Stabilization Problems of Unforeseen Difficulty

The IHP Coron Conference on Nonlinear Partial Differential Equations and Applications
June 21, 2016

Roger Brockett
John A. Paulson School of Engineering and Applied Science
Harvard University
We are here to honor Jean-Michel

Let the festivities continue with a little WHOOP DE DOO on feedback stabilization!
A little dictum from many years ago

As a fourth year college student I took a course called “Scientific Writing”. For the most part this consisted of submitting essays and receiving criticism in response. However the professor did insist on one general principle. It takes the form of something like a contract between the author and the reader and I will try to observe it today. Here it is:

Never overestimate the prior knowledge of the reader and never underestimate the reader’s willingness to follow a logical argument. Shall I explain whoop de doo?
Sufficient conditions for the existence of a continuous, autonomous, stabilizing control

\[ \dot{x} = f(x, u) \; ; \; f(0, 0) = 0 \]

or

\[ \dot{x} = f(x) + \sum u_i g_i(x) \; ; \; f(0) = 0 \]

There are several versions depending on the type of feedback allowed and the type of stability of interest. Here we discuss the existence and construction of continuous, time invariant feedback control giving rise to (local) asymptotic stability.

This is the only problem I intend to talk about!
Notation

If \( u \) is a scalar sometimes it is convenient to write

\[
x^{(n)} + f(x, x^{(1)}, \ldots, x^{(n-1)}) + u g(x, x^{(1)}, \ldots, x^{n-1})
\]

corresponding to

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ f(x) \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(x) \end{bmatrix}
\]

Most of our examples are of this type.
Many questions, and a few answers

1. Is there an algorithmic approach to finding a stabilizing control?

2. Can known topological conditions such as, the index of a fixed point, etc. be extended in the direction of the Routh-Hurwitz test?

3. Can the connection with optimal regulation be made more useful?

4. Does there exist a useful classification in terms of normal forms?

5. What is the best rate of convergence in a given setting, 1/t, etc.? 

6. Assuming asymptotic stability, polynomial Liapunov functions exist but what is the lowest degree possible?

7. Is there a way to make simulation decisive for convergence to 0, even for a single trajectory?
Some (now quite old) background

At one time many suspected that the fact that for linear time invariant systems, controllability is sufficient to assure the existence of a continuous, time invariant stabilizing control law would remain true in something like full generality for nonlinear systems, at least as a local property.

The system
\[
\begin{align*}
\dot{x}_1 &= u_1 \\
\dot{x}_2 &= u_2 \\
\dot{x}_3 &= x_1 u_2 - x_2 u_1
\end{align*}
\]
provides a counter example and, with modification, this fact has found a number of applications, especially in the field of mobile robotics.

It has been known for several decades that there is an additional necessary condition, namely that for \( \dot{x} = f(u, u) \), the \( f \) appearing on the right-hand side should define a map that is onto the origin.
We do not wish to discuss linear or near linear cases

To investigate the local stability of the null solution of \( \dot{x} = f(x) \) it is often proposed to use \( v(x) = x^T Q x \) where \( A^T Q + Q A = -I \) and \( A \) is the Jacobian of \( f \) evaluated at zero. If \( \dot{x} = Ax + bu \) and \( (A, b) \) is controllable then it is known that one can select a vector \( c \) such that the solution of 
\[
(A + bc^T)^T Q + Q (A + bc^T) = -I \quad \text{with} \quad Q > 0.
\]

In fact, quadratic Lyapunov functions always exist for noncritical systems if the null solution is stable.
Restrict attention to the critical cases

The critical case for a differential equation
\[ \dot{x} = f(x) \; ; \; f(0) = 0 \; ; \; \frac{\partial f}{\partial x} |_0 = A \]
All the eigenvalues of \( A \) are in the closed left half-plane and one or more are on the imaginary axis.

The control version of “critical”

Critical cases for stabilization of
\[ \dot{x} = f(x) + g(x)u \; ; \; f(0,0) = 0 \]
When \( \dot{x} = f(x,0) \) is critical as above and the ad chain, \( g, [f,g], [f, [f,g]], \cdots \) does not span.
(Similar for multi input case.)

Critical cases for ODEs in the Russian literature of the 1950s
I will assume $u(0)=0$; this is not quite standard

As discussed above, we will assume that $u$ is continuous and does not depend explicitly on time. In addition, we focus on the case $u(0) = 0$ because if $u(x) = u_0 + u_1(x)$ with $u_1(0) = 0$ then we can replace the original system by

$$\dot{x} = f(x) + g(x)u_0 + g(x)u_1(x)$$

Moreover, there are often technological reasons for wanting $u$ to go to zero when $x$ does. These could relate to power consumption and/or robustness.

After choosing the constant we still have the problem of finding $u$. 
“Obvious” possibilities for attacking the problem

1. Use insight from Routh-Hurwitz

2. Consider an optimal control problem
   \[ \dot{x} = f(x, u) ; \quad \eta = \int_0^\infty L(x, u) \, dt \]
   if \( L(x, u) \) is positive for \( x, u \neq 0 \) then then if
   a solution exists \( x \) and \( u \) will need to go to 0
   If the optimal control in feedback form depends
   continuously on \( x \) then this would be a stabilizing
   solution.

3. Lyapunov functions (related to HJB as above or
   through passivity such as in KYP lemma, or \( \cdots \))
Does Routh-Hurwitz help in this setting?

Recall that the third order system
\[ x^{(3)} + ax^{(2)} + bx^{(1)} + cx = 0 \]
is asymptotically stable if \( a, b, c, \) are positive and \( ab - c > 0 \)

What, if anything, can this tell us about stabilizing
\[ x^{(3)} + ux = 0 \]
\[ \frac{d^3 x}{dt^3} + x^2 \left( \ddot{x} + 10 \dot{x} + x \right) = 0 \]

Period about 3.3

-0.5 to 0.5
\[
\frac{d^3 x}{dt^3} + x^2 (\ddot{x} + \dot{x} + x) = 0
\]
\[
\frac{d^3 x}{dt^3} + x^2 (\ddot{x} + 0.1 \dot{x} + x) = 0
\]
None of these work!
Optimal Regulation and Factorization
In the context of the Euler-Lagrange equations formulated on the tangent bundle, finding a stabilizing control law comes down to “factoring” the Euler-Lagrange equations in such a way get an equation of order $n$ rather than $2n$. In the context of linear time invariant systems the factorization amounts to the use of spectral factorization pioneered by Norbert Wiener.

\[ \dot{x} = -x + u \; ; \; L(x, u) = x^2 + u^2 \]
\[ \frac{d^2 x}{dt^2} + 2x = 0 \]
\[ (D + 1)(D - 1) + 1 = (D + \sqrt{2} - 1)(D - \sqrt{2} - +1) \]

Finding an invariant submanifold for the flow, which contains $0$. 

Difficulties

Example:
\[ \ddot{x} = ux ; \quad L = x^2 + u^2 \]
Euler-Lagrange is fourth order and can be factored to get the optimal regulator equation in the form
\[ \ddot{x} + ux^2 = 0 \]
with \( u(x, \dot{x}) \) taking on only the values \( \pm 1 \).

Surprisingly (\?), the optimal control is not continuous even though the description looks quite friendly.
Smooth solutions of the regulator problem

Again, by the regulator problem we mean the problem of minimizing
\[ \eta = \int_0^\infty L(x, u) \, dt \]
for the system \( \dot{x} = f(x, u) \). Notice that if there were
to be a smoothly differentiable value function \( v(x) \)
expressing the minimum value of \( \eta \) as a function of
the initial state \( x \) then \( v(x) \) would be a Lyapunov
function with derivative \(-L\). Thus the value function
can not be smooth if \( f \) is not onto a neighborhood of
the origin.
Organizing for a war of attrition

Suppose that $f$ and $g$ are analytic with first order terms given by

$$f(x) = Ax + \phi(x) ; \quad g(x) = Bx + \psi(x)$$

The closed loop system takes the form

$$\dot{x} = Ax + \phi(x) + (Bx + \psi(x))u$$

Thus we see that while the higher order terms in $f$ will need to be accounted for, the lowest order terms in $g(x)u(x)$ come from $Bxu(x)$. In particular, if $f(x)$ is linear then the lowest order approximation is given by $\dot{x} = Ax + uBx$
Focus on a more specific class of problems

Given system of the form
\[ \dot{x} = Ax + \left( \sum u_i B_i \right) x \]
when does there exist \( u_i(x) \), continuous and vanishing at zero, such that the null solution is asymptotically stable, and if such a solution is known to exist, give a procedure for finding it.

A satisfactory answer must be \( \mathcal{G}l(n) \) invariant in the sense that \( (A, B_i) \) and \( PAP^{-1}, PB_iP^{-1} \) are equivalent. It is necessary that the joint null space of \( A \) and \( B_1, B_2, \cdots B_m \) should be zero dimensional.
How hard is it to find a suitable Liapunov function?

The multiplicity of the eigenvalues on the imaginary axis relates to the possible Lyapunov functions as follows. If $A$ and $Q \geq 0$ are related by

$$QA + A^TQ \leq 0$$

and if all the eigenvalues of $A$ have negative real parts except for an eigenvalue of geometric multiplicity $k$ at 0, then the dimension of the null space of $Q$ is at least $k - 1$. To prove this observe that one may as well take $A$ to be in Jordan normal formant even restrict $A$ and $Q$ to the block corresponding to the block associated with $\lambda = 0$. 
The multiplicity one case follows a known pattern

Suppose that \( \dot{x} = Ax + f(x) + ug(x) \) such that the eigenvalues if \( A \) that are on the imaginary axis all being of geometric multiplicity one. Then there exist \( Q = Q^T \geq 0 \) of rank \( n - k \) where \( n \) is the dimension of \( x \) and \( k \) is the number of zero eigenvalues of \( A \). If there is a single zero eigenvalue and \( A \) is in Jordan form

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots \\
0 & 0 & 1 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots 
\end{bmatrix}
\]

Then it is natural to look for a vector \( h \) such that \( x^TQx + \int_0^x h^T f(x)dx \) is path independent and positive definite and to let \( u = -ke_n^Tg \)

Recall the Lur’e problem in feedback stability
The higher multiplicity situation

The higher geometric multiplicity cases pose a more difficult situation because it is necessary to incorporate additional nonlinear terms into the Lyapunov function. More concretely, compare $\ddot{x} = u x$ and $x^{(3)} = u x$. For the second order case we can stabilize with $u = -x(\dot{x} + x)$ which gives $\ddot{x} + (\dot{x} + x)x^2 = 0$.

$v(x) = \dot{x}^2 + (1/2)x^4$ and $\dot{v} = -2\dot{x}^2$

The third order system requires two nonlinear terms but we will provide a “non Lyapunov” approach later.
Standard forms: The invariants of matrix pairs

There is considerable work relating to the \(Gl(n)\) classification of matrix pencils and even families such as \(\{A, B_1, B_2 < \cdots , B_m\}\) but here we cannot allow linear combinations, just changes of basis. In that setting it seems difficult to find anything of general utility. In the case of a matrix pair, \((A, B)\) one has the possibility to factor \(B\) as \(B = GH\), with the ranks of \(G\) and \(H\) being equal and equal to the rank of \(B\). Then the quantity
\[G(s) = G(I_s - A)^{-1}H\]
is invariant under \(A \mapsto PAP^{-1}\) and \(B \mapsto PBP^{-1}\), although different factorizations of \(B\) give different values for \(G\) except in the rank one case.
Standard forms: B of rank one

If $f$ and $g$ are analytic then the system has a bilinear approximation, $\dot{x} = Ax + uBx$ and it makes sense to study the stabilization of $\dot{x} = Ax + uBx$. If the rank of $B$ is one and if $B$ is written as $B = bc^T$ then the rational function $g(s) = c^T(Is - A)^{-1}b$ is a characterizing invariant under a change of coordinates $x \mapsto Px$. At present we have only partial answers to the question of when, for a given $g(s)$, there exists a continuous feedback control law $u(x)$ which stabilizes $\dot{x} = Ax + u(x)bc^Tx$. 
What does the KYP lemma tell us?

Suppose that \((A, b, c)\) is controllable and observable
Consider \(\dot{x} = Ax + ubc^T x\) with \(c^T (Is - A)^{-1} b = g(s)\)
The control law
\(u = -(c^T x)^2\)
makes the null solution asymptotically stable
provided that for some \(\alpha > 0\) and \(\beta \geq 0\)
\(\Re(\alpha i\omega + \beta)g(i\omega) \geq 0\)
for all \(\omega\).

No restriction on the order but multiplicity <3
Stability preserving enlargements (SPE)

Let \((x, y) \in \mathbb{R}^n \times \mathbb{R}^1\) satisfy
\[
\dot{x} = f(x) + g(x)y \\
\dot{y} = -r(x)y
\]
with \(f\) Lipschitz and \(g(x)\) bounded in a neighborhood of 0. Suppose that \(f(0) = 0\) and that \(x \equiv 0\) is an asymptotically stable solution of \(\dot{x} = f(x)\). If \(r(x) \geq 0\) and not identically zero along any nonzero solution of
\[
\dot{x} = f(x) + g(x)\beta
\]
with \(\beta\) constant then the null solution of the \((x, y)\) system is asymptotically stable.
The Lemma in pictures

\[ \dot{x} = f(x) + g(x)y \]

\[ \dot{y} = -r(x)y \]

\( r(x) \) nonnegative
Proof of Lemma part I

1. From the Kurzweil-Messara theorem on the existence of Liapunov functions there exists \( v \geq 0 \) such that its derivative along solutions of \( \dot{x} = f(x) \) is strictly negative except when \( x = 0 \) and because \( g(x) \) is bounded we may conclude, that given \( \epsilon > 0 \) there exists \( \beta > 0 \) such that for \( x \in \{ x \mid v(x) = \epsilon \} \) the derivative of \( v \) along \( \dot{x} = f(x) + g(x)y \) is negative for all \( y \) such that \( |y| \leq \beta \).

2. Clearly \( |y(t)| \) is monotone decreasing and bounded from below by 0. Thus \( y \) has a limit as \( t \) goes to infinity. Denote the limit as \( y^* \).
Proof of Lemma part II

3. If $y^* = 0$ then $y(t)$ remains 0 and $x$ approaches zero.

4. Let $\Gamma$ be the positive limiting set (the $\omega$ set) for $\dot{x} = f(x) + g(x)y$; $\dot{y} = -r(x)y$
   As in Krasovskii-LaSalle, invoking Birkhoff’s theorem, $(x, y)$ approaches $\Gamma$. If, as we have assumed, there is no solution in $\Gamma$ with $r \equiv 0$ except the null solution then $(x, y)$ approaches the null solution.
Applying the lemma

Consider the $n^{th}$ order differential equation in a factored form
\[
\left( \frac{d}{dt} + r(z) \right) \left( \frac{d^{n-1}z}{dt^{n-1}} + \phi(d^{n-2}z/dt^{n-2}, \ldots, z) \right) = 0
\]

If $r(z)$ is as above and if the equation
\[
\frac{d^k z}{dt^k} + \phi(d^{k-1}z/dt^{k-1}, \ldots, z) = 0
\]
is asymptotically stable then so is the enlarged system.
Applying the lemma

Building on the fact that the control
\( u = x(\dot{x} + x) \) stabilizes the null solution of
\( \ddot{x} + ux = 0 \) we see that
\[
\frac{d}{dt} + x^2 \left( \ddot{x} + (\dot{x} + x)x^2 = 0 \right) = 0
\]
has an asymptotically stable null solution and takes the form
\[
x^{(3)} + 2x^2\ddot{x} + (2x\dot{x} + 3x^2 + x^4)\dot{x} + x^5 = 0
\]

Theorem: for any \( k \geq 2 \) the null solution of
\( x^{(k)} + ux = 0 \) can be stabilized by a polynomial feedback Law.
Summary

\[ g(s) = \text{tr}B(\text{Is} - A)^{-1} \]
Conditions under which we know that there exits a stabilizing control law for \( \dot{x} = Ax + uBx \) when \( B \) is rank one and the degree of \( g \) is the dimension of \( A \).

1. \( (\alpha + \beta s)g(s) \) is positive real

2. \( g(s) = (\alpha + \beta s)/s^n \)
Power law solutions provide insight on scale

What is the structure of a higher order scalar polynomial differential equation which admits a solution of the form $at^\alpha$? Observe that the $i^{th}$ derivative is a multiple of $t^{\alpha - i}$ and that the $k^{th}$ power has exponent $k(\alpha - i)$. We will say that a term in a $n^{th}$ order equation has an $\alpha$-weight of $r$ according to

$$x^{k_0} \ddot{x}^{k_1} \dot{x}^{k_2} \cdots \mapsto r = k_0\alpha + k_1(\alpha_1 - 1) + k_2(\alpha_2 - 2) \cdots$$

is solved for the highest derivative is $\alpha$-homogenous of weight $r$ if all the terms are of equal weight.

Example of a homogenous equation of weight = -2.

$$x^{(3)} + |x|\ddot{x} + x^2 \dot{x} + .1|x|\dot{x}^3 = 0$$
A numerical solution of

\[ x^{(3)} + |x|\ddot{x} + x^2 \dot{x} + 0.1|x|x^3 = 0 \]
Observe that one can give this an algebraic flavor

If we regard $x$, $\dot{x}$, $\ddot{x}$ etc. as being algebraically independent and consider the set of all real real linear combinations of monomials the set is closed under addition and multiplication and can be considered to be a ring. The set of possible controls for $x^{(3)} + u = 0$ are limited to the ideal consisting those elements containing $x$ as a factor.
Recall the index calculation

It is known that the index of a critical point defined by an analytic vector field can be computed on the basis its jet if the critical point is isolated. (Eisenbud et al.) In the special case of $\dot{x} = Ax + uBx$ in which $B$ is rank one, the control is able to get rid of terms that are divisible by its multiplier. This suggests that the quotient ring should contain all the information needed to understand the problem.
Many questions, and a few answers

1. Is there an algorithmic approach to finding a stabilizing control?

2. Can known topological conditions such as, the index of a fixed point, etc. be extended in the direction of the Routh-Hurwitz test?

3. Can the connection with optimal regulation be made more useful?

4. Does there exist a useful classification in terms of normal forms?

5. What is the best rate of convergence in a given setting, 1/t, etc.?

6. Assuming asymptotic stability, polynomial Lyapunov functions exist but what is the lowest degree possible?

7. Is there a way to make simulation decisive for convergence to 0, even for a single trajectory?
Jean-Michel

Congratulations on your many achievements!

Best wishes for the years ahead!