_3 e_3 + k_4 \psi(t) e_4 - k_1 e_1 + k_2 \psi(t) e_2),$$

where the control gains $k_i$ satisfy the following inequalities:

$$k_3 \geq d_2 - d_1,$$
$$k_4 = \frac{m k_2 (k_1 + k_3 + d_1 - d_2)}{d_2 k_2 + m k_1},$$
$$0 < k_1 < (k_3 + d_1 - d_2) \frac{d_2}{m},$$
$$k_2 > 0.$$

If $\psi(t)$ satisfies the assumptions of proposition 3.1, then system $\Sigma_1$ in closed-loop with (78) is globally asymptotically stable.

A change of coordinates

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & \frac{d_1}{m} & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \lambda & 0 & 1 & 0 \\ \mu & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & \frac{-1}{\mu - \lambda} \\ 0 & 0 & \frac{1}{\mu - \lambda} & 0 \\ 0 & 0 & \mu - \lambda & 0 \\ 0 & 0 & 0 & \mu - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$
Fixed point stabilization of the slider

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\dot{x}_4 \\
\end{pmatrix} = A_1(t) \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{pmatrix}
\]

\[
A_1(t) = \begin{pmatrix}
0 & 0 & \frac{m\mu-d_2}{m(\mu-\lambda)}\psi(t) & \frac{d_2-m\lambda}{m(\mu-\lambda)}\psi(t) \\
0 & -\frac{d_2}{m}\psi(t) & \frac{\mu}{(\mu-\lambda)}\psi(t) & -\mu \\
\frac{k_2+m\lambda}{d_2}\psi(t) & \frac{m\mu(k_2+m\lambda)(d_2-m\lambda)}{d_2(m^2\lambda\mu+d_2k_2)}\psi(t) & \frac{\mu}{(\mu-\lambda)}\psi(t) & -\frac{d_2-m\lambda}{m(\mu-\lambda)}\psi(t) \\
\frac{k_2+m\mu}{d_2}\psi(t) & -\frac{m\lambda(k_2+m\mu)(\mu-d_2)}{d_2(m^2\lambda\mu+d_2k_2)}\psi(t) & 0 & -\lambda \\
\end{pmatrix}
\]

A Lyapunov function candidate:

\[
V(x_1, x_2, x_3, x_4) = \frac{m}{2d_2}x_1^2 + \frac{m(d_2-m\lambda)(\mu-d_2)}{2d_2(m^2\lambda\mu+d_2k_2)}x_2^2 + \frac{(\mu-d_2)}{2(k_2+m\lambda)(\mu-\lambda)}x_3^2 + \frac{(d_2-m\lambda)}{2(k_2+m\mu)(\mu-\lambda)}x_4^2
\]

Its time-derivative:

\[
\dot{V}(x_1, x_2, x_3, x_4) = -\frac{(d_2-m\lambda)(\mu-d_2)}{m^2\lambda\mu+d_2k_2}x_2^2 - \frac{\mu(m\mu-d_2)}{(k_2+m\lambda)(\mu-\lambda)}x_3^2 - \frac{\lambda(d_2-m\lambda)}{(k_2+m\mu)(\mu-\lambda)}x_4^2
\]
Fixed point stabilization of the slider

\[ \dot{V} = -x^T C^T(t) C(t) x \leq 0 \]

\[ C = \begin{pmatrix} 0 & \frac{\sqrt{(m\mu - d_2)(d_2 - m\lambda)}}{m^2 \lambda \mu + d_2 k_2} & \frac{\sqrt{\mu (m\mu - d_2)}}{(\mu - \lambda)(k_2 + m\lambda)} & \frac{\sqrt{\lambda (d_2 - m\lambda)}}{(\mu - \lambda)(k_2 + m\mu)} \end{pmatrix} \]

globally asymptotically stable if the pair \( (A_1(\psi(t)), C) \) is uniformly completely observable

By duality, OK if:

pair \( (A_1^T(\psi(t)), C^T) \) satisfies the Kalman controllability condition if \( \psi(t) \neq 0 \) and if \( \psi(t) \) satisfies the assumptions i), ii) and iii) of proposition 3.1.

For the complete system, the previous law $u_1$ and the following yaw rate viewed as virtual Input (completed by backstepping technique) ensure the fixed point stabilization:

$$\psi = \frac{-K_1 (\psi - \psi_r + Ke_2 \sin(\omega t)) - Ke_4 \sin(\omega t) + K_\omega e_2 \cos(\omega t)}{1 - Ke_1 \sin(\omega t)}$$

Proof: Lyapunov function candidate:

$$\dot{V} = V + \frac{1}{2} (\psi - \psi_r + Ke_2 \sin(\omega t))^2$$

Time-derivative of the Lyapunov function:

$$\dot{V} = -\frac{(d_2 - M\lambda)(M\mu - d_2)}{(M^2\lambda\mu + d_2k_2)} \chi_2^2 - \frac{\mu (M\mu - d_2)}{(k_2 + M\lambda)(\mu - \lambda)} \chi_3^2 - \frac{\lambda (d_2 - M\lambda)}{(k_2 + M\mu)(\mu - \lambda)} \chi_4^2 - K_1 (\psi - \psi_r + Ke_2 \sin(\omega t))^2$$
LaSalle arguments:

\[ \mathcal{I} \subseteq \{ \dot{V} = 0 \} = \{ x_2 = 0, x_3 = 0, x_4 = 0, \psi = \psi_r - Ke_2 \sin(\omega t) \} \]

On this invariant subset we can prove that \( e_2 \) remains constant and that the yaw rate satisfies the assumptions of proposition 3.1 if \( \omega \) is sufficiently small:

\[ \psi = -K \omega e_2 \cos(\omega t) \]

From proposition 4.4, \( x = 0 \) is globally asymptotically stable which implies that \( e_2 \) tends to zero. \( \psi \) tends to zero and \( \psi \) tends to the desired reference value \( \psi_r \), which concludes
Fixed point stabilization of the slider

Figure 14. Slider point stabilization, $P_{ref} = (x_{ref} = 0.5, y_{ref} = 0.5, \psi_{ref} = -0.75)$, $P_{init} = (x_{init} = -1.6, y_{init} = 0, \psi_{init} = -1.218)$, gains are $k_1 = 10$, $k_3 = 3$, $k_4 = 15$, $K_1 = 1$, $K = 1$, $\omega = 3$. a) slider trajectory on the $x/y$ plane, b) associated Lyapunov function.
FIGURE 15. Slider point stabilization, $P_{\text{ref}} = (x_{\text{ref}} = 0.5, y_{\text{ref}} = 0.5, \psi_{\text{ref}} = -0.75)$, $P_{\text{ini}} = (x_{\text{ini}} = -1.6, y_{\text{ini}} = 0, \psi_{\text{ini}} = -1.218)$, gains are $k_1 = 10$, $k_3 = 3$, $k_4 = 15$, $K_1 = 1$, $K = 1$, $\omega = 3$. 
A hybrid control strategy for unicycle and slider type robots.

How to extend these results in the context of finite-time stabilization?

We collaborate in a recent research project on FTS funded by ANR with colleagues from INRIA Lille (W. Perruquetti, A. Poliakov, D. Efimov, J.P. Richard ...), UPMC (J.M. Coron, E. Trélat ...) and L. Rosier
Conclusion and perspectives

Inspired from [37, Prop. 5], let us consider the following cascaded system:

\[
\begin{align*}
X &= f(X, y) \\
y &= v
\end{align*}
\] (19)

**Theorem 3.3.** Let us consider system (19) where \( X \in \mathbb{R}^n \) and \( y \in \mathbb{R} \). Let us suppose that:

- the vector-field \( f \) is \( r \)-homogeneous of degree \( \kappa \in \mathbb{R} \), with \( r = (r_1, \cdots, r_{n+1}) \).
- \( \kappa \) satisfies the following:
  \[
  \kappa > -r_{n+1}
  \] (20)
- there exists a continuous feedback law \( \tilde{y} : \mathbb{R}^n \rightarrow \mathbb{R} \), \( r \)-homogeneous of degree \( r_{n+1} \) with \( \tilde{r} = (r_1, \cdots, r_n) \), such that the origin of \( \tilde{X} = f(X, \tilde{y}(X)) \) is asymptotically stable,
- there exists an odd integer \( l > \max\left(\frac{r_i}{r_{n+1}}\right) \) such that \( \tilde{y}^l \) is \( C^1 \).

Then the control law:

\[
v(X, y) = -K(y - \tilde{y}(X))^{\kappa + r_{n+1}}
\] (21)

with a sufficiently high gain \( K > 0 \), makes the origin of the cascaded system (19) asymptotically stable. Moreover, if \( \kappa < 0 \), this asymptotic stability is a finite-time stability.

B. d’A-N, J-M. Coron, W. Perruquetti, « FTS of nonholonomic or underactuated mechanical systems: the examples of the unicycle and the slider », in preparation

Conclusion and perspectives

The case of the double integrator

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u
\end{aligned}
\]  

(48)

**Theorem 3.6.** The origin of the double integrator system (48) is GFTS with the following feedback law:

\[
u = -K_2 (r_2 - \bar{r}_2)^{\frac{x_1}{r_1}}
\]

where \(K_2 > 0\) is a sufficiently high gain,

\[x_2 = -\{x_1\}^\alpha_1, \quad \alpha_1 \in ]1/2; 1[\]

and

\[r_2 = \alpha_1 r_1 = r_1 + \kappa_1, \quad r_1 > 0,\]

\(r_1\) and \(r_2\) being the weights of the dilatation.

\[\{x\}^\alpha = \text{sgn}(x)|x|^\alpha, \text{ for any real number } \alpha \geq 0\]

E. Bernuau, W. Perruquetti, D. Efimov, E. Moulay, « Robust FT ouput feedback stabilization of the double intergrator »,

Y. Hong, « FT stabilization and stabilizability of a class of controllable systems », SCL, 2002

The case of the slider

\[ \begin{align*}
  x_1 &= x_2 \\
  \dot{x}_2 &= u_1 \\
  x_3 &= x_4 \\
  \dot{x}_4 &= u_1 x_5 \\
  x_5 &= x_6 \\
  \dot{x}_6 &= v_2
\end{align*} \]

Choose for \( t > \frac{T}{2}, u_1 \) to stabilize the double integrator

1) \begin{align*}
  u_1 \text{ and } v_2 \text{ in a first step} \\
  x_3(t) = x_4(t) = x_5(t) = x_6(t) = 0 \text{ for all } t \geq \frac{T}{2}
\end{align*}

Then if we impose \( v_2 = 0 \) for \( t \geq \frac{T}{2} \),
\( x_3, x_4, x_5 \) and \( x_6 \) will remain zero.

\[ u_1 = \frac{r_1}{m}, u_2 = \psi \text{ and } v_2 = \frac{\dot{r}_2}{l} \]

\[ X = (x_1 = x, x_2 = \dot{x}, x_3 = y, x_4 = \dot{y}, x_5 = u_2 = \psi, x_6 = \dot{u}_2 = \dot{\psi})^T \]

We conclude by time-periodicity.
Some works on hybrid systems

Stabilization of dynamical systems modeled by hyperbolic PDEs coupled with Boundary Conditions modeled by non linear ODEs.

Some examples:

**Overhead crane with flexible cable:**


Some works on hybrid systems

Irrigation canals:


Some works on hybrid systems

Wind Instruments:


Some works on hybrid systems
Some works on hybrid systems

- We have considered the simpler problem of controlling a slide flute: a cylindrical recorder without finger holes but with a piston mechanism to modify the length.

- From a mathematical point of view: a system of conservation laws leading to hyperbolic PDEs coupled with a nonlinear ODE with delay.

\[
\begin{align*}
\frac{\partial \tilde{\alpha}}{\partial t} (\sigma, t) + \left( \frac{c-i\sigma}{L} \right) \frac{\partial \tilde{\alpha}}{\partial x} (\sigma, t) &= 0 \\
\frac{\partial \tilde{\beta}}{\partial t} (\sigma, t) - \left( \frac{c+i\sigma}{L} \right) \frac{\partial \tilde{\beta}}{\partial x} (\sigma, t) &= 0
\end{align*}
\]

\[
x = L\sigma
\]
Model of the jet channel and the mouth

The ideal boundary condition \( p(0, t) = 0 \) at \( x=0 \) is too simple!

flue channel

Bernoulli equations

\[
\rho_0 l_c \frac{dU_j}{dt} + \frac{1}{2} \rho_0 U_j^2 = p_j - p_m
\]

\[
p_m - p_0 = c_1 \dot{u}_0 - \Delta p
\]

The pressure in the mouth is related to the output flow by the radiation impedance

\[
p_m = c_2 Q_{out} - c_3 Q_{out}
\]

- Interaction jet/labium, jet/popaque, jet drive mechanism [Cremer, Ising 1968; Coltman 1976].
- Vortex shedding at the labium [Verge, Fabre et al. 1994; Verge, Hirschberg, Caussé 1996].
- \( U_j \) is the jet velocity and \( \Delta p \) the pressure jump responsible of the sound production, mainly composed of \( \Delta p_{jd} \) the pressure jump due to the jet drive mechanism, and \( \Delta p_a \) due to the vortex shedding.
Model of the jet channel and the mouth for the flute

- The pressure term due to the dipolar jet-drive:

\[ \Delta p_{jd} = \text{cste.} \frac{dQ_{in}}{dt} \]

Jet position

Spatial amplification of the jet

Mouth cross section at the flue exit

Spatial amplification of the jet

\[ \eta(t) = 2he^{tW} \frac{u_0(t - \tau)}{\pi S_m U_j}, \quad \tau = \frac{W}{0.3U_j} \]

Labium delay

- The dissipative pressure term due to vortex shedding:

\[ \Delta p_d = -\frac{1}{2} \rho_0 \left( \frac{u_0}{\alpha_v S_m} \right)^2 \text{sign}(u_0) \]

Vena-contracta factor of the flow
pf = 55 Pa

pf = 300 Pa

f1s = 310 Hz (close to 290.8 Hz)

f2s = 925 Hz (close to 903.7 Hz)
A downscale

pressure signal for a down scale

pipe length for a down scale
Control algorithm

- The note being chosen, we solve the linearized B.C. w.r.t. L and U₀. For example, to obtain Fe ideal \((c/4L) = 324\) Hz, this leads to:

\[
L_r = 0.242 \, m, \quad U_{0r} = 8.61 \, m/s
\]

- This is our feedforward control algorithm.

- Feedback:

\[
\begin{cases}
\dot{L} = -k(L - L_r) \\
p_f = PID(p_f - p_{fr}) \text{ with } p_{fr} = \frac{1}{2}\rho_0 U_{0r}^2
\end{cases}
\]
Control

Target pitch \rightarrow modal analysis \rightarrow PID \rightarrow \text{Slide + Artificial mouth}

feedback

feedforward
A graphical interface
A new mechatronic project
HAPPY BIRTHDAY

JEAN-MICHEL !!!