A uniform controllability result for the Keller-Segel system

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Chemotaxis: *description of the change of motion when a population formed of individuals (such as amoebae, bacteria, endothelial cells, etc.) reacts in response (taxis) to an external chemical stimulus spread in the environment where they reside.*

- tumor angiogenesis
- inflammatory response
- wound healing
- embryology
- bacterial colony growth, pattern formation
- ...

The (one species) Keller-Segel system of chemotaxis:

$$\left|\begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{in } \ Q := \Omega \times (0, T), \\ \epsilon v_t - \Delta v = au - bv & \text{in } \ Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \ \Sigma := \partial\Omega \times (0, T), \\ u(x,0) = u_0; \ v(x,0) = v_0 & \text{in } \ \Omega, \end{array}\right. \tag{1}$$

where $a, b > 0$ and $0 < \epsilon \leq 1$.

Here:

- $u = u(x, t)$: population density;
- $v = v(x, t)$: density of the chemical.

We assume:

$$\Omega \subset \mathbb{R}^N, \ N = 2, 3.$$

For all $t > 0$, we have the mass preservation:

$$\frac{1}{|\Omega|} \int_{\Omega} u(x,t)dx = \frac{1}{|\Omega|} \int_{\Omega} u_0(x)dx. \tag{2}$$

An important aspect of system (1) is the expected onset of chemotactic collapse. In other words, under suitable circumstances, the whole population concentrate in a single point (spora) in finite time.

In mathematical terms, this means formation of a Dirac delta-type singularity in finite time, i.e.,

$$u(x,t) \longrightarrow M\delta(x_0) \quad \text{as} \quad t \longrightarrow T, \tag{3}$$

for some $T < \infty$, where x_0 is the point where the spora develops, and

$$M = \frac{1}{|\Omega|}\int_\Omega u(x,t)dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx.$$

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Singularities (infinite mass at a point in space) occur in finite time for large data, while smooth solutions exist globally for small data.

Control problem

Given $0 < \epsilon \leq 1$, $T > 0$ and $(\bar{u}, \bar{v})$, solution of (1), find a control g^ϵ such that the associated solution (u, v) of

$$\left| \begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{in} \quad Q, \\ \epsilon v_t - \Delta v = au - bv + g^\epsilon \chi & \text{in} \quad Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \Sigma \\ u(x,0) = u_0; \ v(x,0) = v_0 & \text{in} \quad \Omega, \end{array} \right. \tag{4}$$

satisfies

$$u(T) = \bar{u}(T), \ v(T) = \bar{v}(T) \tag{5}$$

and

$$g^\epsilon \text{ is bounded independently from } \epsilon. \tag{6}$$

Here, $\chi : \mathbb{R}^N \to \mathbb{R}$ is a C^∞ function such that *supp* $\chi \subset\subset \omega$, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in ω', for some $\omega' \subset\subset \omega \subset \Omega$.

The importance of (6) is due to the fact that, in many applications[1], for sufficiently small ϵ, system (1) is approximated by

$$\left| \begin{array}{ll} u_t - \Delta u = -\nabla \cdot (u \nabla v) & \text{in} \quad Q, \\ -\Delta v = au - bv & \text{in} \quad Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \Sigma, \\ u(x,0) = u_0; \;\; v(x,0) = v_0 & \text{in} \quad \Omega. \end{array} \right. \tag{7}$$

[1] See [1] and [7].

Main Result

Theorem (C.-S. & Guerrero)

Let $0 < \epsilon \leq 1$ and let $(M_1, M_2) \in \mathbb{R}^2_+$ be such that $aM_1 - bM_2 = 0$. Then, there exists $\delta > 0$ such that, for any $(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega)$ with $u_0, v_0 \geq 0$, satisfying $\frac{1}{|\Omega|}\int_\Omega u_0 dx = M_1$, $\frac{\partial v_0}{\partial \nu} = 0$ on $\partial\Omega$ and $||(u_0 - M_1, v_0 - M_2)||_{H^1(\Omega)\times H^2(\Omega)} \leq \delta$, there exists $g^\epsilon \in L^2(0, T; H^1(\Omega))$, with $\|g^\epsilon\|_{L^2(0,T;H^1(\Omega))}$ bounded independently from ϵ, such that the associated solution (u, v) of (4) satisfies:

$$(u(T), v(T)) = (M_1, M_2) \text{ in } \Omega.$$

How do we do?

Linearization + (Uniform) Inverse mapping theorem

Linearization

We linearize system (4) around (M_1, M_2):

$$\left| \begin{array}{ll} u_t - \Delta u = -M_1 \Delta v + h_1 & \text{in} \quad Q, \\ \epsilon v_t - \Delta v = au - bv + g^\epsilon \chi + h_2 & \text{in} \quad Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on} \quad \Sigma, \\ u(x,0) = u_0; \;\; v(x,0) = v_0 & \text{in} \quad \Omega, \end{array} \right. \tag{8}$$

where h_1 and h_2 are given exterior forces belonging to appropriate Banach spaces and having exponential decay at $t = T$.

We prove the following result for (8):

Theorem (2)

Let $0 < \epsilon \leq 1$ *and* $(M_1, M_2) \in \mathbb{R}^2$ *be such that* $aM_1 - bM_2 = 0$. *Assume that:*

$$(u_0, v_0) \in H^1(\Omega) \times H^2(\Omega), \ \int_\Omega u_0 dx = 0, \ \frac{\partial v_0}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{9}$$

and

$$h_1, h_2 \text{ have "good" exponential decay at } t = T. \tag{10}$$

Then there exists a control $g^\epsilon \in L^2(0, T; H^1(\Omega))$, *bounded independently from* ϵ, *such that, if* (u, v) *is the associated solution of* (8), *then*

$$u(T) = v(T) = 0.$$

Theorem (2) is consequence of a Carleman inequality for the adjoint system of (8).

The adjoint system of (8) reads:

$$\left|\begin{array}{ll} -\varphi_t - \Delta\varphi = a\xi + f_1 & \text{in } Q, \\ -\epsilon\xi_t - \Delta\xi = -b\xi - M_1\Delta\varphi + f_2 & \text{in } Q, \\ \frac{\partial\varphi}{\partial\nu} = \frac{\partial\xi}{\partial\nu} = 0 & \text{on } \Sigma, \\ \varphi(x,T) = \varphi_T;\ \xi(x,T) = \xi_T & \text{in } \Omega. \end{array}\right. \tag{11}$$

The (uniform) Carleman inequality we prove for (11) is the following.

Theorem

Given $0 < \epsilon \leq 1$, there exists $C = C(\Omega, \omega)$ such that, for any $(\varphi_T, \xi_T) \in L^2_0(\Omega) \times L^2(\Omega)$ and any $f_1, f_2 \in L^2(Q)$, the solution (φ, ξ) of system (11) satisfies:

$$\iint_Q e^{-C/t^m}|\varphi - (\varphi)_\Omega(t)|^2 dxdt + \iint_Q e^{-C/t^m}|\xi|^2 dxdt$$

$$\leq C\left(\iint_{\omega\times(0,T)} e^{-C/t^m}|\xi|^2 dxdt + \iint_Q e^{-C/t^m}(|f_1|^2 + |f_2|^2)dxdt\right),$$

for some $m > 1$, where $(\varphi)_\Omega(t) = \frac{1}{|\Omega|}\int_\Omega \varphi(x,t)dx$.

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$$\iint_Q e^{-C/t^m} |\varphi - (\varphi)_\Omega(t)|^2 dxdt + \iint_Q e^{-C/t^m} |\xi|^2 dxdt$$

$$\leq C\left(\iint_{\omega\times(0,T)} e^{-C/t^m} |\xi|^2 dxdt + \iint_Q e^{-C/t^m} (|f_1|^2 + |f_2|^2) dxdt \right),$$

for some $m > 1$, *where* $(\varphi)_\Omega(t) = \frac{1}{|\Omega|} \int_\Omega \varphi(x,t) dx$.

The proof of this Carleman inequality is divided into 3 steps.

Step 1. Carleman Inequality for $\Delta\varphi$.

We write $e^{-C/t^m}\varphi = \eta + \psi$, where η solves a heat equation with f_1 as a right-hand side term and ψ solves a heat equation with a right-hand side in $H^1(0, T; H^2(\Omega))$. We consider the equation satisfied by ψ and apply the usual Carleman inequality for the heat equation with Neumann boundary for ψ and combine it with the usual energy estimates for the equation satisfied by η. This gives a global estimate of $\Delta\varphi$ in terms of a local integral of $\Delta\psi$ and global integrals of $\Delta\xi$ and f_1.

Step 2. Carleman inequality for ξ.

In the second step we obtain a Carleman inequality for ξ, with a precise dependence of the degenerating parameter multiplying the time derivative and combine it with the Carleman inequality obtained in step 1. This gives a global estimate of ξ and $\Delta\varphi$ in terms of local integrals of ξ and $\Delta\psi$ and global integrals of f_1 and f_2.

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Step 3. Estimate of the local integral of $\Delta\psi$.

In the last step we estimate a local integral of $\Delta\psi$ in terms of a local integral of ξ and global integrals of f_1 and f_2. Combining steps 2 and 3 we finish the proof of the Carleman inequality.

Inverse mapping theorem

We change our controllability problem for the Keller-Segel system into a null controllability problem by writing the solution (u,v) of the Keller-Segel system as $u = M_1 + z$ and $v = M_2 + w$, where (z,w) solves

$$\left|\begin{array}{ll} L(z,w) = (-\nabla\cdot(z\nabla w), g\chi) & \text{in } Q, \\ \frac{\partial z}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0 & \text{on } \Sigma, \\ z(x,0) = u_0 - M_1;\ w(x,0) = v_0 - M_2 & \text{in } \Omega \end{array}\right. \tag{12}$$

with

$$L(u,v) = (u_t - \Delta u + M_1\Delta v, \epsilon v_t - \Delta v + bv - au). \tag{13}$$

We have $(u(T), v(T)) = (M_1, M_2)$ if and only if $(z(T), w(T)) = 0$.

We apply Liusternik's inverse mapping theorem:

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We have $(u(T), v(T)) = (M_1, M_2)$ if and only if $(z(T), w(T)) = 0$.

We apply Liusternik's inverse mapping theorem:

Theorem

Let E and G be two Banach spaces and let $\mathcal{A} : E \to G$ be a continuous function from E to G defined in $B_\eta(0)$ for some $\eta > 0$ and $\mathcal{A}(0) = 0$. Let Λ be a continuous and linear operator from E onto G and suppose there exists $C_0 > 0$ such that

$$||e||_E \leq C_0||\Lambda(e)||_G \tag{14}$$

and that there exists $\delta < C_0^{-1}$ such that

$$||\mathcal{A}(e_1) - \mathcal{A}(e_2) - \Lambda(e_1 - e_2)|| \leq \delta||e_1 - e_2|| \tag{15}$$

whenever $e_1, e_2 \in B_\eta(0)$. Then the equation $\mathcal{A}(e) = h$ has a solution $e \in B_\eta(0)$ whenever $||h||_G \leq c\eta$, where $c = C_0^{-1} - \delta$.

Remark

In the case where $\mathcal{A} \in C^1(E;G)$,*using the mean value theorem, it can be shown, that for any* $\delta < C_0^{-1}$, *inequality* (15) *is satisfied with* $\Lambda = \mathcal{A}'(0)$ *and* $\eta > 0$ *the continuity constant at zero, i. e.,*

$$||\mathcal{A}'(e) - \mathcal{A}'(0)||_{\mathcal{L}(E;G)} \leq \delta \qquad (16)$$

whenever $||e||_E \leq \eta$.

We define:

$$G = X \times Y,$$

$$X = \{(h_1, h_2);\ e^{C/t^m} h_1 \in L^2(Q), e^{C/t^m} h_2 \in L^2(0, T; H^1(\Omega)) \\ \text{and } \int_\Omega h_1(x,t)dx = 0 \text{ a. e. } t \in (0,T)\}, \tag{17}$$

$$Y = \{(z_0, w_0) \in H^1(\Omega) \times H^2(\Omega);\ \int_\Omega z_0 dx = 0 \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega\}, \tag{18}$$

$$E = \Big\{(u, v, g) : e^{C/t^m} g\chi, e^{C/t^m} \big(L(u,v)\big)_1 \in L^2(Q), \\ e^{C/t^m}\Big(\big(L(u,v)\big)_2 - g\chi\Big) \in L^2(0, T; H^1(\Omega)), \\ \int_\Omega \big(L(u,v)\big)_1 dx = 0 \text{ and } \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \Sigma\Big\}$$

and, for each $0 < \epsilon \leq 1$, the operator

$$\mathcal{A}(z, w, g) = (L(u,v) + ((\nabla \cdot (z\nabla w), -g\chi)), z(.,0), w(.,0))\ \forall (z, w, g) \in E.$$

We have

$$\mathcal{A}'(0,0,0) = (L(u,v) + (0, -g\chi)), z(.,0), w(.,0)) \ \forall (z,w,g) \in E.$$

It is not difficult to show that the operator $\mathcal{A}$ fits the regularity required to apply Liusternik's theorem and hence obtain the uniform null controllability of the Keller-Segel system around (M_1, M_2).

Comments

- We are able to avoid the blow up.
- It would be interesting to analyze the controllability of the Keller-Segel system (1) around non constant trajectories, even for a fixed ϵ. The linearization around $(\bar{u}, \bar{v})$ reads:

$$\left| \begin{array}{ll} u_t - \Delta u = \nabla \cdot (\bar{u} \nabla v) + \nabla \cdot (u \nabla \bar{v}) + h_1 & \text{in } Q, \\ \epsilon v_t - \Delta v = au - bv + g^\epsilon \chi + h_2 & \text{in } Q, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 & \text{on } \Sigma, \\ u(x,0) = u_0; \; v(x,0) = v_0 & \text{in } \Omega. \end{array} \right. \tag{19}$$

Other related systems

Keller-Segel + Stokes:

$$\left| \begin{array}{lll} u_t + y \cdot \nabla u - \Delta u = \nabla \cdot (u \nabla v) & \text{in} & Q, \\ v_t + y \cdot \nabla v - \Delta v = au - bv & \text{in} & Q, \\ y_t - \Delta y + \nabla \Pi = (n, 0, 0) & \text{in} & Q \\ \nabla \cdot y = 0 & \text{in} & Q. \end{array} \right. \tag{20}$$

Two species Keller-Segel:

$$\left| \begin{array}{lll} u^1_t - \Delta u^1 = -\nabla \cdot (u^1 \nabla v) & \text{in} & Q, \\ u^2_t - \Delta u^2 = -\nabla \cdot (u^2 \nabla v) & \text{in} & Q, \\ v_t - \Delta v = a^1 u^1 + a^2 u^2 - bv & \text{in} & Q. \end{array} \right. \tag{21}$$

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Thank you!!