Time optimal control problems:

Bang-bang property and observability estimates from measurable sets

Can ZHANG (zhangcansx@163.com)

Paris, June 2016

In honor of Prof. J. M. Coron's 60th birthday occasion

1. Motivations
   - Observability estimates over measurable sets implies the bang-bang property
2. Main results
3. Conclusion and further research

Bang-bang property of time optimal controls

We introduce the time optimal control problem for the heat equation. Let $\omega \subset \Omega$ be an open subset with its characteristic function χ_ω. Consider the time optimal control problem:

$$(TP)_1^M: \qquad T(M) \triangleq \inf_{u \in \mathcal{U}^M} \{t > 0 : y(t; u) = 0\},$$

Bang-bang property of time optimal controls

We introduce the time optimal control problem for the heat equation. Let $\omega \subset \Omega$ be an open subset with its characteristic function χ_ω. Consider the time optimal control problem:

$$(TP)_1^M: \qquad T(M) \triangleq \inf_{u \in \mathcal{U}^M} \left\{ t > 0 : \ y(t;u) = 0 \right\},$$

where $\mathcal{U}^M$ is the control constraint given by

$$\mathcal{U}^M = \left\{ u \in L^\infty(\Omega \times \mathbb{R}^+) : \ |u(x,t)| \le M \ \text{ for a.e. } (x,t) \in \Omega \times \mathbb{R}^+ \right\}, \ M > 0,$$

and $y(\cdot;u)$ solves the non-homogeneous heat equation

$$\begin{cases} y_t - \Delta y = \chi_\omega u & \text{in } \Omega \times \mathbb{R}^+, \\ y = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{cases}$$

• We call $T(M)$ the optimal time (if it exists) and u^* a time optimal control if $y(T(M); u^*) = 0$.

• We call $T(M)$ the optimal time (if it exists) and u^* a time optimal control if $y(T(M); u^*) = 0$. In fact, the existence of optimal time and time optimal controls are guaranteed by the observability inequality and the decay of energy of the heat equation.

• We call $T(M)$ the optimal time (if it exists) and u^* a time optimal control if $y(T(M); u^*) = 0$. In fact, the existence of optimal time and time optimal controls are guaranteed by the observability inequality and the decay of energy of the heat equation.

• In the state space $L^2(\Omega)$, since the reachable set

$$\{y(T(M); u) : u \in \mathcal{U}^M\}$$

has no interior point in $L^2(\Omega)$, to the best of our knowledge, we do not know how to separate this set from the target set $\{0\}$ by a hyperplane in $L^2(\Omega)$. Thus, we do not know how to get the Pontryagin maximum principle for Problem $(TP)_1^M$ by the way used in the case of O.D.E..

• We call $T(M)$ the optimal time (if it exists) and u^* a time optimal control if $y(T(M);u^*) = 0$. In fact, the existence of optimal time and time optimal controls are guaranteed by the observability inequality and the decay of energy of the heat equation.

• In the state space $L^2(\Omega)$, since the reachable set

$$\{y(T(M);u) : u \in \mathcal{U}^M\}$$

has no interior point in $L^2(\Omega)$, to the best of our knowledge, we do not know how to separate this set from the target set $\{0\}$ by a hyperplane in $L^2(\Omega)$. Thus, we do not know how to get the Pontryagin maximum principle for Problem $(TP)_1^M$ by the way used in the case of O.D.E..

• It is natural to ask if the bang-bang property (for simplicity B-B-P) holds for $(TP)_1^M$: Any time optimal control u^* for $(TP)_1^M$ verifies $|u^*(x,t)| = M$ for a.e. $(x,t) \in \omega \times (0, T(M))$.

- The time optimal control is unique.

The argument is very simple. Assume u^* and v^* are two time optimal controls. Since

$$\Big|\frac{u^*(x,t)-v^*(x,t)}{2}\Big|^2=\frac{|u^*(x,t)|^2+|v^*(x,t)|^2}{2}-\Big|\frac{u^*(x,t)+v^*(x,t)}{2}\Big|^2,$$

and $(u^*+v^*)/2$ is also a time optimal control, by the B-B-P, we get that $u^*=v^*$ a.e. in $\omega\times(0,T(M))$.

The advantage of B-B-P

- The time optimal control is unique.

The argument is very simple. Assume u^* and v^* are two time optimal controls. Since

$$\left|\frac{u^*(x,t)-v^*(x,t)}{2}\right|^2 = \frac{|u^*(x,t)|^2+|v^*(x,t)|^2}{2} - \left|\frac{u^*(x,t)+v^*(x,t)}{2}\right|^2,$$

and $(u^*+v^*)/2$ is also a time optimal control, by the B-B-P, we get that $u^*=v^*$ a.e. in $\omega\times(0,T(M))$.

- The equivalence between the time optimal control problem and norm optimal control problem, and so provide a necessary and sufficient condition for the time optimal control problem.

G. Wang, E. Zuazua, On the equivalence of minimal time and minimal norm controls for internally controlled heat equations, SICON, 2012.

Observability inequalities from measurable sets

To present our motivations, we begin with the simplest situation. Let $T > 0$ and Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Consider the heat equation

$$\begin{cases} \partial_t u - \Delta u = 0, & \text{in } \Omega \times (0, T), \\ u = 0, & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 \in L^2(\Omega). \end{cases}$$

Two important a priori estimates for the above equation are as follows.
Interior case:

$$\|u(T)\|_{L^2(\Omega)} \leq N(\Omega, T, \mathcal{D}) \int_{\mathcal{D}} |u(x,t)|\, dxdt, \ \forall\ u_0 \in L^2(\Omega), \qquad (1)$$

where $\mathcal{D}$ is a measurable subset of $\Omega \times (0, T)$.

Boundary case:

$$\|u(T)\|_{L^2(\Omega)} \leq N(\Omega, T, \mathcal{J}) \int_{\mathcal{J}} |\frac{\partial}{\partial \nu} u(x,t)| \, d\sigma dt, \ \forall \ u_0 \in L^2(\Omega), \quad (2)$$

where $\mathcal{J}$ is a measurable subset of $\partial\Omega \times (0, T)$.

Boundary case:

$$\|u(T)\|_{L^2(\Omega)} \leq N(\Omega, T, \mathcal{J}) \int_{\mathcal{J}} |\frac{\partial}{\partial\nu} u(x,t)|\, d\sigma dt, \ \forall\ u_0 \in L^2(\Omega), \quad (2)$$

where $\mathcal{J}$ is a measurable subset of $\partial\Omega \times (0, T)$.

• Such a priori estimates are usually called observability inequalities in Control Theory when the observation regions are open subsets.

Boundary case:

$$\|u(T)\|_{L^2(\Omega)} \leq N(\Omega, T, \mathcal{J}) \int_{\mathcal{J}} |\frac{\partial}{\partial \nu} u(x,t)| \, d\sigma dt, \ \forall \ u_0 \in L^2(\Omega), \quad (2)$$

where $\mathcal{J}$ is a measurable subset of $\partial\Omega \times (0, T)$.

• Such a priori estimates are usually called observability inequalities in Control Theory when the observation regions are open subsets. Our aim is to build up estimates (1) and (2) when $\mathcal{D}$ and $\mathcal{J}$ are subsets of positive measure and positive surface measure, respectively.

Observability estimates from measurable sets implies the B-B-P

Main idea: By contradiction, we would suppose that there were a constant $\varepsilon \in (0, M)$ and a subset of positive measure $\mathcal{D} \subset \omega \times (0, T(M))$ such that

$$|u^*(x,t)| \leq M - \varepsilon, \quad \forall\, (x,t) \in \mathcal{D}.$$

It provides a "room" for constructing another control (by the duality) such that there exist $\delta \in (0, T(M))$ and $v \in L^\infty(\Omega \times \mathbb{R}^+)$, with $\|v\|_{L^\infty} \leq M$, such that

$$\begin{cases} \partial_t y - \Delta y = \chi_\omega v & \text{in} \quad \Omega \times (0, T^* - \delta), \\ y = 0 & \text{on} \quad \partial\Omega \times (0, T^* - \delta), \\ y(x,0) = y_0(x) & \text{in} \quad \Omega, \\ y(x, T^* - \delta) = 0 & \text{in} \quad \Omega. \end{cases}$$

This leads to a contradiction with the time optimality of $T(M)$.

V. J. Mizel, T. I. Seidman, An abstract bang-bang principle and time optimal boundary control of the heat equation, SICON, 1997.

G. Wang, L^∞-Null controllability for the heat equation and its consequences for the time optimal control problem, SICON, 2008.

1. Motivations
2. Main results
   - Heat equations
   - Abstract evolution equations
     - An specific example
3. Conclusion and further research

Space-time analyticity estimate

Recall the following result in the last talk given by M. Santiago:

Theorem

Assume that $\triangle_{4R}(q_0)$ (when non-empty) is real-analytic. Then, there are constants N and ρ, with $0 < \rho \leq 1$, such that

$$|\partial_x^\alpha \partial_t^\beta e^{t\Delta} f(x)| \leq \frac{N\,(t-s)^{-\frac{n}{4}}\; e^{8R^2/(t-s)} |\alpha|!\,\beta!}{(R\rho)^{|\alpha|}\,((t-s)\,/4)^\beta} \|e^{s\Delta} f\|_{L^2(\Omega)},$$

when $x \in B_{2R}(q_0) \cap \overline{\Omega}$, $0 \leq s < t$, $\alpha \in \mathbb{N}^n$ and $\beta \geq 0$.

Here $\triangle_{4R}(q_0) \triangleq B_{4R}(q_0) \cap \partial\Omega$.

Propagation of estimate for real-analytic functions

Theorem

Assume that $f : B_{2R} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ *is real-analytic in* B_{2R} *verifying*

$$|\partial_x^\alpha f(x)| \leq \frac{M|\alpha|!}{(\rho R)^{|\alpha|}} \ , \ \textit{when} \ \ x \in B_{2R}, \ \alpha \in \mathbb{N}^n,$$

for some $M > 0$ *and* $0 < \rho \leq 1$. *Let* $E \subset B_R$ *be a measurable set with positive measure. Then, there are positive constants* $N = N(\rho, |E|/|B_R|)$ *and* $\theta = \theta(\rho, |E|/|B_R|)$, *with* $\theta \in (0, 1)$, *such that*

$$\|f\|_{L^\infty(B_R)} \leq N \left(\fint_E |f| \, dx \right)^\theta M^{1-\theta}.$$

S. Vessella, A continuous dependence result in the analytic continuation problem, Forum Math., 11 (1999), 695–703.

Main results

Based on these two theorems and a telescoping series method introduced in the last talk, we have

The observability estimates (1) and (2) over measurable sets of positive measure in both space and time variables are true.

Main results

Based on these two theorems and a telescoping series method introduced in the last talk, we have

The observability estimates (1) and (2) over measurable sets of positive measure in both space and time variables are true.

Consequently,

$(TP)_1^M$ has B-B-P, i.e., any time optimal control u^* satisfies $|u^*(x,t)| = M$ for a.e. $(x,t) \in \omega \times (0, T(M))$.

Main results

Based on these two theorems and a telescoping series method introduced in the last talk, we have

The observability estimates (1) and (2) over measurable sets of positive measure in both space and time variables are true.

Consequently,

$(TP)_1^M$ has B-B-P, i.e., any time optimal control u^* satisfies $|u^*(x,t)| = M$ for a.e. $(x,t) \in \omega \times (0, T(M))$.

Remark. One can also consider the time optimal boundary control problem and obtain the corresponding bang-bang property.

J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, Observability inequalities and measurable sets, J. Eur. Math. Soc., 16 (2014) 2433-2475.

Extensions

These observability estimates from measurable subsets can also be extended to solutions of the general higher-order parabolic equations with both space and time analytic coefficients.

L. Escauriaza, S. Montaner, C. Zhang, Observation from measurable sets for parabolic analytic evolutions and applications, J. Math. Pures Appl., 104 (2015) 837-867.

L. Escauriaza, S. Montaner, C. Zhang, Analyticity of solutions to parabolic evolutions and applications. Submitted.

Abstract time optimal control problem

- X and U are two Hilbert spaces
- A generate a C_0 semigroup $\{S(t); t \geq 0\}$ on X
- $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $\{S(t);\ t \geq 0\}$, where X_{-1} is the dual of $D(A^*)$ with respect to the pivot space X

The controlled equation reads:

$$\frac{dz}{dt} = Az + Bf,\ t > 0,\ \ z(0) = z_0 \in X,\ \ f \in L^2_{loc}(\mathbb{R}^+; U). \qquad (3)$$

The time optimal control problem is as

$$(TP)^M_2: \qquad T(M) \triangleq \inf_{f \in \mathcal{U}_M} \left\{ t > 0 \ :\ z(t; f, 0, z_0) = z_1 \right\},$$

where $z_1 \in X$ is the target which differs from z_0 and

$$\mathcal{U}_M = \left\{ f : \mathbb{R}^+ \to U \ \text{measurable} \ :\ \|f(t)\|_U \leq M,\ \text{a.e. } t > 0 \right\},$$

with $M > 0$.

In this problem, $T(M)$ is called the optimal time (if it exists), $f^* \in \mathcal{U}_M$ is called an optimal control if $z(T(M); f^*, 0, z_0) = z_1$.

Theorem

Let A generate an analytic semigroup $\{S(t); t \geq 0\}$ in X. Assume that (A^*, B^*) satisfies the observability inequality from time intervals:

$$\|S(L)^*\varphi_0\|_X^2 \leq e^{\frac{d}{L^k}} \int_0^L \|B^*S(t)^*\varphi_0\|_U^2 \, dt,$$

for all $\varphi_0 \in D(A^*)$ and $L \in (0, 1]$, where positive constants d and k are independent of L and φ_0. Then the problem $(TP)_2^M$ holds the bang-bang property.

In this problem, $T(M)$ is called the optimal time (if it exists), $f^* \in \mathcal{U}_M$ is called an optimal control if $z(T(M); f^*, 0, z_0) = z_1$.

Theorem

Let A generate an analytic semigroup $\{S(t); t \geq 0\}$ in X. Assume that (A^*, B^*) satisfies the observability inequality from time intervals:

$$\|S(L)^*\varphi_0\|_X^2 \leq e^{\frac{d}{L^k}} \int_0^L \|B^* S(t)^* \varphi_0\|_U^2 \, dt,$$

for all $\varphi_0 \in D(A^*)$ and $L \in (0,1]$, where positive constants d and k are independent of L and φ_0. Then the problem $(TP)_2^M$ holds the bang-bang property.

Here the bang-bang property means that: any optimal control f^* satisfies $\|f^*(t)\|_U = M$ for a.e. $t \in (0, T(M))$.

G. Wang and C. Zhang. Observability estimate from measurable sets in time for some evolution equations. Submitted.

An Specific Example: Time-optimal controlled Stokes equations

Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with a smooth boundary $\partial\Omega$. Consider the controlled Stokes system

$$\begin{cases} y_t - \Delta y + \nabla p = f, & \text{in } \Omega \times \mathbb{R}^+, \\ \operatorname{div} y = 0, & \text{on } \Omega \times \mathbb{R}^+, \\ y = 0, & \text{in } \partial\Omega \times \mathbb{R}^+, \\ y(\cdot, 0) = y_0, & \text{in } \Omega, \end{cases} \tag{4}$$

where y_0 is arbitrarily fixed in the usual space:

$$L^2_\sigma(\Omega) \triangleq \{y \in (L^2(\Omega))^3 : \operatorname{div} y = 0,\ y \cdot \nu = 0 \text{ on } \partial\Omega\},$$

and f is taken from the control constraint set:

$$\mathcal{U}_M \triangleq \left\{ f = (0, f_2, f_3) \in L^\infty(\mathbb{R}^+; (L^2(\omega))^3) : \ \|f(t)\|_{(L^2(\omega))^3} \leq M, \forall t > 0 \right\},$$

with $M > 0$, $\omega \subset \Omega$ a nonempty open subset.

The time optimal control problem now reads:

$$(TP)_3^M: \qquad T(M) \triangleq \inf_{f \in \mathcal{U}_M} \left\{ t > 0 : \ y(t; f) = 0 \right\}.$$

The time optimal control problem now reads:

$$(TP)_3^M : \qquad T(M) \triangleq \inf_{f \in \mathcal{U}_M} \{ t > 0 : \ y(t; f) = 0 \}.$$

To apply the above-mentioned abstract result to obtain the bang-bang property for $(TP)_3^M$, we write $X = L^2_\sigma(\Omega)$ and $U = \{0\} \times L^2(\omega) \times L^2(\omega)$.

The time optimal control problem now reads:

$$(TP)_3^M: \qquad T(M) \triangleq \inf_{f\in\mathcal{U}_M} \big\{t>0:\ y(t;f)=0\big\}.$$

To apply the above-mentioned abstract result to obtain the bang-bang property for $(TP)_3^M$, we write $X=L^2_\sigma(\Omega)$ and $U=\{0\}\times L^2(\omega)\times L^2(\omega)$. Define the operator A on X by

$$\begin{cases} D(A)=\big(H^2(\Omega)\bigcap H^1_0(\Omega)\big)^3\bigcap L^2_\sigma(\Omega), \\ Ay=P(\Delta y)\ \text{ for all }\ y\in D(A), \end{cases}$$

where P is the Helmholtz projection operator from $(L^2(\Omega))^3$ into X.

The time optimal control problem now reads:

$$(TP)_3^M: \qquad T(M) \triangleq \inf_{f \in \mathcal{U}_M} \left\{ t > 0 : \ y(t; f) = 0 \right\}.$$

To apply the above-mentioned abstract result to obtain the bang-bang property for $(TP)_3^M$, we write $X = L^2_\sigma(\Omega)$ and $U = \{0\} \times L^2(\omega) \times L^2(\omega)$. Define the operator A on X by

$$\begin{cases} D(A) = \left(H^2(\Omega) \bigcap H^1_0(\Omega)\right)^3 \bigcap L^2_\sigma(\Omega), \\ Ay = P(\Delta y) \ \text{ for all } \ y \in D(A), \end{cases}$$

where P is the Helmholtz projection operator from $(L^2(\Omega))^3$ into X. Let $B \in \mathcal{L}(U, X)$ be defined by $Bf = Pf$ for all $f \in U$.

The time optimal control problem now reads:

$$(TP)_3^M: \qquad T(M) \triangleq \inf_{f \in \mathcal{U}_M} \{t > 0 : \ y(t; f) = 0\}.$$

To apply the above-mentioned abstract result to obtain the bang-bang property for $(TP)_3^M$, we write $X = L^2_\sigma(\Omega)$ and $U = \{0\} \times L^2(\omega) \times L^2(\omega)$. Define the operator A on X by

$$\begin{cases} D(A) = \left(H^2(\Omega) \bigcap H^1_0(\Omega)\right)^3 \bigcap L^2_\sigma(\Omega), \\ Ay = P(\Delta y) \ \text{ for all } \ y \in D(A), \end{cases}$$

where P is the Helmholtz projection operator from $(L^2(\Omega))^3$ into X. Let $B \in \mathcal{L}(U, X)$ be defined by $Bf = Pf$ for all $f \in U$. Then, A is self-adjoint and generates an analytic semigroup in X;

The time optimal control problem now reads:

$$(TP)_3^M: \qquad T(M) \triangleq \inf_{f \in \mathcal{U}_M} \{t > 0: \ y(t; f) = 0\}.$$

To apply the above-mentioned abstract result to obtain the bang-bang property for $(TP)_3^M$, we write $X = L^2_\sigma(\Omega)$ and $U = \{0\} \times L^2(\omega) \times L^2(\omega)$. Define the operator A on X by

$$\begin{cases} D(A) = \left(H^2(\Omega) \bigcap H^1_0(\Omega)\right)^3 \bigcap L^2_\sigma(\Omega), \\ Ay = P(\Delta y) \ \text{ for all } \ y \in D(A), \end{cases}$$

where P is the Helmholtz projection operator from $(L^2(\Omega))^3$ into X. Let $B \in \mathcal{L}(U, X)$ be defined by $Bf = Pf$ for all $f \in U$. Then, A is self-adjoint and generates an analytic semigroup in X; B is an admissible control operator for $\{e^{tA}; t \geq 0\}$ and $B^*: X \to U$ is given by

$$B^* \varphi = (0, \chi_\omega \varphi_2, \chi_\omega \varphi_3) \quad \text{for all } \ \varphi = (\varphi_1, \varphi_2, \varphi_3) \in X;$$

The time optimal control problem now reads:

$$(TP)_3^M: \qquad T(M) \triangleq \inf_{f\in\mathcal{U}_M} \{t>0:\ y(t;f)=0\}.$$

To apply the above-mentioned abstract result to obtain the bang-bang property for $(TP)_3^M$, we write $X = L^2_\sigma(\Omega)$ and $U = \{0\}\times L^2(\omega)\times L^2(\omega)$. Define the operator A on X by

$$\begin{cases} D(A) = \left(H^2(\Omega)\bigcap H^1_0(\Omega)\right)^3 \bigcap L^2_\sigma(\Omega), \\ Ay = P(\Delta y) \ \text{ for all } \ y\in D(A), \end{cases}$$

where P is the Helmholtz projection operator from $(L^2(\Omega))^3$ into X. Let $B\in\mathcal{L}(U,X)$ be defined by $Bf = Pf$ for all $f\in U$. Then, A is self-adjoint and generates an analytic semigroup in X; B is an admissible control operator for $\{e^{tA}; t\ge 0\}$ and $B^*: X\to U$ is given by

$$B^*\varphi = (0, \chi_\omega\varphi_2, \chi_\omega\varphi_3) \quad \text{for all} \ \ \varphi=(\varphi_1,\varphi_2,\varphi_3)\in X;$$

The equation (4) can be rewritten as

$$\frac{dy}{dt} = Ay + Bf, \quad t>0. \quad y(0)=y_0.$$

On the other hand, by Coron-Guerrero's work

J. M. Coron, S. Guerrero, Null controllability of the N-dimensional Stokes system with $N-1$ scalar controls, Journal of Differential Equations, 246 (2009), 2908-2921.

there exists a positive constant $C = C(\Omega, \omega)$ such that for each $L \in (0,1]$,

$$\sum_{j=1}^{3} \int_{\Omega} |\varphi_j(x,L)|^2 \, dx \leq e^{\frac{C}{L^9}} \int_0^L \int_{\omega} |\varphi_2(x,t)|^2 + |\varphi_3(x,t)|^2 \, dxdt$$

for all $\varphi_0 \in L^2_\sigma(\Omega)$, where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ solves the equation

$$\begin{cases} \varphi_t - \Delta\varphi + \nabla p = 0, & \text{in } \Omega \times (0,L), \\ \operatorname{div} \varphi = 0, & \text{in } \Omega \times (0,L), \\ \varphi = 0, & \text{in } \partial\Omega \times (0,L), \\ \varphi(\cdot, 0) = \varphi_0. \end{cases}$$

In other words, the pair (A^*, B^*) satisfies observability inequality:

$$\|e^{LA^*}\varphi_0\|_X^2 \leq e^{\frac{C}{L^9}} \int_0^L \|B^* e^{tA^*}\varphi_0\|_U^2 \, dt \;\text{ for all }\; \varphi_0 \in X \;\text{ and }\; L \in (0,1].$$

In other words, the pair (A^*, B^*) satisfies observability inequality:

$$\|e^{LA^*}\varphi_0\|_X^2 \leq e^{\frac{C}{L^9}} \int_0^L \|B^* e^{tA^*}\varphi_0\|_U^2 \, dt \ \text{ for all } \ \varphi_0 \in X \ \text{ and } \ L \in (0,1].$$

Therefore, we have

Corollary

Problem $(TP)_3^M$ has the bang-bang property.

In other words, the pair (A^*, B^*) satisfies observability inequality:

$$\|e^{LA^*}\varphi_0\|_X^2 \leq e^{\frac{C}{L^9}} \int_0^L \|B^* e^{tA^*}\varphi_0\|_U^2\, dt \ \text{ for all } \ \varphi_0 \in X \ \text{ and } \ L \in (0,1].$$

Therefore, we have

Corollary

Problem $(TP)_3^M$ has the bang-bang property.

Remark. In this time-optimal control problem, when the control constraint set is replaced by the pointwise-type:

$$\left\{f \in L^\infty(\Omega \times \mathbb{R}^+)^3 : \ |f_i(x,t)| \leq M, \ (x,t) \in \Omega \times \mathbb{R}^+, i = 1,2,3\right\},$$

the corresponding bang-bang property is much more difficult. This work is now in process co-worked with Felipe and Diego.

Including remarks

Different kinds of observability inequality from measurable subsets imply different versions of bang-bang property for time optimal control problems, as well as norm optimal control problems.

Including remarks

Different kinds of observability inequality from measurable subsets imply different versions of bang-bang property for time optimal control problems, as well as norm optimal control problems.

When the controlled system is not "time-invariant", we still do not know how to derive the bang-bang property even if we have established the corresponding observability inequality from measurable subsets.

For the progress of B-B-P, we refer the following recent work, which in particular derive, but from another aspect, the bang-bang property of time optimal control problem for the heat equation with a potential $a(x,t) = a(x) + b(t) \in L^\infty$.

G. Wang, Y. Xu, and Y. Zhang. Attainable subspaces and the bang-bang property of time optimal controls for heat equations. SIAM J. Control Optim., 2015.

Study the bang-bang property of optimal control problems to the general parabolic equation with some nonlinearity terms. On this issue, the bang-bang property for the semilinear heat equation with global Lipschitz nonlinearity and a good sign condition is proved. For the general case, it is still open.

K. D. Phung, L. Wang and C. Zhang, Bang-bang property for time optimal control of semilinear heat equation. Annales de l'Institut Henri Poincaré (C) Non Linear Analysis, 31 (2014), 477-499.

K. Kunisch, L. Wang, Bang-bang property of time optimal controls of semilinear parabolic equation. Discrete Contin. Dyn. Syst. 36 (2016), 279-302.

Thank you for your attention!